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INTEGRATION INTRODUCTION

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STANDARD INTEGRATION PRACTICE

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Question 1

Integrate the following expressions with respect to x .

a) $\int 2\sqrt{x} - \frac{1}{x^2} dx$

$$\frac{4}{3}x^{\frac{3}{2}} + \frac{1}{x} + C$$

b) $\int 4\sqrt{x} - 2\sqrt{x^5} dx$

$$\frac{8}{3}x^{\frac{3}{2}} - \frac{4}{7}x^{\frac{7}{2}} + C$$

c) $\int \frac{3}{4\sqrt{x}} + \frac{1}{x^3} dx$

$$\frac{3}{2}\sqrt{x} - \frac{1}{2x^2} + C$$

d) $\int 2x\sqrt{x} - \frac{4}{3x^2} dx$

$$\frac{4}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{-1} + C$$

e) $\int 2\sqrt{x} + \frac{1}{2\sqrt{x}} dx$

$$\frac{4}{3}x^{\frac{3}{2}} + \sqrt{x} + C$$

Handwritten solutions for Question 1:

- a) $\int 2\sqrt{x} - \frac{1}{x^2} dx = \int 2x^{\frac{1}{2}} - x^{-2} dx = \frac{2}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{1}{-1}x^{-1} + C = \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{x} + C$
- b) $\int 4\sqrt{x} - 2\sqrt{x^5} dx = \int 4x^{\frac{1}{2}} - 2x^{\frac{5}{2}} dx = \frac{4}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{2}{\frac{7}{2}}x^{\frac{7}{2}} + C = \frac{8}{3}x^{\frac{3}{2}} - \frac{4}{7}x^{\frac{7}{2}} + C$
- c) $\int \frac{3}{4\sqrt{x}} + \frac{1}{x^3} dx = \int \frac{3}{4}x^{-\frac{1}{2}} + x^{-3} dx = \frac{3}{4} \cdot \frac{2}{\frac{1}{2}}x^{\frac{1}{2}} + \frac{1}{-2}x^{-2} + C = \frac{3}{2}\sqrt{x} - \frac{1}{2x^2} + C$
- d) $\int 2x\sqrt{x} - \frac{4}{3x^2} dx = \int 2x^{\frac{3}{2}} - \frac{4}{3}x^{-2} dx = \frac{2}{\frac{5}{2}}x^{\frac{5}{2}} - \frac{4}{3} \cdot \frac{1}{-1}x^{-1} + C = \frac{4}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{-1} + C$
- e) $\int 2\sqrt{x} + \frac{1}{2\sqrt{x}} dx = \int 2x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} dx = \frac{2}{\frac{3}{2}}x^{\frac{3}{2}} + \frac{1}{2} \cdot \frac{2}{\frac{1}{2}}x^{\frac{1}{2}} + C = \frac{4}{3}x^{\frac{3}{2}} + \sqrt{x} + C$

Question 2

Integrate the following expressions with respect to x .

a) $\int \frac{4}{x^3} + 12x^{\frac{2}{3}} dx$

$$-2x^{-2} + \frac{36}{5}x^{\frac{5}{3}} + C$$

b) $\int 14x^{\frac{3}{4}} - \frac{3}{2x^4} dx$

$$8x^{\frac{7}{4}} + \frac{1}{2}x^{-3} + C$$

c) $\int 4x - \frac{6}{x^3} + 4\sqrt{x} - 1 dx$

$$2x^2 - 3x^{-2} + \frac{8}{3}x^{\frac{3}{2}} - x + C$$

d) $\int \sqrt[3]{x^2} - \frac{4}{x^2} dx$

$$\frac{3}{5}x^{\frac{5}{3}} + \frac{4}{x} + C$$

e) $\int 6\sqrt{x^3} - \frac{1}{2x^6} dx$

$$\frac{12}{5}x^{\frac{5}{2}} + \frac{1}{10}x^{-5} + C$$

Handwritten solutions for Question 2:

