1st ORDER O.D.E.

EXAM QUESTIONS
Question 1 (***)

\[ \frac{dy}{dx} + \frac{4y}{x} = 6x - 5, \quad x > 0. \]

Determine the solution of the above differential equation subject to the boundary condition is \( y = 1 \) at \( x = 1 \).

Give the answer in the form \( y = f(x) \).

\[ y = x^2 - x + \frac{1}{x^2}, \quad y = x^2 - x + \frac{1}{x^2} \]
Question 2 \textcolor{red}{(**+)}

\[ \frac{dy}{dx} + y \tan x = e^{2x} \cos x, \quad y(0) = 2. \]

Show that the solution of the above differential equation is

\[ y = \frac{1}{2}(e^{2x}+3)\cos x. \]
Question 3  (**+)**

The velocity of a particle \( v \) ms\(^{-1} \) at time \( t \) s satisfies the differential equation

\[
\frac{dv}{dt} = v + t, \quad t > 0.
\]

Given that when \( t = 2 \), \( v = 8 \), show that when \( t = 8 \)

\[
v = 16(2 + \ln 2).
\]

**proof**

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Question 4  (**+)**

\[
x \frac{dy}{dx} + 4y = 8x^4, \quad \text{subject to } y = 1 \text{ at } x = 1.
\]

Show that the solution of the above differential equation is

\[
y = x^4.
\]

**proof**
Question 5  (***)

\[ \frac{dy}{dx} \sin x = \sin x \sin 2x + y \cos x. \]

Given that \( y = \frac{3}{2} \) at \( x = \frac{\pi}{6} \), find the exact value of \( y \) at \( x = \frac{\pi}{4} \).

\[ 1 + \sqrt{2} \]

Question 6  (***)

\[ x \frac{dy}{dx} + 2y = 9x \left( x^3 + 1 \right)^{\frac{1}{2}}, \text{ with } y = \frac{27}{2} \text{ at } x = 2. \]

Show that the solution of the above differential equation is

\[ y = \frac{2}{x^2} \left( x^3 + 1 \right)^{\frac{3}{2}}. \]

\[ \text{proof}\]
A trigonometric curve $C$ satisfies the differential equation
\[ \frac{dy}{dx} \cos x + y \sin x = \cos^3 x . \]

a) Find a general solution of the above differential equation.

b) Given further that the curve passes through the Cartesian origin $O$, sketch the graph of $C$ for $0 \leq x \leq 2\pi$.

The sketch must show clearly the coordinates of the points where the graph of $C$ meets the $x$ axis.

\[ y = \sin x \cos x + A \cos x \]
Question 8  (***)

20 grams of salt are dissolved into a beaker containing 1 litre of a certain chemical.

The mass of salt, \( M \) grams, which remains undissolved \( t \) seconds later, is modelled by the differential equation

\[
\frac{dM}{dt} + \frac{2M}{20-t} + 1 = 0, \quad t \geq 0.
\]

Show clearly that

\[
M = \frac{1}{10}(10-t)(20-t).
\]

**proof**
Question 9 (***+)

Given that $z = f(x)$ and $y = g(x)$ satisfy the following differential equations

$$\frac{dz}{dx} + 2z = e^{-2x} \quad \text{and} \quad \frac{dy}{dx} + 2y = z,$$

a) Find $z$ in the form $z = f(x)$

b) Express $y$ in the form $y = g(x)$, given further that at $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

$$z = (x + C)e^{-2x} \quad \text{and} \quad y = \left(\frac{1}{2}x^2 + 2x + 1\right)e^{-2x}.$$
Question 10 \((***)\)

\[ x \frac{dy}{dx} = \sqrt{y^2 + 1}, \quad x > 0, \text{ with } y = 0 \text{ at } x = 2. \]

Show that the solution of the above differential equation is

\[ y = \frac{x}{4} - \frac{1}{x}. \]

\[ \text{proof} \]

Question 11 \((***)\)

\[ (x + 1) \frac{dy}{dx} = y + x + x^2, \quad x > -1. \]

Given that \(y = 2\) at \(x = 1\), solve the above differential equation to show that

\[ y = 4(3 - \ln 2) \text{ at } x = 3. \]

\[ \text{proof} \]
Question 12 (***+)

$$\frac{dy}{dx} + ky = \cos 3x, \ k \ is \ a \ non \ zero \ constant.$$ 

By finding a complimentary function and a particular integral, or otherwise, find the general of the above differential equation.

$$y = Ae^{-x} + \frac{k}{9+k^2} \cos 3x + \frac{3}{9+k^2} \sin 3x$$
By reversing the role of $x$ and $y$ in the above differential equation, or otherwise, find its general solution.

$$xy^2 = y^4 + C$$
Question 14  (****)

The curve with equation \( y = f(x) \) satisfies

\[
x \frac{dy}{dx} + (1 - 2x) y = 4x, \quad x > 0, \quad f(1) = 3e^2 - 1.
\]

Determine an equation for \( y = f(x) \).
Question 15  (***)

A curve $C$, with equation $y = f(x)$, passes through the points with coordinates $(1,1)$ and $(2,k)$, where $k$ is a constant.

