DISCRIMINANT
EXAM
QUESTIONS
Question 1  (**)
Show by using the discriminant that the graph of the curve with equation

\[ y = x^2 - 4x + 10, \]

does not cross the x-axis.

\[ \text{proof} \]

Question 2  (**)
Show that the quadratic equation

\[ x^2 + (2k + 3)x + k^2 + 3k + 1 = 0 \]

has two distinct real roots in \( x \), for all values of the constant \( k \).

\[ \text{proof} \]
Question 3 (**+)**
Find the range of values of the constant $k$ so that the equation

$$x^2 + kx + 16 = 0,$$

has no real roots.

$$-8 < k < 8$$

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Question 4 (***)
Find the range of values of the constant $k$ so that the graph of the curve with equation

$$y = x^2 + 5x + k,$$

does not cross the $x$ axis.

$$x \geq \frac{25}{4}$$
Question 5 (**+)

Use an algebraic method to show that the graphs
\[ y = 1 - x \quad \text{and} \quad y = x^2 - 6x + 10, \]
do not intersect.

\[ \text{proof} \]

Question 6 (***)

Find the range of values of the constant \( m \) so that the graph of the curve with equation
\[ y = 2x^2 + mx + 2, \]
does not cross the \( x \) axis.

\[ -4 \leq m \leq 4 \]
Question 7  (***)

The following quadratic equation, where \( m \) is a constant, has two distinct real roots.

\[
x^2 + (m+2)x + 4m - 7 = 0, \quad x \in \mathbb{R}
\]

Determine the range of the possible values of \( m \).

\[ m < 4 \cup m > 8 \]

Question 8  (***)

Show that the quadratic equation

\[
(k+1)x^2 + 2kx + k = 1
\]

has two distinct real roots for all real values of the constant \( k \), except for one value which must be stated.

\[ k \neq -1 \]
Question 9 (***)

Find the range of the values of the constant $m$ so that the equation

$$x^2 + (m + 2)x + 3m = 2,$$

has two distinct real roots.

\[
m < 2 \text{ or } m > 6
\]
Question 10  (***)

The straight line \( L \) and the curve \( C \) have respective equations

\[
L: \quad 2y = 7x + 10, \\
C: \quad y = x(6 - x).
\]

a) Show that \( L \) and \( C \) do not intersect.

b) Find the coordinates of the maximum point of \( C \).

c) Sketch on the same diagram the graph of \( L \) and the graph of \( C \), showing clearly the coordinates of any points where each of the graphs meet the coordinate axes.

\[
\text{max}(3, 9)
\]
Question 11 (***+)

The quadratic curves with equations

\[ y = x^2 - 4x + 5 \quad \text{and} \quad y = m + 2x - x^2, \]

where \( m \) is a constant, touch each other at the point \( P \).

Determine the coordinates of \( P \).

\[ P \left( \frac{3}{2}, \frac{5}{4} \right) \]

Question 12 (***+)

Use the discriminant of a suitable quadratic equation to show that the graphs of the curves with equations

\[ y = 2 - \frac{1}{x} \quad \text{and} \quad y = \frac{1}{2 - x}, \]

touch each other.

\[ \text{(The discriminant for the quadratic equation is zero.)} \]

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Question 13  (***)

A quadratic curve has equation
\[ f(x) = 12x^2 + 4x - 161, \ x \in \mathbb{R}. \]

Express the above equation as the product of two linear factors.

A detailed method must be shown in this question.

\[ f(x) = (6x + 23)(2x - 7) \]
Question 14  (***)
Find the possible solutions of the quadratic equation

\[ x^2 + (k-1)x + k + 2 = 0, \]

where \( k \) is a constant, given that the equation has repeated roots.

\[ x = 1 \cup x = -3 \]

Question 15  (****)
The quadratic curves with equations

\[ y = k(2x^2 + 1) \quad \text{and} \quad y = x^2 - 2x, \]

where \( k \) is a constant, touch each other.

