

Created by T. Madas

CONSISTENCY OF EQUATIONS

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Question 1 (***)

The system of simultaneous equations

$$x + 2y + z = 1$$

$$2x + 3y + z = 3$$

$$3x + 4y + z = k$$

where k is a scalar constant, does not have a unique solution, but is consistent.

- a) Determine the value of k .
- b) Show that the general solution of the system can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t+3 \\ -t-1 \\ t \end{pmatrix},$$

where t is a scalar parameter.

$$k=5$$

(a) $\begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 2 & 3 & 1 & | & 3 \\ 3 & 4 & 1 & | & k \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & -1 & 1 & | & 1 \\ 0 & -2 & k-3 & | & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & -1 & 1 & | & 1 \\ 0 & 0 & k-5 & | & -2 \end{bmatrix}$

THE LAST ROW BECOMES 0 WHEN $k=5$

(b) CONTINUE THE ROW REDUCTION WITH $k=5$

$\begin{bmatrix} 1 & 2 & 1 & | & 1 \\ 0 & -1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 0 & -1 & 1 & | & 1 \\ 1 & 2 & 1 & | & 1 \end{bmatrix} \Rightarrow \begin{matrix} x - z = 1 \\ y + z = 1 \end{matrix}$

Then $\begin{cases} x - z = 1 \\ y + z = 1 \\ z = z \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ z \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ OR $z = t$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+t \\ 1-t \\ t \end{pmatrix}$ AS EUPHONIC

Question 2 (***)

$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}.$$

Show that the above simultaneous equations ...

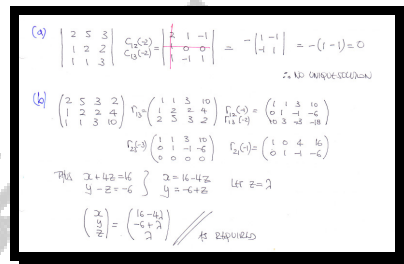
a) ... do **not** have a unique solution.

b) ... are **consistent** and their general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 - 4\lambda \\ \lambda - 6 \\ \lambda \end{pmatrix},$$

where λ is a scalar parameter.

proof



(a) $\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{C_2 - 2C_1} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{C_3 - C_1} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{C_3 + C_2} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
 \therefore No Unique Solution

(b) $\begin{pmatrix} 2 & 5 & 3 & 2 \\ 1 & 2 & 2 & 4 \\ 1 & 1 & 3 & 10 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 5 & 3 & 2 \\ 1 & 1 & 3 & 10 \end{pmatrix} \xrightarrow{C_2 - 2C_1} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & -1 & -6 \\ 1 & 1 & 3 & 10 \end{pmatrix} \xrightarrow{C_3 - C_1} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & -1 & -6 \\ 0 & -1 & 1 & 6 \end{pmatrix} \xrightarrow{C_3 + C_2} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
 \therefore Consistent
 $\begin{cases} x + 2y + 2z = 4 \\ y - z = -6 \end{cases} \Rightarrow \begin{cases} x + 2(-6) + 2z = 4 \\ y - z = -6 \end{cases} \Rightarrow \begin{cases} x - 12 + 2z = 4 \\ y - z = -6 \end{cases} \Rightarrow \begin{cases} x + 2z = 16 \\ y - z = -6 \end{cases}$
Let $z = \lambda$
 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 - 4\lambda \\ \lambda - 6 \\ \lambda \end{pmatrix}$
As Required

Question 3 (***)

The system of simultaneous equations

$$x + y + 2z = 2$$

$$x + 2y + z = 2$$

$$2x + ay + 5z = b$$

where a and b are constants, does **not** have a unique solution, but it is **consistent**.

- a) Determine the value of a and the value of b .
- b) Show that the general solution of the system can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-3t \\ t \\ t \end{pmatrix},$$

where t is a parameter.

$$\boxed{a=1}, \boxed{b=4}$$

(a) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & a & 1 & b \end{array} \right) \xrightarrow{R_2 - R_1, R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2-a \\ 0 & a-2 & 1 & b-4 \end{array} \right)$

\therefore If no unique solution $a-1=0 \therefore a=1$

$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 5 & b \end{array} \right) \xrightarrow{R_2 - R_1, R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2-a \\ 0 & -1 & 1 & b-4 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2-a \\ 0 & 0 & 0 & b-4 \end{array} \right)$

If consistent let last row be zero $\Rightarrow b=4$

(b) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2-a \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 2-a \\ 0 & 1 & -1 & 2-a \\ 0 & 0 & 0 & 0 \end{array} \right)$ If $2-a=2 \Rightarrow a=1$

let $z=t$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-3t \\ t \\ t \end{pmatrix}$

Question 4 (***)

$$x + 2y + z = 1$$

$$x + y + 3z = 2$$

$$3x + 5y + 5z = 4$$

Show that the solution of the above simultaneous equations is

$$x = 3 - 5t, \quad y = 2t - 1, \quad z = t$$

where t is a parameter.