- a) $\int \frac{4}{x^3} + 12x^{\frac{2}{3}} dx = \int 4x^{-3} + 12x^{\frac{2}{3}} dx = -\frac{4}{2}x^{-2} + \frac{36}{5}x^{\frac{5}{3}} + C = -2x^{-2} + \frac{36}{5}x^{\frac{5}{3}} + C //$
- b) $\int 14x^{\frac{3}{4}} - \frac{3}{2x^4} dx = \int 14x^{\frac{3}{4}} - \frac{3}{2}x^{-4} dx = \frac{14}{\frac{3}{4}+1}x^{\frac{3}{4}+1} - \frac{3}{2 \cdot (-3)}x^{-3} + C = 8x^{\frac{7}{4}} + \frac{1}{2}x^{-3} + C //$
- c) $\int 4x - \frac{6}{x^3} + 4\sqrt{x} - 1 dx = \int 4x - 6x^{-3} + 4x^{\frac{1}{2}} - 1 dx = 2x^2 - \frac{6}{-2}x^{-2} + \frac{4}{\frac{1}{2}+1}x^{\frac{1}{2}+1} - x + C = 2x^2 - 3x^{-2} + \frac{8}{3}x^{\frac{3}{2}} - x + C //$
- d) $\int \sqrt[3]{x^2} - \frac{4}{x^2} dx = \int x^{\frac{2}{3}} - 4x^{-2} dx = \frac{3}{\frac{2}{3}+1}x^{\frac{2}{3}+1} - \frac{4}{-1}x^{-1} + C = \frac{3}{5}x^{\frac{5}{3}} + \frac{4}{x} + C //$
- e) $\int 6\sqrt{x^3} - \frac{1}{2x^6} dx = \int 6x^{\frac{3}{2}} - \frac{1}{2}x^{-6} dx = \frac{6}{\frac{3}{2}+1}x^{\frac{3}{2}+1} - \frac{1}{2 \cdot (-5)}x^{-5} + C = \frac{12}{5}x^{\frac{5}{2}} + \frac{1}{10}x^{-5} + C //$

Question 3

Integrate the following expressions with respect to x .

a) $\int \frac{2+5x}{3x^3} dx$

$$-\frac{1}{3x^2} - \frac{5}{3x} + C$$

b) $\int \frac{2x^2+x}{2\sqrt{x}} dx$

$$\frac{2}{5}x^{\frac{5}{2}} + \frac{1}{3}x^{\frac{3}{2}} + C$$

c) $\int \frac{2x+\sqrt{x}}{3x^3} dx$

$$-\frac{2}{3}x^{-1} - \frac{2}{9}x^{-\frac{3}{2}} + C$$

d) $\int \frac{\sqrt{x}(4-x)}{2x^2} dx$

$$-\sqrt{x} - \frac{4}{\sqrt{x}} + C$$

e) $\int \frac{(x+1)(2x-1)}{2x^5} dx$

$$-\frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{8x^4} + C$$

Handwritten solutions for Question 3:

- a) $\int \frac{2+5x}{3x^3} dx = \int \frac{2}{3x^3} + \frac{5x}{3x^3} dx = \int \frac{2}{3}x^{-3} + \frac{5}{3}x^{-2} dx = -\frac{2}{3}x^{-2} - \frac{5}{3}x^{-1} + C = -\frac{2}{3x^2} - \frac{5}{3x} + C$
- b) $\int \frac{2x^2+x}{2\sqrt{x}} dx = \int \frac{2x^2}{2\sqrt{x}} + \frac{x}{2\sqrt{x}} dx = \int x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{3}x^{\frac{3}{2}} + C$
- c) $\int \frac{2x+\sqrt{x}}{3x^3} dx = \int \frac{2x}{3x^3} + \frac{\sqrt{x}}{3x^3} dx = \int \frac{2}{3}x^{-2} + \frac{1}{3}x^{-\frac{5}{2}} dx = -\frac{2}{3}x^{-1} - \frac{2}{9}x^{-\frac{3}{2}} + C = -\frac{2}{3x} - \frac{2}{9\sqrt{x}} + C$
- d) $\int \frac{\sqrt{x}(4-x)}{2x^2} dx = \int \frac{4\sqrt{x}}{2x^2} - \frac{x\sqrt{x}}{2x^2} dx = \int \frac{2}{x^{\frac{3}{2}}} - \frac{1}{2}x^{-\frac{1}{2}} dx = -\frac{4}{x^{\frac{1}{2}}} - \frac{1}{2}x^{\frac{1}{2}} + C = -\frac{4}{\sqrt{x}} - \frac{1}{2}\sqrt{x} + C$
- e) $\int \frac{(x+1)(2x-1)}{2x^5} dx = \int \frac{2x^2+x-2x-1}{2x^5} dx = \int \frac{2x^2}{2x^5} + \frac{x}{2x^5} - \frac{2x}{2x^5} - \frac{1}{2x^5} dx = \int x^{-3} + \frac{1}{2}x^{-4} - \frac{1}{x^4} - \frac{1}{2}x^{-5} dx = -\frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{8x^4} + C$

Question 4

Integrate the following expressions with respect to x .

a) $\int \frac{x+x^2}{\sqrt{x}} dx$

$$\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$$

b) $\int \frac{4x^3 + \sqrt{x}}{2x^2} dx$

$$x^2 - x^{-\frac{1}{2}} + C$$

c) $\int \frac{x^2+2}{x^4} dx$

$$-x^{-1} - \frac{2}{3}x^{-3} + C$$

d) $\int \frac{1-\sqrt{x}}{4x^3} dx$

$$-\frac{1}{8}x^{-2} + \frac{1}{6}x^{-\frac{3}{2}} + C$$

e) $\int \frac{\sqrt[3]{x^5} - 2x\sqrt{x}}{3x} dx$

$$\frac{1}{5}x^{\frac{5}{3}} - \frac{4}{9}x^{\frac{3}{2}} + C$$

Handwritten solutions for Question 4:

- a) $\int \frac{x+x^2}{\sqrt{x}} dx = \int \frac{x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}} dx = \int x^{\frac{1}{2}} + x^{\frac{3}{2}} dx = \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$
- b) $\int \frac{4x^3 + \sqrt{x}}{2x^2} dx = \int \frac{4x^3}{2x^2} + \frac{\sqrt{x}}{2x^2} dx = \int 2x + \frac{1}{2}x^{-\frac{3}{2}} dx = x^2 - x^{-\frac{1}{2}} + C$
- c) $\int \frac{x^2+2}{x^4} dx = \int \frac{x^2}{x^4} + \frac{2}{x^4} dx = \int x^{-2} + 2x^{-4} dx = -x^{-1} - \frac{2}{3}x^{-3} + C$
- d) $\int \frac{1-\sqrt{x}}{4x^3} dx = \int \frac{1}{4x^3} - \frac{\sqrt{x}}{4x^3} dx = \int \frac{1}{4}x^{-3} - \frac{1}{4}x^{-\frac{5}{2}} dx = -\frac{1}{8}x^{-2} + \frac{1}{6}x^{-\frac{3}{2}} + C$
- e) $\int \frac{\sqrt[3]{x^5} - 2x\sqrt{x}}{3x} dx = \int \frac{x^{\frac{5}{3}} - 2x^{\frac{3}{2}}}{3x} dx = \int \frac{x^{\frac{5}{3}}}{3x} - \frac{2x^{\frac{3}{2}}}{3x} dx = \int \frac{1}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{1}{2}} dx = \frac{1}{5}x^{\frac{5}{3}} - \frac{4}{9}x^{\frac{3}{2}} + C$