Determine the possible values of \( k \).

\[ k = -\frac{1}{2}, k = 1 \]
Question 16  (****)
Find the range of values that the constant $k$ can take so that

$$2x^2 + (k + 2)x + k = 0$$

has two distinct real roots.

$$k \in \mathbb{R}, k \neq 2$$

Question 17  (****)
Find the possible solutions of the quadratic equation

$$x^2 + (3 - m)x + 5 = m^2,$$

where $m$ is a constant, given that the equation has repeated roots.

$$x = -2 \cup x = -\frac{2}{3}$$
Question 18  (****)

\[ f(x) = px^2 + 4x(p + 3) + 5p, \]

where \( p \) is a non zero constant.

The equation \( f(x) = -19 \) has two distinct real roots.

Find the range of the possible values of \( p \).

\[ -4 < p < 0 \cup 0 < p < 9 \]
The above quadratic equation, where $a$ and $b$ are constants, has no real solutions.

Show clearly that

$$b > \frac{1}{2}(2a+1)(2a-1).$$
Question 20  (***)

The curve $C$ has equation

$$y = 4x^2 - 7x + 11.$$

The straight line $L$ has equation

$$y = 5x + k,$$

where $k$ is a constant.

Given that $C$ and $L$ intersect at two distinct points, show that $k > 2$. 

\[\boxed{\text{proof}}\]
Question 21  (***)
The straight line $L$ has equation

$$y = kx - 9,$$

where $k$ is a constant.

The curve $C$ has equation

$$y = 3(x+1)^2.$$

It is further given that $L$ is a tangent to $C$ at the point $P$.

Determine the possible coordinates of $P$.

\[ (-2,3) \text{ or } (2,27) \]
Question 22  (****)
The curve $C$ has equation

$$y = 3x^2 - 4x + 7.$$  

The straight line $L$ has equation

$$y = 2x + k.$$  

where $k$ is a constant.

Given that $C$ and $L$ do not intersect, show that $k < 4$
The straight line with equation

\[ y = 2x + c \]

is a tangent to the curve with equation

\[ y = x^2 + 6x + 7. \]

By using the discriminant of a suitable quadratic, determine the value of the constant \( c \) and find the point of contact between the tangent and the curve.

\[ \text{Answer: } c = 3, \ (\ -2, -1) \]
Question 24  (****)

A circle has equation
\[ x^2 + y^2 = 8y. \]

a) Find the coordinates of the centre of the circle and the size of its radius.

b) Sketch the circle.

The line with equation \( x + y = k \), where \( k \) is a constant, is a tangent to this circle.

c) Determine, as exact surds, the possible values of \( k \).

\[ (0,4), \quad r = 4, \quad k = 4 \pm 4\sqrt{2} \]
Question 25 (****)

\[ f(n) = n^2 - 2kn + k + 12, \quad n \in \mathbb{N}, \]

where \( k \) is a constant.

Given that \( f(n) = n^2 - 2kn + k + 12 \) is a square number for all values of \( n \), determine the possible values of the constant \( k \).

\[ k = -3 \quad \cup \quad k = 4 \]
Question 26     (****)
The straight line with equation

\[ y = 2x + k, \]

where \( k \) is constant, is a tangent to the curve with equation

\[ y = x^2 - 8x + 1. \]

By using the discriminant of a suitable quadratic, determine the value of the constant \( k \) and hence find the point of contact between the tangent and the curve.

\[ k = -24, \quad \left(5, -14\right) \]
Question 27 (****)

Find, in surd form, the range of values of $m$ for which the quadratic equation

$$x^2 + (3 - m)x + 10 = 3$$

has no real roots.

$$3 - 2\sqrt{7} < m < 3 + 2\sqrt{7}$$
**Question 28  (**)**

The cubic curve with equation

$$y = ax^3 + bx^2 + cx + d,$$

where $a$, $b$, $c$ are non zero constants and $d$ is a constant, has one local maximum and one local minimum.