 , proof

FIND THE SOLUTION BY THE JORDAN-GAUSS ALGORITHM

$$\begin{cases} x + 2y + z = 1 \\ x + y + 3z = 2 \\ 3x + 5y + 5z = 4 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 3 & 5 & 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

EXTRACT THE SOLUTION

$$\begin{cases} x - 3z = -1 \\ y + 2z = 1 \end{cases} \Rightarrow \begin{cases} x = -1 + 3z \\ y = 1 - 2z \end{cases}$$

Let $z = t$

$$\begin{cases} x = -1 + 3t \\ y = 1 - 2t \\ z = t \end{cases}$$

As required

Question 5 (***)

$$x + y + 2z = 2$$

$$2x - y + z = -2$$

$$3x + y + 4z = 2$$

Show, by reducing the augmented matrix of the above system of equations into row echelon form, that the solution can be written as

$$x = -t, \quad y = 2 - t, \quad z = t$$

where t is a scalar parameter.

 , proof

• FOR THE SYSTEM OF EQUATION INTO A MATRIX

$$\begin{cases} x + y + 2z = 2 \\ 2x - y + z = -2 \\ 3x + y + 4z = 2 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

• APPLY ELEMENTARY ROW OPERATIONS

$$\begin{aligned} r_2(-2) &= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -6 \\ 0 & -2 & -2 & -4 \end{bmatrix} & r_3(-3) &= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -6 \\ 0 & -2 & -2 & -4 \end{bmatrix} \\ r_3(2) &= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

• CONTINUE THE REDUCTION, LEAVING THE BOTTOM ROW

$$r_2(-1/3) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

• SO WE HAVE

$$\begin{cases} x + z = 0 \\ y + z = 2 \end{cases} \Rightarrow \begin{cases} x = -z \\ y = 2 - z \end{cases} \quad \text{let } z = t$$

$$\begin{aligned} x &= -t \\ y &= 2 - t \\ z &= t \end{aligned}$$

KEY TO ROW OPERATIONS

- $r_2 = \text{sum row 1 \& 2}$
- $r_3(2) = \text{multiply row 3 by } \frac{1}{2}$
- $r_3(-2) = \text{multiply row 1 by } -2, \text{ and add it to row 3}$

Question 6 (*)**

A system of equation is given in matrix form below

$$\begin{pmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where t is an integer constant, and a , b and c are real constants.

The system of equations does not have a unique solution, but it is consistent.

Show clearly that

$$a + b = c.$$

proof

No unique solution \Rightarrow Determinant is zero
 $\begin{vmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{vmatrix} = t(3-t) - 2(2t+3) + 3(2t+15) = t(3-t-2t-6) + 3(2t+15) = t(-t-3) + 6t+45 = -t^2-3t+6t+45 = -t^2+3t+45$
 $= -(t^2-3t-45) = -(t-9)(t+5)$
 Since for zero $(t-9)(t+5) = 0$
 $t = 9$ or $t = -5$
 Now Row Reducing
 $\begin{pmatrix} 1 & 2 & 3 & a \\ 2 & 3 & -t & b \\ 3 & 5 & t+1 & c \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1, R_3 \leftarrow R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 & a \\ 0 & -1 & -t-4 & b-2a \\ 0 & -1 & -t-4 & c-3a \end{pmatrix}$
 $R_3 \leftarrow R_3 - R_2 \Rightarrow \begin{pmatrix} 1 & 2 & 3 & a \\ 0 & -1 & -t-4 & b-2a \\ 0 & 0 & 0 & c-a-b \end{pmatrix}$
 For system to be consistent $\Rightarrow c-a-b = 0$
 $\Rightarrow a+b=c$
 As Required

Question 7 (***)

A system of equations is given below in terms of the scalar parameters t and s .

$$2x + y + 3z = t + 1$$

$$5x - 2y + (t+1)z = 3$$

$$tx + 2y + 4z = s$$

- a) Show that if $t = -5$ or $t = 2$, the system does not have a unique solution.
- b) Determine the value of s is the system is to have infinite solutions with $t = 2$.

$$s = 4$$

6) $\begin{vmatrix} 2 & 1 & 3 \\ 5 & -2 & t+1 \\ t & 2 & 4 \end{vmatrix} = \begin{vmatrix} r_1(z) \\ r_2(z) \\ r_3(z) \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 9 & 0 & t+7 \\ t+4 & 0 & -2 \end{vmatrix} = \text{EXPANDED BY MINOR (COLUMN 2)}$

$$= - \begin{vmatrix} 9 & t+7 \\ t+4 & -2 \end{vmatrix} = \begin{vmatrix} t+4 & -2 \\ 9 & t+7 \end{vmatrix} = (t+4)(t+7) + 18 = t^2 + 3t - 10$$

NO UNIQUE SOLUTION $\Rightarrow t^2 + 3t - 10 = 0$
 $(t-2)(t+5) = 0 \quad \therefore t = 2 \text{ or } t = -5$

6) $\begin{bmatrix} 2 & 1 & 3 & | & t+1 \\ 5 & -2 & t+1 & | & 3 \\ t & 2 & 4 & | & s \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 5 & -2 & t+1 & | & 3 \\ 2 & 1 & 3 & | & t+1 \\ t & 2 & 4 & | & s \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 1 & 3 & | & t+1 \\ 5 & -2 & t+1 & | & 3 \\ t & 2 & 4 & | & s \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 1 & 3 & | & t+1 \\ 5 & -2 & t+1 & | & 3 \\ t & 2 & 4 & | & s \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 1 & 3 & | & t+1 \\ 5 & -2 & t+1 & | & 3 \\ t & 2 & 4 & | & s \end{bmatrix}$

NON INFINITE SOLUTION \Rightarrow YES NO
 $\therefore s = 4$

Question 8 (***)

The three planes defined by the equations

$$x + 2y + z = 2$$

$$2x + ay + z = 2$$

$$x + y + 2z = b$$

where a and k are constants, intersect along a straight line L .