Given further that the equation of $C$ satisfies the differential equation

$$x^2 \frac{dy}{dx} + xy(x+3) = 1,$$

determine the exact value of $k$.

$$k = \frac{e + 1}{8e}$$
Question 16 (****)

\[(1 - x^2) \frac{dy}{dx} + y = (1 - x^2) \left(1 - x\right)^{\frac{1}{2}}, -1 < x < 1.\]

Given that \( y = \frac{\sqrt{2}}{2} \) at \( x = \frac{1}{2} \), show that the solution of the above differential equation can be written as

\[y = \frac{2}{3} \sqrt{(1 - x^2)(1 + x)}.\]
A curve \( C \), with equation \( y = f(x) \), meets the \( y \) axis at the point with coordinates \((0,1)\).

It is further given that the equation of \( C \) satisfies the differential equation

\[
\frac{dy}{dx} = x - 2y.
\]

a) Determine an equation of \( C \).

b) Sketch the graph of \( C \).

The graph must include in exact simplified form the coordinates of the stationary point of the curve and the equation of its asymptote.

\[ y = \frac{x}{2} - \frac{1}{4} + \frac{5}{4} e^{-2x} \]
Question 18 (****)

\[
\frac{dy}{dx} + \frac{y}{x} = \frac{5}{(x^2 + 2)(4x^2 + 3)}, \quad x > 0.
\]

Given that \( y = \frac{1}{2} \ln \left( \frac{7}{6} \right) \) at \( x = 1 \), show that the solution of the above differential equation can be written as

\[
y = \frac{1}{2x} \ln \left( \frac{4x^2 + 3}{2x^2 + 4} \right).
\]

\[\text{proof}\]
Question 19  (****)

\[ \frac{dy}{dx} + 3y = xe^{-x^2}, \quad x > 0. \]

Show clearly that the general solution of the above differential equation can be written in the form

\[ 2yx^2 + (x^2 + 1)e^{-x^2} = \text{constant}. \]

proof
Question 20  (***)

The general point $P$ lies on the curve with equation $y = f(x)$.

The gradient of the curve at $P$ is 2 more than the gradient of the straight line segment $OP$.

Given further that the curve passes through $Q(1, 2)$, express $y$ in terms of $x$.

$$y = 2x(1 + \ln x)$$
Question 21 \quad (****+)

A curve with equation \( y = f(x) \) passes through the origin and satisfies the differential equation

\[
2y \left(1 + x^2\right) \frac{dy}{dx} + xy^2 = \left(1 + x^2\right)^{\frac{3}{2}}.
\]

By finding a suitable integrating factor, or otherwise, show clearly that

\[
y^2 = \frac{x^3 + 3x}{3\sqrt{x^2 + 1}}.
\]
The curve with equation \( y = f(x) \) passes through the origin, and satisfies the relationship

\[
\frac{dy}{dx} \left[ y(x^2 + 1) \right] = x^5 + 2x^3 + x + 3xy.
\]

Determine a simplified expression for the equation of the curve.

\[
y = \frac{1}{3}(x^2 + 1)^2 - \frac{1}{3}(x^2 + 1)^3
\]
Question 23  (****+)

A curve with equation \( y = f(x) \) passes through the point with coordinates \((0,1)\) and satisfies the differential equation

\[
y^2 \frac{dy}{dx} + y^3 = 4e^x.
\]

By finding a suitable integrating factor, solve the differential equation to show that

\[
y^3 = 3e^x - 2e^{-3x}.
\]

proof
Question 24  (****+)

It is given that a curve with equation \( y = f(x) \) passes through the point \( \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \) and satisfies the differential equation

\[
\left( \frac{dy}{dx} - \sqrt{\tan x} \right) \sin 2x = y.
\]

Find an equation for the curve in the form \( y = f(x) \).

\[
y = x\sqrt{\tan x}
\]
Question 25
The variables $x$ and $y$ satisfy

$$(2y-x)\frac{dy}{dx} = y, \quad y > 0, \quad x > 0.$$ 

If $y = 1$ at $x = 2$, show that $x = y + \frac{1}{y}$. 

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Question 26

The variables \( x \) and \( y \) satisfy

\[
\frac{dy}{dx} = \frac{y(y+1)}{y-x-xy-1}, \quad y > 0.
\]

If \( y = 1 \) at \( x = 1 - \ln 4 \), show that \( y + \ln(y+1) = 0 \) at \( x = 3 \).
Question 27  (****)

The curve with equation \( y = f(x) \) has the line \( y = 1 \) as an asymptote and satisfies the differential equation

\[
x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0.
\]

Solve the above differential equation, giving the solution in the form \( y = f(x) \).

\[
, \quad y = e^{-\frac{1}{x}} - \frac{1}{x}
\]
Question 28  (***)

It is given that a curve with equation \( x = f(y) \) passes through the point \( (0, \frac{1}{2}) \) and satisfies the differential equation

\[
(2y + 3x) \frac{dy}{dx} = y.
\]

Find an equation for the curve in the form \( x = f(y) \).

\[
, \quad x = 4y^3 - y
\]
Question 29 (****)

Use suitable manipulations to solve this exact differential equation.

\[ 4x \frac{dy}{dx} + \sin 2y = 4\cos^2 y, \quad y\left(\frac{1}{4}\right) = 0. \]

Given the answer in the form \( y = f(x) \).

\[ y = \arctan \left( 2 - \frac{1}{\sqrt{x}} \right) \]

\[ \text{SPX-P}, \quad \arctan \left( \frac{2}{\sqrt{x}} \right) = - \]