Question 5

Integrate the following expressions with respect to x .

a) $\int x(\sqrt{x} + x^{-4}) dx$

$$\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{2}x^{-2} + C$$

b) $\int \frac{1}{\sqrt{x}} \left(\frac{2}{x} - \frac{3}{4x^2} \right) dx$

$$-4x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} + C$$

c) $\int 4x^{\frac{7}{2}} \left(\frac{6}{x^2} - \frac{5}{\sqrt{x}} \right) dx$

$$\frac{48}{5}x^{\frac{5}{2}} - 5x^4 + C$$

d) $\int 2\sqrt{x} \left(\frac{5}{x} + x^2 \right) dx$

$$20x^{\frac{1}{2}} + \frac{4}{7}x^{\frac{7}{2}} + C$$

e) $\int \frac{2}{x^{\frac{3}{2}}} \left(\frac{7x^3 - 5x^2}{4x} \right) dx$

$$\frac{7}{3}x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + C$$

$$\begin{aligned}
 \text{a) } \int x(\sqrt{x} + x^{-4}) dx &= \int x(x^{\frac{1}{2}} + x^{-4}) dx = \int x^{\frac{3}{2}} + x^{-3} dx = \left(\frac{\frac{3}{2}}{\frac{3}{2}+1} x^{\frac{3}{2}+1} + \frac{1}{-2} x^{-2} \right) + C \\
 &= \frac{2}{5} x^{\frac{5}{2}} - \frac{1}{2} x^{-2} + C \\
 \text{b) } \int \frac{1}{\sqrt{x}} \left(\frac{2}{x} - \frac{3}{4x^2} \right) dx &= \int x^{-\frac{1}{2}} \left(2x^{-1} - \frac{3}{4} x^{-2} \right) dx = \int 2x^{-\frac{3}{2}} - \frac{3}{4} x^{-\frac{5}{2}} dx \\
 &= \left(\frac{2}{-\frac{3}{2}+1} x^{-\frac{3}{2}+1} - \frac{3}{4} \frac{x^{-\frac{5}{2}+1}}{-\frac{5}{2}+1} \right) + C = -4x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}} + C \\
 \text{c) } \int 4x^{\frac{7}{2}} \left(\frac{6}{x^2} - \frac{5}{\sqrt{x}} \right) dx &= \int 4x^{\frac{7}{2}} \left(6x^{-2} - 5x^{-\frac{1}{2}} \right) dx = \int 24x^{\frac{3}{2}} - 20x^3 dx \\
 &= \left(\frac{24}{\frac{3}{2}+1} x^{\frac{3}{2}+1} - \frac{20}{4} x^4 \right) + C = \frac{48}{5} x^{\frac{5}{2}} - 5x^4 + C \\
 \text{d) } \int 2\sqrt{x} \left(\frac{5}{x} + x^2 \right) dx &= \int 2x^{\frac{1}{2}} \left(5x^{-1} + x^2 \right) dx = \int 10x^{-\frac{1}{2}} + 2x^{\frac{5}{2}} dx \\
 &= \left(\frac{10}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + \frac{2}{\frac{5}{2}+1} x^{\frac{5}{2}+1} \right) + C = 20x^{\frac{1}{2}} + \frac{4}{7} x^{\frac{7}{2}} + C \\
 \text{e) } \int \frac{2}{x^{\frac{3}{2}}} \left(\frac{7x^3 - 5x^2}{4x} \right) dx &= \int 2x^{-\frac{3}{2}} \left(\frac{7x^2 - 5x}{4} \right) dx = \int \frac{1}{2} x^{\frac{1}{2}} (7x^2 - 5x) dx \\
 &= \int \left(\frac{7}{2} x^{\frac{5}{2}} - \frac{5}{2} x^{\frac{3}{2}} \right) dx = \left(\frac{7}{2} \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{5}{2} \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right) + C \\
 &= \frac{7}{3} x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + C
 \end{aligned}$$