Determine an equation of L .

$$\boxed{}, \quad \mathbf{r} = (6-3t)\mathbf{i} + (t-2)\mathbf{j} + t\mathbf{k}$$

• FIRSTLY IF THERE IS NO UNIQUE SOLUTION, THE DETERMINANT OF THE MATRIX $\begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{bmatrix}$ MUST BE ZERO

• EXPAND BY TOP ROW

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} a & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & a \\ 1 & 1 \end{vmatrix} = 0$$

$$(2a-1) - (2 \times 3) + (2-a) = 0$$

$$2a-1-6+2-a=0$$

$$a-5=0$$

$$\underline{a=5}$$

• SIMPLY ROW REDUCING TO OBTAIN A "BOTTOM ZERO ROW" IF THE SYSTEM IS TO BE INCONSISTENT

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 2 \\ 1 & 1 & 2 & b \end{bmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & b-2 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & b-4 \end{bmatrix} \quad \therefore \underline{b=4}$$

• CONTINUING ROW REDUCING, KNOWING THE BOTTOM ROW

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -2 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

• EXTRACTING THE SOLUTION, WE HAVE

$$\begin{aligned} 2 + 3z &= 6 \\ y - z &= -2 \end{aligned}$$

• LET $z=t$, SOME PARAMETER

$$\begin{aligned} x &= 6-3t \\ y &= -2+t \\ z &= t \end{aligned} \quad \text{ie } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

OR

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6-3t \\ -2+t \\ t \end{pmatrix} //$$

Question 9 (***)

$$x + y - 2z = 2$$

$$3x - y + 6z = 2$$

$$6x + 5y - 9z = k$$

- a) Show that the system of equations does not have a unique solution.
- b) Show that there exists a value of k for which the system is consistent.
- c) Show, by reducing the system into row echelon form, that the consistent solution of the system can be written as

$$x = 1 - t, \quad y = 3t + 1, \quad z = t$$

where t is a scalar parameter.

$$k = 11$$

Handwritten solution for Question 9:

(a) $\begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & 6 \\ 6 & 5 & -9 \end{pmatrix} \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 6R_1}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -4 & 18 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & -4 & 18 \end{pmatrix} \xrightarrow{R_2 \times (-1)} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & -4 & 18 \end{pmatrix} \xrightarrow{R_3 + 4R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \times \frac{1}{2}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & 6 \\ 6 & 5 & -9 \end{pmatrix} \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 6R_1}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -4 & 18 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & -4 & 18 \end{pmatrix} \xrightarrow{R_2 \times (-1)} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & -4 & 18 \end{pmatrix} \xrightarrow{R_3 + 4R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \times \frac{1}{2}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & 6 \\ 6 & 5 & -9 \end{pmatrix} \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 6R_1}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -4 & 18 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & -4 & 18 \end{pmatrix} \xrightarrow{R_2 \times (-1)} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & -4 & 18 \end{pmatrix} \xrightarrow{R_3 + 4R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \times \frac{1}{2}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Question 10 (***)

$$3x - y + 5z = 5$$

$$2x + y - 5z = 10$$

$$x + y + kz = 7$$

where k is a constant.

- a) Given that $k \neq -5$ find the unique solution of the system of equations.
- b) Given instead that $k = -5$ show, by reducing the system into row echelon form, that the consistent solution of the system can be written as

$$x = 3, \quad y = 5t + 4, \quad z = t.$$

$$x = 3, \quad y = 4, \quad z = 0$$

Handwritten solution for Question 10b, showing the reduction of the system into row echelon form for $k = -5$.

Initial augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -1 & 5 & 5 \\ 2 & 1 & -5 & 10 \\ 1 & 1 & -5 & 7 \end{array} \right)$$

Row operations:

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -5 & 7 \\ 3 & -1 & 5 & 5 \\ 2 & 1 & -5 & 10 \end{array} \right)$$

$$\xrightarrow{\substack{r_2 - 3r_1 \\ r_3 - 2r_1}} \left(\begin{array}{ccc|c} 1 & 1 & -5 & 7 \\ 0 & -4 & 20 & -16 \\ 0 & -1 & 5 & 6 \end{array} \right)$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -5 & 7 \\ 0 & -1 & 5 & 6 \\ 0 & -4 & 20 & -16 \end{array} \right)$$

$$\xrightarrow{r_2 \times (-1)} \left(\begin{array}{ccc|c} 1 & 1 & -5 & 7 \\ 0 & 1 & -5 & -6 \\ 0 & -4 & 20 & -16 \end{array} \right)$$

$$\xrightarrow{r_3 + 4r_2} \left(\begin{array}{ccc|c} 1 & 1 & -5 & 7 \\ 0 & 1 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

