1^{st} ORDER ORDINARY DIFFERENTIAL EQUATIONS

Question 1

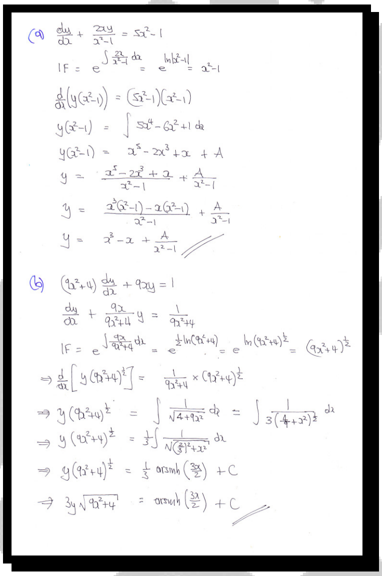
Find a general solution for each of the following differential equations.

a) $\dfrac{dy}{dx} + \dfrac{2xy}{x^2-1} = 5x^2 - 1$

b) $\left(9x^2+4\right)\dfrac{dy}{dx} + 9xy = 1$

c) $x\dfrac{dy}{dx} + 5y = \dfrac{\ln x}{x}, \quad x > 0$

$\boxed{y = x^3 - x + \dfrac{A}{x^2-1}}$, $\boxed{3y\sqrt{9x^2+4} = \operatorname{arsinh}\left(\dfrac{3}{2}x\right) + C}$, $\boxed{y = \dfrac{\ln x}{4x} - \dfrac{1}{16x} + \dfrac{A}{x^5}}$



(a) $\frac{dy}{dx} + \frac{2xy}{x^2-1} = 5x^2-1$

$IF = e^{\int \frac{2x}{x^2-1}dx} = e^{\ln|x^2-1|} = x^2-1$

$\frac{d}{dx}\left(y(x^2-1)\right) = (5x^2-1)(x^2-1)$

$y(x^2-1) = \int 5x^4 - 6x^2 + 1 \, dx$

$y(x^2-1) = x^5 - 2x^3 + x + A$

$y = \frac{x^5-2x^3+x}{x^2-1} + \frac{A}{x^2-1}$

$y = \frac{x^3(x^2-1) - x(x^2-1)}{x^2-1} + \frac{A}{x^2-1}$

$y = x^3 - x + \frac{A}{x^2-1}$

(b) $(9x^2+4)\frac{dy}{dx} + 9xy = 1$

$\frac{dy}{dx} + \frac{9x}{9x^2+4}y = \frac{1}{9x^2+4}$

$IF = e^{\int \frac{9x}{9x^2+4}dx} = e^{\frac{1}{2}\ln(9x^2+4)} = e^{\ln(9x^2+4)^{\frac{1}{2}}} = (9x^2+4)^{\frac{1}{2}}$

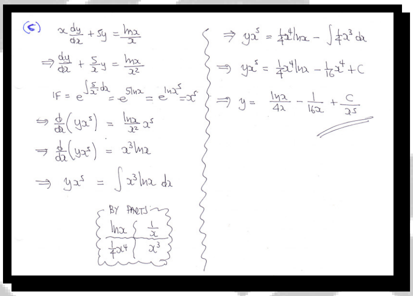
$\Rightarrow \frac{d}{dx}\left[y(9x^2+4)^{\frac{1}{2}}\right] = \frac{1}{9x^2+4} \times (9x^2+4)^{\frac{1}{2}}$

$\Rightarrow y(9x^2+4)^{\frac{1}{2}} = \int \frac{1}{\sqrt{4+9x^2}}dx = \int \frac{1}{3\left(\frac{4}{9}+x^2\right)^{\frac{1}{2}}}dx$

$\Rightarrow y(9x^2+4)^{\frac{1}{2}} = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2+x^2}}dx$

$\Rightarrow y(9x^2+4)^{\frac{1}{2}} = \frac{1}{3}\operatorname{arsinh}\left(\frac{3x}{2}\right) + C$

$\Rightarrow 3y\sqrt{9x^2+4} = \operatorname{arsinh}\left(\frac{3x}{2}\right) + C$



(c) $x\frac{dy}{dx} + 5y = \frac{\ln x}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{5}{x}y = \frac{\ln x}{x^2}$