Question 6

Integrate the following expressions with respect to x .

a) $\int \frac{(2x-1)(2x-3)}{2x^{\frac{3}{2}}} dx$

$$\frac{4}{3}x^{\frac{3}{2}} - 8x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} + C$$

b) $\int \frac{(1+2\sqrt{x})^2}{2x^3} dx$

$$-\frac{1}{4}x^{-2} - \frac{4}{3}x^{-\frac{3}{2}} - 2x^{-1} + C$$

c) $\int \frac{2x^3 + \sqrt{x^3}}{\sqrt{x}} dx$

$$\frac{4}{7}x^{\frac{7}{2}} + \frac{1}{2}x^2 + C$$

d) $\int \frac{(1+\sqrt{x})(3-\sqrt{x})}{x^4} dx$

$$-x^{-3} - \frac{4}{5}x^{-\frac{5}{2}} + \frac{1}{2}x^{-2} + C$$

e) $\int \frac{(2x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})}{3x^5} dx$

$$-\frac{1}{3}x^{-2} + \frac{1}{3}x^{-4} - \frac{1}{9}x^{-3} + \frac{2}{15}x^{-5} + C$$

Handwritten solutions for Question 6:

a) $\int \frac{(2x-1)(2x-3)}{2x^{\frac{3}{2}}} dx = \int \frac{4x^2 - 8x + 3}{2x^{\frac{3}{2}}} dx = \int \frac{4x^2}{2x^{\frac{3}{2}}} - \frac{8x}{2x^{\frac{3}{2}}} + \frac{3}{2x^{\frac{3}{2}}} dx$
 $= \int 2x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} dx = \frac{4}{3}x^{\frac{3}{2}} - 8x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} + C$

b) $\int \frac{(1+2\sqrt{x})^2}{2x^3} dx = \int \frac{1+4\sqrt{x}+4x}{2x^3} dx = \int \frac{1}{2x^3} + \frac{2\sqrt{x}}{x^3} + \frac{2x}{x^3} dx$
 $= \int \frac{1}{2}x^{-3} + 2x^{-\frac{5}{2}} + 2x^{-2} dx = -\frac{1}{4}x^{-2} - \frac{4}{3}x^{-\frac{3}{2}} - 2x^{-1} + C$

c) $\int \frac{2x^3 + \sqrt{x^3}}{\sqrt{x}} dx = \int \frac{2x^3 + x^{\frac{3}{2}}}{x^{\frac{1}{2}}} dx = \int 2x^{\frac{5}{2}} + 2x dx$
 $= \frac{4}{7}x^{\frac{7}{2}} + \frac{1}{2}x^2 + C$

d) $\int \frac{(1+\sqrt{x})(3-\sqrt{x})}{x^4} dx = \int \frac{3+2\sqrt{x}-x-\frac{1}{2}}{x^4} dx = \int \frac{5}{2}x^{-4} + \frac{2\sqrt{x}}{x^4} - \frac{x}{x^4} - \frac{1}{2x^4} dx$
 $= \int \frac{5}{2}x^{-4} + 2x^{-\frac{7}{2}} - x^{-3} - \frac{1}{2}x^{-4} dx = -x^{-3} - \frac{4}{5}x^{-\frac{5}{2}} + \frac{1}{2}x^{-2} + C$

e) $\int \frac{(2x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})}{3x^5} dx = \int \frac{2x^2 - 4x + 1 - 2x^{-2}}{3x^5} dx = \int \frac{2x^2}{3x^5} - \frac{4x}{3x^5} + \frac{1}{3x^5} - \frac{2x^{-2}}{3x^5} dx$
 $= \int \frac{2}{3}x^{-3} - \frac{4}{3}x^{-4} + \frac{1}{3}x^{-5} - \frac{2}{3}x^{-7} dx = -\frac{1}{3}x^{-2} + \frac{1}{3}x^{-4} - \frac{1}{9}x^{-3} + \frac{2}{15}x^{-5} + C$