From the second row:

$$y - 5z = -6 \Rightarrow y = 5z - 6$$

From the first row:

$$x + y - 5z = 7 \Rightarrow x + (5z - 6) - 5z = 7 \Rightarrow x - 6 = 7 \Rightarrow x = 13$$

Let $z = t$, then:

$$y = 5t - 6$$

Final solution:

$$x = 13, \quad y = 5t - 6, \quad z = t$$

Question 11 (***)

A system of equations is given below

$$x + 2y - z = 4$$

$$2x - y + z = 5$$

$$4x - 7y + 5z = 7$$

- a) Show that the system does not have a unique solution but is consistent.
- b) Show that the general solution of the system can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-t \\ 3+3t \\ 4+5t \end{pmatrix},$$

where t is a parameter.

, proof

a) • WRITE THE SYSTEM IN MATRIX FORM

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 4 & -7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

• CALCULATE THE DETERMINANT OF THE MATRIX BY THE FIRST ROW

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 4 & -7 & 5 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 4 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 4 & -7 \end{vmatrix}$$

$$= -5 + 4 - 2(14 - 4) + (-14 + 4)$$

$$= -5 + 4 - 20 + 10$$

$$= -11$$

∴ NO UNIQUE SOLUTION

[THIS COULD ALSO BE DONE BY OBTAINING A ZERO ROW IN THE FIRST 3 COLUMNS OF AN AUGMENTED MATRIX]

• PROCEED TO FIND THE SOLUTION IF CONSISTENT

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 1 & 5 \\ 4 & -7 & 5 & 7 \end{bmatrix} \xrightarrow{\substack{R_2(-2) \\ R_3(-4)}} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & -3 \\ 0 & -15 & 9 & -9 \end{bmatrix} \xrightarrow{R_3 \times \frac{1}{3}} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & -3 \\ 0 & -5 & 3 & -3 \end{bmatrix}$$

"TOTAL ZERO ROW"

∴ CONSISTENT

b) • CONTINUE THE ROW REDUCTION TO REDUCE TO REF FORM

$$\xrightarrow{R_2 \times (-1/5)} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -3/5 & 3/5 \\ 0 & -5 & 3 & -3 \end{bmatrix}$$

• EXTRACT THE SOLUTION

$$\begin{cases} x + 2y - z = 4 \\ y - \frac{3}{5}z = \frac{3}{5} \end{cases} \Rightarrow \begin{cases} x = 4 - 2y + z \\ y = \frac{3}{5} + \frac{3}{5}z \end{cases}$$

• LET $z = 4 + 5t$

$$\Rightarrow \begin{cases} x = 4 - 2\left(\frac{3}{5} + \frac{3}{5}(4+5t)\right) + (4+5t) \\ y = \frac{3}{5} + \frac{3}{5}(4+5t) \end{cases}$$

$$\Rightarrow \begin{cases} x = 4 - \frac{6}{5} - \frac{6}{5}(4+5t) + 4 + 5t \\ y = \frac{3}{5} + 3 + 3t \end{cases}$$

$$\Rightarrow \begin{cases} x = 2 - t \\ y = 3 + 3t \\ z = 4 + 5t \end{cases}$$

OR

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-t \\ 3+3t \\ 4+5t \end{pmatrix}$$

Question 12 (***)

$$A = \begin{pmatrix} 9 & 2 & k \\ 1 & -1 & -3 \\ k-1 & 1 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Consider the homogeneous system of simultaneous equations

$$A\mathbf{x} = \mathbf{0}$$

- a) Find the values of k for which the system has a **non trivial** solution.
 b) If $k \neq 0$ find the general, non trivial solution of the system.

$$\boxed{k=0, 6}, \quad \boxed{x=0, y=-3\lambda, z=\lambda}$$

a) CONSIDER THE DETERMINANT OF A

$$\begin{vmatrix} 9 & 2 & k \\ 1 & -1 & -3 \\ k-1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 9 & 2 & k \\ 1 & -1 & -3 \\ k & 0 & 0 \end{vmatrix} \quad \therefore \dots \text{EXPANDED BY THE BOTTOM ROW}$$

$$= k \begin{vmatrix} 9 & 2 \\ 1 & -1 \end{vmatrix} = k(-9+2) = k(-7) = -7k$$

FOR A NON TRIVIAL SOLUTION $|A| = 0$ SO $k = 0$ OR $k = 6$

b) LET $k=6$

$$\begin{cases} \text{I) } 9x + 2y + 6z = 0 \\ \text{II) } x - y - 3z = 0 \\ \text{III) } 5x + y + 3z = 0 \end{cases} \quad \text{ADDING II & III GIVES } x = 0$$

HENCE THE FIRST TWO EQUATIONS REDUCE TO

$$\begin{cases} 2y + 6z = 0 \\ -y - 3z = 0 \end{cases} \rightarrow y = -3z$$

LET $z = \lambda$
 $y = -3\lambda$
 $x = 0$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3\lambda \\ \lambda \end{pmatrix}$$

Question 13 (***)

Consider the system of simultaneous equations

$$kx + ky - z = -1$$

$$ky + 2z = 2k$$

$$x + 2y + z = 1$$

where the constant k can **only** take the values 0, 1 and 2.Determine for each of the possible values of k whether the system ...