Show clearly that

$$b^2 > 3ac$$

(proof)
Question 29   (****)
The straight line with equation
\[ y = k(4x - 17), \]
does **not** intersect with the quadratic with equation
\[ y = 13 - 8x - x^2. \]

a) Show clearly that
\[ 4k^2 + 33k + 29 < 0. \]

b) Hence find the range of possible values of \( k \).

\[ -\frac{29}{4} < k < -1 \]
Question 30  (****)

A straight line crosses the \( y \) axis at \((0, -5)\) and does not cross the curve \( y = 3x^2 - 2 \).

Find the range of the possible values of the gradient of the line.

\[-6 < \text{gradient} < 6\]

Question 31  (****)

The straight line with equation

\[ y = 3(2x + 1) \]

meets the curve with equation

\[ y = k(x^2 + 2) \].

By using the discriminant of a suitable quadratic, determine the range of the possible values of the constant \( k \).

\[ k \leq -\frac{3}{2} \quad \text{or} \quad k \geq 3 \]
Question 32  (****)

Show that the curves with equations

\[ y = x^4 - 4 \quad \text{and} \quad y = kx^2 \]

intersect for all values of the constant \( k \).

\[ \text{proof} \]

Question 33  (****)

\[ (x-a)(x-b) = m^2, \]

where \( a, b \) and \( m \) are constants.

By using discriminant considerations, show that the above quadratic equation will always have real solutions.

\[ \text{proof} \]
Question 34  (****)

The curve \( C \) and the straight line \( L \) have respective equations

\[
x^2 - \frac{y^2}{2} = 1 \quad \text{and} \quad y = x + c,
\]

where \( c \) is a constant.

Show that \( C \) and \( L \), intersect for all values of \( c \).

\[
\text{proof}
\]
Question 35  (****)

A curve $C$ has equation

$$y = \frac{1}{x-1}, \quad x \neq 1.$$  

a) Sketch the graph of $C$, clearly labelling its asymptotes and the coordinates of any point where $C$ meets the coordinate axes.

The line with equation $y = a - 2x$, where $a$ is a constant, does not meet $C$.

b) Show clearly that

$$2 - 2\sqrt{2} < a < 2 + 2\sqrt{2}.$$

asymptotes $x = 1, \quad y = 0$, $(0, -1)$
Question 36  

A circle $C$ has equation

$$x^2 + y^2 + 2x - 4y + 1 = 0$$

The straight line $L$ with equation $y = mx$ is a tangent to $C$.

Find the possible values of $m$ and hence determine the possible coordinates at which $L$ meets $C$.

$$m = 0, \quad m = \frac{4}{3}, \quad (-1,0), \quad \left(\frac{2}{3}, \frac{4}{3}\right)$$
Question 37  (****+)

The straight line $L$ crosses the $y$ axis at $(0, -1)$.

The curve with equation

$$y = x^2 + 2x$$

has no intersections with $L$.

Determine the range of the possible values of the gradient of $L$.

$\boxed{0 < m < 4}$
Question 38  (****+)

The equation of a quadratic curve $C$ is

$$y = k \left( 2x^2 - x + 1 \right) - 5x^2 + x - 2,$$

where $k$ is a constant.

Given that the graph of $C$ lies below the $x$ axis, determine the range of the possible values of $k$.

$$\square, \; k < \frac{13}{7}$$
Question 39  (***)

\[ f(x) = k + 12x - 4x^2, \]

where \( k \) is a constant.

It is further given that \( f(x) > 5 \) for some values of \( x \).

Show by suitable discriminant calculations, or otherwise, that

\[ k > -4. \]
The figure above shows the graph of the curve \( C \) with equation

\[ y = \sqrt{2x - 4}, \quad x \geq 2. \]

The point \( P \) lies on \( C \), so that the tangent to \( C \) at \( P \) passes through the origin \( O \).

Determine the coordinates of \( P \).

\[ \boxed{P(4,2)} \]

You may not use calculus in this question.
A curve \( C \) has equation

\[
y = 2x^2 + 4(p+2)x + 8p + q + 8,
\]

where \( p \) and \( q \) are constants.

The curve meets the \( y \) axis at \( y = 18 \).