$IF = e^{\int \frac{5}{x}dx} = e^{5\ln x} = e^{\ln x^5} = x^5$

$\Rightarrow \frac{d}{dx}(yx^5) = \frac{\ln x}{x^2}x^5$

$\Rightarrow \frac{d}{dx}(yx^5) = x^3\ln x$

$\Rightarrow yx^5 = \int x^3 \ln x \, dx$

BY PARTS

$\ln x$	$\frac{1}{x}$
$\frac{1}{4}x^4$	x^3

$\Rightarrow yx^5 = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3 \, dx$

$\Rightarrow yx^5 = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$

$\Rightarrow y = \frac{\ln x}{4x} - \frac{1}{16x} + \frac{C}{x^5}$

Question 2

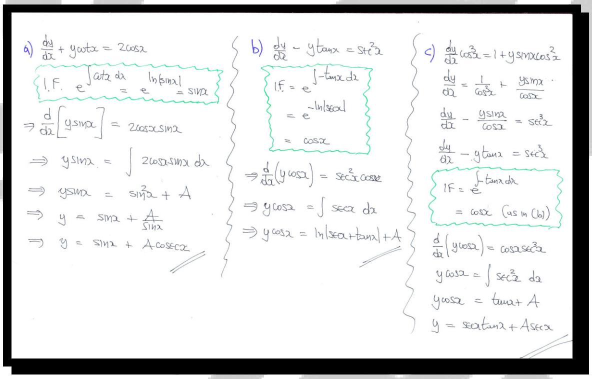
Find a general solution for each of the following differential equations.

a) $\dfrac{dy}{dx} + y\cot x = 2\cos x$

b) $\dfrac{dy}{dx} - y\tan x = \sec^2 x$

c) $\dfrac{dy}{dx}\cos^3 x = 1 + y\sin x\cos^2 x$

$\boxed{y = \sin x + \operatorname{cosec} x}$, $\boxed{y = \sec x\left(\ln\left|\sec x + \tan x\right| + C\right)}$, $\boxed{y = \sec x\tan x + C\sec x}$



Question 3

Find a general solution for each of the following differential equations.

a) $\frac{dy}{dx} - y\tanh x = \sinh 2x$

b) $\frac{dy}{dx} - \frac{y}{x} = x\tanh x$

c) $x\frac{dy}{dx} + \frac{xy}{\coth x} = \text{sech}\, x$

$\boxed{y = 2\cosh^2 x + A\cosh x}$, $\boxed{y = x\ln(\cosh x) + Cx}$, $\boxed{y = \ln|x| + C\,\text{sech}\, x}$

Question 4

Find a solution for each of the following differential equations, subject to the boundary conditions given.

a) $\frac{dy}{dx}+\frac{y}{x}=x$, subject to $x=1$, $y=1$

b) $\frac{dy}{dx}+\frac{y}{x}=\frac{\ln x}{x}$, subject to $x=1$, $y=0$

c) $\frac{dy}{dx}+\frac{y}{x}=e^{x^2}$, subject to $x=1$, $y=\frac{1}{2}e$

$\boxed{y=\frac{1}{3}\left(x^2+\frac{2}{x}\right)}$, $\boxed{y=\frac{1}{x}-1+\ln x}$, $\boxed{y=\frac{e^{x^2}}{2x}}$

Question 5

Find a solution for each of the following differential equations, subject to the boundary conditions given.

a) $\frac{dy}{dx} - \frac{y}{x} = x^2$, subject to $x = 1$, $y = 1$

b) $\frac{dy}{dx} + \frac{3y}{x} = \left(x^4 + 3\right)^{\frac{1}{2}}$, subject to $x = 1$, $y = \frac{1}{5}$

c) $\frac{dy}{dx} + 2y = e^{-3x}$, subject to $x = 0$, $y = 1$

$\boxed{y = \frac{1}{2}x(x+1)}$, $\boxed{10yx^3 = \left(x^4 + 3\right)^{\frac{5}{2}} - 30}$, $\boxed{y = 2e^{-2x} - e^{-3x}}$

Question 6

Find a solution for each of the following differential equations, subject to the boundary conditions given.

a) $x\frac{dy}{dx}+2y=\frac{\sin 2x}{x}$, subject to $x=\frac{\pi}{4}$, $y=\frac{8}{\pi^2}$

b) $\frac{dy}{dx}+y\tan x=\sin x$, subject to $x=0$, $y=0$

c) $\frac{dy}{dx}-y\tan x=4x^3\sec x$, subject to $x=0$, $y=1$

$\boxed{y=\frac{1}{2x^2}(1-\cos 2x)=\frac{\sin^2 x}{x^2}}$, $\boxed{y=\cos x\ln|\sec x|}$, $\boxed{y=\left(x^4+1\right)\sec x}$