- i. ... has a unique solution
- ii. ... has no unique solution, but it is consistent.
- iii. ... is inconsistent.

$$k = 0 \Rightarrow \text{inconsistent}, \quad k = 1 \Rightarrow \text{no unique solution/consistent},$$

$$k = 2 \Rightarrow \text{unique solution}$$

Handwritten solution for Question 13:

System of equations in matrix form:

$$\begin{pmatrix} k & k & -1 \\ 0 & k & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2k \\ 1 \end{pmatrix}$$

Row reduction for $k=0$:

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \times (-1)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Since R_2 is all zeros, the system is inconsistent for $k=0$.

Row reduction for $k=1$:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{R_3 \times (-1/3)} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + 3R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unique solution for $k=1$.

Row reduction for $k=2$:

$$\begin{pmatrix} 2 & 2 & -1 \\ 0 & 2 & 4 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 2 & 4 \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & 2 & -1 \\ 0 & 2 & 4 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \times (1/2)} \begin{pmatrix} 1 & 1 & -1/2 \\ 0 & 2 & 4 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 1 & -1/2 \\ 0 & 2 & 4 \\ 0 & 1 & 3/2 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 2 & 4 \\ 0 & 1 & 3/2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{R_2 \times (2/3)} \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 2/3 & 1 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{R_2 \times (3/2)} \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + (5/2)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - (3/2)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unique solution for $k=2$.

Question 14 (***)

$$\mathbf{A} = \begin{pmatrix} k+1 & 1 & k \\ 1 & 2 & k \\ 2 & k & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Show that the system of equations $\mathbf{Ax} = \mathbf{b}$ does not have a unique solution, but is consistent if $k=1$, and its general solution in this case can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 1+t \\ 1-3t \end{pmatrix}.$$

where t is a scalar parameter.

, proof

• FIRSTLY IF $k=1$

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} \quad |\Delta| = 2 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (2 \times 1) - (-1) + (-2)$$

$$= 2 + 1 - 2$$

$$= 0$$

∴ IF $k=1$, THERE IS NO UNIQUE SOLUTION

• DON'T WRITE THE SYSTEM AS AN AUGMENTED MATRIX

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -1 & -4 \\ 0 & -3 & -1 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \times (-1/3)} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

• EXTEND THE SOLUTION

$$\begin{cases} x + \frac{1}{3}z = \frac{4}{3} \\ y + \frac{1}{3}z = \frac{4}{3} \end{cases} \rightarrow \begin{cases} x = \frac{4}{3} - \frac{1}{3}z \\ y = \frac{4}{3} - \frac{1}{3}z \end{cases}$$

LET $z = 2$

$$\rightarrow \begin{cases} x = \frac{4}{3} - \frac{1}{3}(2) \\ y = \frac{4}{3} - \frac{1}{3}(2) \\ z = 2 \end{cases}$$

• FINALLY TO GET IT TO THE REQUIRED FORM

$$t = \frac{1}{3} - \frac{1}{3}z$$

$$3t = 1 - z$$

$$z = 1 - 3t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3} - \frac{1}{3}(1-3t) \\ \frac{4}{3} - \frac{1}{3}(1-3t) \\ 1-3t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} - \frac{1}{3} + t \\ \frac{1}{3} - \frac{1}{3} + t \\ 1-3t \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 1+t \\ 1-3t \end{bmatrix}$$

As Required

Question 15 (***)

A system of equations is given below

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 4 & 7 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix}.$$

Show that the system does not have a unique solution but is consistent and its general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8t \\ t-1 \\ 5t+2 \end{pmatrix},$$

where t is a parameter.

, proof

• CHECKING THE DETERMINANT FOR UNIQUENESS OF SOLUTION

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 4 & 7 & 5 \end{vmatrix} = 2 \begin{vmatrix} -2 & 2 \\ 7 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} + 3 \begin{vmatrix} 1 & -2 \\ 4 & 7 \end{vmatrix}$$

$$= 2(-24) - (-3) + 3(15)$$

$$= -48 + 3 + 45$$

$$= 0 \quad \therefore \text{No unique solution}$$

• WRITE THE SYSTEM AS AN AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 1 & -2 & 2 & 6 \\ 4 & 7 & 5 & 3 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 2 & 1 & 3 & 5 \\ 4 & 7 & 5 & 3 \end{array} \right] \xrightarrow{\substack{r_2 \leftarrow r_2 - 2r_1 \\ r_3 \leftarrow r_3 - 4r_1}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 5 & -1 & -7 \\ 0 & 15 & -3 & -21 \end{array} \right]$$

$$\xrightarrow{r_3 \leftarrow \frac{1}{3}r_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 5 & -1 & -7 \\ 0 & 5 & -1 & -7 \end{array} \right] \xrightarrow{r_3 \leftarrow r_3 - r_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 5 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