Given further that \( C \) has no \( x \) intercepts, show that

\[ 2 < q < 50. \]
Question 42     (****+)

The curve $C$ has equation

$$y = \frac{x+1}{x^2+3}, \ x \in \mathbb{R}.$$ 

By considering the discriminant of a suitable quadratic equation, determine the range of the possible values of $y$.

$\quad \quad \quad \quad \quad \quad \quad [-\frac{1}{6}, \frac{1}{2}]$
Question 43  (***)

The curve \( C \) has equation

\[
y = 1 - \frac{3x}{x^2 - 2x + 4}, \quad x \in \mathbb{R}.
\]

Use a non differentiation method to find the coordinates of the stationary points of \( C \).

\((-2, \frac{3}{2}), (2, -\frac{1}{2})\)
Question 44  (***)

A quadratic curve has equation

$$f(x) = 2x^2 + (4k + 3)x + (2k - 1)(k + 2), \quad x \in \mathbb{R},$$

where \( k \) is a constant.

a) Evaluate the discriminant of \( f(x) \).

b) Express \( f(x) \) as the product of two linear factors.

$$\quad \quad b^2 - 4ac = 25, \quad f(x) = (2x + 2k - 1)(x + k + 2)$$
where $k$ is a real constant.

Given that the above equation has distinct real roots, determine the nature of the roots of the following equation

$$(k + 2)\left(x^2 + 2x + 1 + k\right) = 2k\left(x^2 + 1\right).$$

**Answer:** no real solutions.
The figure above shows the graph of the curve \( C \) and the straight line \( L \) with respective equations

\[
\frac{x^2}{5} + \frac{y^2}{4} = 1 \quad \text{and} \quad y = x - 5.
\]

When \( C \) is translated in the positive \( x \) direction, \( L \) becomes a tangent to \( C \), at some point \( P \).

Determine the exact coordinates of \( P \).

\[
\left( \frac{11}{3}, -\frac{4}{3} \right) \quad \text{or} \quad \left( \frac{19}{3}, \frac{4}{3} \right)
\]
Question 47  (*****)

\[ ax^3 + ax^2 + ax + b = 0, \]

where \( a \) and \( b \) are non zero real constants.

Given that \( x = b \) is a root of the above equation, determine the range of the possible values of \( a \).
It is given that the above nested radical converges to a limit \( L, \ L \in \mathbb{R} \).

Determine the range of possible values of \( x \).
The straight line $L$ is a tangent at the point $P$ to the curve with equation

$$y^2 = 8x.$$ 

The straight line $L$ is also a tangent at the point $Q$ to the curve with equation

$$y = -64x^2.$$ 

Determine the exact area of the triangle $POQ$, where $O$ is the origin.

$$\text{area} = \frac{3}{256}.$$
Question 50  (*****)

Find in exact form the equations of the common tangents to the curves with equations

\[ (x - 2)^2 + (y + 1)^2 = 4 \quad \text{and} \quad y = x^2 - 4x + 11. \]

\[ y = 2\sqrt{2}(x - 2) + 5, \quad y = -2\sqrt{2}(x - 2) + 5, \quad \text{and} \quad y = 2\sqrt{30}(x - 2) - 23. \]

\[ y = -2\sqrt{30}(x - 2) - 23. \]
Question 51  (*****)

The following quadratic in $x$ is given below

$$x^2 + 3kx + k^2 = 7x + 3k,$$

where $k$ is a constant.

Show that the above quadratic has real solutions whose difference is at least 2.
A circle with equation

\[ x^2 + (y-1)^2 = 1. \]

Two tangents to the circle are drawn so both are passing through the point \( (0,3) \).

Determine in exact simplified form the value of the finite region between the circle and the two tangents, shown shaded in the figure above.

\[
\text{area} = \frac{1}{3} (3\sqrt{3} - \pi)
\]
Question 53  (*****)

The points $P$ and $Q$ are the points of tangency of the common tangent to each of the curves with equations

$$y^2 = 4ax \quad \text{and} \quad ay = 2x^2,$$

where $a$ is a positive constant.

Show that $|PQ|$ is $\frac{7}{2}$ times the distance of the common tangent from the origin $O$.