• FINDING THE ZERO ROW AND CONTINUING THE ROW REDUCTION

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 5 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{r_2 \leftarrow \frac{1}{5}r_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 1 & -\frac{1}{5} & -\frac{7}{5} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

EXTRACTING THE SOLUTION

$$\left. \begin{array}{l} x - 2y + 2z = 6 \\ y - \frac{1}{5}z = -\frac{7}{5} \end{array} \right\} \Rightarrow \begin{array}{l} x = 6 + 2y - 2z \\ y = -\frac{7}{5} + \frac{1}{5}z \end{array}$$

• NOW $z = \lambda$, so the solution can be written as

$$\begin{cases} x = 6 + 2y - 2\lambda \\ y = -\frac{7}{5} + \frac{1}{5}\lambda \\ z = \lambda \end{cases}$$

• LOOKING AT THE EXPRESSION FROM LET $\lambda = 5t+2$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 - 2(-\frac{7}{5} + \frac{1}{5}(5t+2)) \\ -\frac{7}{5} + \frac{1}{5}(5t+2) \\ 5t+2 \end{pmatrix} = \begin{pmatrix} 6 - 2(-\frac{7}{5} + t + \frac{2}{5}) \\ -\frac{7}{5} + t + \frac{2}{5} \\ 5t+2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8t \\ t-1 \\ 5t+2 \end{pmatrix}$$

Question 16 (***)

$$\begin{pmatrix} 3 & k & -1 \\ 2k & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

- a) Determine the values of k for which the above system of equations does not have a unique solution.
- b) Show, that one of these values of k leads to inconsistency and the other produces a general solution of the form

$$x = -t, \quad y = 4t + 1, \quad z = 5t - 1$$

where t is a scalar parameter.

$$\boxed{}, \quad k = -1, 2$$

a) EXAMINING THE DETERMINANT - NON UNIQUE SOLUTION, VALUES
A ZERO DETERMINANT

$$\begin{vmatrix} 3 & k & -1 \\ 2k & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \text{EXPANDING BY THE SECOND ROW}$$

$$= -2k \begin{vmatrix} k & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= -2k(k-1) + (4)$$

$$= -2k^2 + 2k + 4$$

SETTING FOR ZERO

$$\Rightarrow -2k^2 + 2k + 4 = 0$$

$$\Rightarrow k^2 - k - 2 = 0$$

$$\Rightarrow (k+1)(k-2) = 0$$

$$\Rightarrow k = -1, 2$$

b) WRITE THE SYSTEM AS AN AUGMENTED MATRIX IN TERMS OF k

$$\left[\begin{array}{ccc|c} 3 & k & -1 & 3 \\ 2k & 1 & 0 & 1 \\ 1 & -1 & 1 & -2 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2k & 1 & 0 & 1 \\ 3 & k & -1 & 3 \end{array} \right]$$

NOW CONTINUE WITH TWO SEPARATE
ROWS PRODUCING ONE FOR EACH
OF THE TWO VALUES OF k

EXTRACT THE SOLUTION FROM THE SYSTEM WHERE $k = 2$

$$\begin{cases} x + \frac{1}{2}z = -\frac{1}{2} \\ y - \frac{1}{2}z = \frac{3}{2} \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} - \frac{1}{2}z \\ y = \frac{3}{2} + \frac{1}{2}z \end{cases}$$

Let $z = 2t$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - \frac{1}{2}(2t) \\ \frac{3}{2} + \frac{1}{2}(2t) \\ 2t \end{pmatrix} = \begin{pmatrix} -1 - t \\ 3 + t \\ 2t \end{pmatrix}$$

Let $z = 5t - 1$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - \frac{1}{2}(5t-1) \\ \frac{3}{2} + \frac{1}{2}(5t-1) \\ 5t-1 \end{pmatrix} = \begin{pmatrix} -t \\ 4t+1 \\ 5t-1 \end{pmatrix}$$

Question 17 (***)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & 7 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$$

Show that the system of equations $\mathbf{Ax} = \mathbf{b}$ does not have a unique solution, but for a certain value of b is consistent and its general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+4\lambda \\ \lambda \\ -1-3\lambda \end{pmatrix}$$

where λ is a parameter.

, proof

● EXAMINING THE DETERMINANT OF \mathbf{A}

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & 7 \end{vmatrix} = -8 - 2(2) + 2(6) = -8 - 4 + 12 = 0$$

∴ NO UNIQUE SOLUTION

● WRITE THE SYSTEM AS AN AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 4 & 5 & 7 & b \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 4R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -3 & -1 & 1 \\ 0 & -3 & -1 & b-4 \end{array} \right]$$

● FOR CONSISTENCY WE REQUIRE A ZERO ROW WHICH EVENLY OCCURS
WHEN $b-4=1$, i.e. $b=5$.

● CONTINUING THE ROW REDUCTIONS WITHOUT THE BOTTOM ROW

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -3 & -1 & 1 \end{array} \right] \xrightarrow{R_2 \times (-1/3)} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 1/3 & -1/3 \end{array} \right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 5/3 \\ 0 & 1 & 1/3 & -1/3 \end{array} \right]$$

EXTRACTING THE SOLUTION

$$\begin{cases} x + \frac{5}{3}z = \frac{5}{3} \\ y + \frac{1}{3}z = -\frac{1}{3} \end{cases} \Rightarrow \begin{cases} x = \frac{5}{3} - \frac{5}{3}z \\ y = -\frac{1}{3} - \frac{1}{3}z \end{cases}$$

⇒ LET $z = t$

$$\Rightarrow \begin{cases} x = \frac{5}{3} - \frac{5}{3}t \\ y = -\frac{1}{3} - \frac{1}{3}t \\ z = t \end{cases}$$

● LOOKING AT THE REQUIRED FORM OF THE SOLUTION, WE LET $t = -1-3\lambda$ (BY LOOKING AT z)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3} - \frac{5}{3}t \\ -\frac{1}{3} - \frac{1}{3}t \\ t \end{pmatrix} = \begin{pmatrix} \frac{5}{3} - \frac{5}{3}(-1-3\lambda) \\ -\frac{1}{3} - \frac{1}{3}(-1-3\lambda) \\ -1-3\lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3} + \frac{5}{3} + 5\lambda \\ -\frac{1}{3} + \frac{1}{3} + \lambda \\ -1-3\lambda \end{pmatrix} = \begin{pmatrix} \frac{10}{3} + 5\lambda \\ \lambda \\ -1-3\lambda \end{pmatrix}$$

As Required

Question 18 (***)

Consider the following matrix equation

$$\begin{pmatrix} k & 1 & 0 \\ 3 & -2 & k-3 \\ 10k & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ 15 \end{pmatrix},$$

where a , b and k are scalar constants.

- a)** Find the values of k for which the equation has a unique solution.

It is further asserted that $k = 2$.

- b)** Express a in terms of b if the matrix equation is to be consistent.

- c) Show that if $a=1$ and $b=4$, the solution of the matrix equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t+1 \\ -2t-1 \\ 7t+1 \end{pmatrix},$$

where t is a scalar parameter.

$$\boxed{}, \boxed{k \neq 2 \cup k \neq \frac{3}{7}}, \boxed{2a+7b=15}$$

[illegible]

Question 19 (***)

$$\begin{aligned} x - 2y + az &= 5 \\ (a+1)x + 3y &= a \\ 2x + y + (a-1)z &= 3 \end{aligned}$$

- a) Determine the two values of the constant a for which the above system of equations does **not** have a unique solution.
- b) Show clearly that the system is consistent for one of these values and inconsistent for the other.

$$a = -1, \frac{5}{3}$$

(a) $\begin{pmatrix} 1 & -2 & a \\ a+1 & 3 & 0 \\ 2 & 1 & a-1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & -2 & a \\ a+1 & 3 & 0 \\ 0 & 5 & -a-1 \end{pmatrix}$ (R2 \leftrightarrow R1)

$$= \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ a+1 & 3 & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - (a+1)R_2} \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ 0 & 3 & a+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ 0 & 3 & a+1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - \frac{3}{5}R_2} \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ 0 & 0 & \frac{5}{5}(a+1) - \frac{3}{5}(-a-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ 0 & 0 & \frac{5}{5}(a+1) - \frac{3}{5}(-a-1) \end{pmatrix} = \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ 0 & 0 & \frac{5}{5}(a+1) + \frac{3}{5}(a+1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & a \\ 0 & 5 & -a-1 \\ 0 & 0 & \frac{8}{5}(a+1) \end{pmatrix}$$

THUS IF NO UNIQUE SOLUTION

IF $a = -1$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{3}R_2} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

INCONSISTENT

IF $a = \frac{5}{3}$

$$\begin{pmatrix} 1 & -2 & \frac{5}{3} \\ 0 & 5 & -\frac{8}{3} \\ 0 & 0 & \frac{8}{5}(\frac{5}{3}+1) \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{5}R_2} \begin{pmatrix} 1 & -2 & \frac{5}{3} \\ 0 & 1 & -\frac{8}{15} \\ 0 & 0 & \frac{8}{5}(\frac{5}{3}+1) \end{pmatrix}$$

THUS ARE MULTIPLES OF EACH OTHER, SO ARE ROWS

\therefore CONSISTENT

Question 20 (***)

A system of equations is given below

$$3x + 2y - z = 10$$

$$5x - y - 4z = 17$$

$$x + 5y + pz = q$$

where p and q are constants.

- Find the value of the constant p so that the system of equations does not have a unique solution.
- Show that for this value of p the system is consistent if $q = 3$.
- Show that the general solution of the system can be written as

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(9\mathbf{i} - 7\mathbf{j} + 13\mathbf{k}),$$

where λ is a scalar parameter.

$$p = 2$$

(a) $\begin{vmatrix} 3 & 2 & -1 \\ 5 & -1 & -4 \\ 1 & 5 & p \end{vmatrix} = \begin{vmatrix} 3 & -1 & -2 \\ 5 & -4 & 1 \\ 1 & p & 1 \end{vmatrix} = 3(-p+2) - 2(5p+4) - (25+1)$
 $= -3p+6 - 10p-8 - 26$
 $= -13p-28$
 No unique solution if $-13p-28=0 \Rightarrow p = -2$

(b) $\begin{pmatrix} 3 & 2 & -1 & 10 \\ 5 & -1 & -4 & 17 \\ 1 & 5 & 2 & q \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 5 & 2 & q \\ 5 & -1 & -4 & 17 \\ 3 & 2 & -1 & 10 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 5R_1, R_3 \leftarrow R_3 - 3R_1} \begin{pmatrix} 1 & 5 & 2 & q \\ 0 & -26 & -14 & 17-5q \\ 0 & -13 & -7 & 10-3q \end{pmatrix}$
 For zero row $17-5q = 0 \Rightarrow q = \frac{17}{5}$
 $10-3q = 10 - 3(\frac{17}{5}) = \frac{50-51}{5} = -\frac{1}{5}$
 $q = \frac{17}{5}$

(c) If $q=3$
 $\begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & -26 & -14 & 2 \\ 0 & -13 & -7 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & -13 & -7 & 1 \\ 0 & -13 & -7 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & -13 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
 $2x - \frac{13}{13}y = \frac{1}{13} \Rightarrow 2x - y = \frac{1}{13}$
 $y = 2x - \frac{1}{13}$
 $z = \frac{1}{13}$
 $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x - \frac{1}{13} \\ \frac{1}{13} \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{13} \\ \frac{1}{13} \end{pmatrix}$
 $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -7 \\ 13 \end{pmatrix}$
 $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -7 \\ 13 \end{pmatrix}$
 $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -7 \\ 13 \end{pmatrix}$

Question 21 (**)**

A system of equation is given below

$$3x - 2y - 18z = 6$$

$$2x + y - 5z = 25$$

- a) Show, by reducing the system into row echelon form, that the solution of the system can be written as

$$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

where λ is a scalar parameter.

A new system is now given

$$3x - 2y - 18z = 6$$

$$2x + y - 5z = 25$$

$$7x + ky + 2z = 20$$

where k is a constant.

- b) Determine if the system has solutions for different values of k .

$$k \neq 10 \Rightarrow \text{unique, otherwise inconsistent}$$

(a) $\begin{pmatrix} 3 & -2 & -18 & 6 \\ 2 & 1 & -5 & 25 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 3 & -2 & -18 & 6 \end{pmatrix} \xrightarrow{r_2 \leftarrow r_2 - 3r_1} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & -5 & 9 & -72 \end{pmatrix} \xrightarrow{r_2 \leftarrow -\frac{1}{5}r_2} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \end{pmatrix} \xrightarrow{r_1 \leftarrow r_1 - r_2} \begin{pmatrix} 2 & 0 & -\frac{1}{5} & \frac{13}{5} \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \end{pmatrix} \xrightarrow{r_1 \leftarrow 5r_1} \begin{pmatrix} 10 & 0 & -1 & 13 \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \end{pmatrix} \xrightarrow{r_1 \leftarrow r_1 + r_2} \begin{pmatrix} 10 & 0 & 0 & \frac{86}{5} \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \end{pmatrix} \xrightarrow{r_1 \leftarrow \frac{1}{10}r_1} \begin{pmatrix} 1 & 0 & 0 & \frac{43}{25} \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \end{pmatrix} \xrightarrow{r_2 \leftarrow r_2 + \frac{9}{5}r_1} \begin{pmatrix} 1 & 0 & 0 & \frac{43}{25} \\ 0 & 1 & 0 & \frac{109}{25} \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -2 & -18 & 6 \\ 2 & 1 & -5 & 25 \\ 7 & k & 2 & 20 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 3 & -2 & -18 & 6 \\ 7 & k & 2 & 20 \end{pmatrix} \xrightarrow{r_2 \leftarrow r_2 - 3r_1} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & -5 & 9 & -72 \\ 7 & k & 2 & 20 \end{pmatrix} \xrightarrow{r_3 \leftarrow r_3 - 7r_1} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & -5 & 9 & -72 \\ 0 & k-7 & -35 & -155 \end{pmatrix} \xrightarrow{r_2 \leftarrow -\frac{1}{5}r_2} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \\ 0 & k-7 & -35 & -155 \end{pmatrix} \xrightarrow{r_3 \leftarrow r_3 + (k-7)r_2} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \\ 0 & 0 & -\frac{35}{5}(k-7) & -\frac{155}{5}(k-7) \end{pmatrix} \xrightarrow{r_3 \leftarrow -\frac{1}{35}(k-7)r_3} \begin{pmatrix} 2 & 1 & -5 & 25 \\ 0 & 1 & -\frac{9}{5} & \frac{72}{5} \\ 0 & 0 & 1 & \frac{155}{35}(k-7) \end{pmatrix}$

Looking at bottom row $30-3k=0 \Rightarrow k=10$ $-36-9k = -36-9 \times 10 \neq 0$

\therefore If $k=10 \Rightarrow$ inconsistent
If $k \neq 10 \Rightarrow$ unique