Question 7

Evaluate the following integrals.

a) $\int_1^4 \frac{2}{\sqrt{x}} dx$

4

b) $\int_1^2 4x^3 + 5 + \frac{2}{x^2} dx$

21

c) $\int_1^2 3x^2 - 1 - \frac{4}{x^2} dx$

4

d) $\int_1^3 \frac{x}{3} + \frac{1}{x^2} dx$

2

e) $\int_0^2 (2x-1)(3x-4) dx$

2

$$\begin{aligned}
 \text{(a)} \quad \int_1^4 \frac{2}{\sqrt{x}} dx &= \int_1^4 2x^{-\frac{1}{2}} dx = \left[\frac{2x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 = \left[4x^{\frac{1}{2}} \right]_1^4 \\
 &= (4 \times 4^{\frac{1}{2}}) - (4 \times 1^{\frac{1}{2}}) = 8 - 4 = 4 \\
 \text{(b)} \quad \int_1^2 4x^3 + \frac{2}{x^2} dx &= \int_1^2 4x^3 + 5 + \frac{2}{x^2} dx = \left[x^4 + 5x - \frac{2}{x} \right]_1^2 \\
 &= \left[x^4 + 5x - \frac{2}{x} \right]_1^2 = (16 + 10 - 1) - (1 + 5 - 2) \\
 &= 25 - 4 = 21 \\
 \text{(c)} \quad \int_1^2 3x^2 - 1 - \frac{4}{x^2} dx &= \int_1^2 3x^2 - 1 - 4x^{-2} dx = \left[x^3 - x + 4x^{-1} \right]_1^2 \\
 &= \left[x^3 - x + \frac{4}{x} \right]_1^2 = (8 - 2 + 2) - (1 - 1 + 4) \\
 &= 8 - 4 = 4 \\
 \text{(d)} \quad \int_1^3 \frac{x}{3} + \frac{1}{x^2} dx &= \int_1^3 \frac{1}{3}x + x^{-2} dx = \left[\frac{1}{6}x^2 - x^{-1} \right]_1^3 \\
 &= \left[\frac{1}{6}x^2 - \frac{1}{x} \right]_1^3 = \left(\frac{1}{6} \times 9 - \frac{1}{3} \right) - \left(\frac{1}{6} - 1 \right) \\
 &= \frac{3}{2} - \frac{1}{3} - \frac{1}{6} + 1 = \frac{9}{6} - \frac{2}{6} + \frac{6}{6} = \frac{13}{6} - \frac{1}{6} + 1 \\
 &= \frac{12}{6} + 1 = 2 \\
 \text{(e)} \quad \int_0^2 (2x-1)(3x-4) dx &= \int_0^2 6x^2 - 11x + 4 dx \\
 &= \left[2x^3 - \frac{11}{2}x^2 + 4x \right]_0^2 \\
 &= \left(16 - 22 + 8 \right) - (0) \\
 &= 2
 \end{aligned}$$

Question 8

Evaluate the following integrals.

a) $\int_1^3 x^2 + \frac{14}{x^2} dx$

18

b) $\int_1^2 x^4 + 3 - \frac{2}{5x^2} dx$

9

c) $\int_1^5 2x - \frac{15}{x^2} dx$

12

d) $\int_1^4 \frac{x^3 + 2\sqrt{x}}{x} dx$

25

e) $\int_1^4 \sqrt{x} (5x - 3) dx$

48

a) $\int_1^3 x^2 + \frac{14}{x^2} dx = \int_1^3 x^2 + 14x^{-2} dx = \left[\frac{1}{3}x^3 - 14x^{-1} \right]_1^3 = \left[\frac{1}{3}x^3 - \frac{14}{x} \right]_1^3$
 $= \left(\frac{1}{3} \times 3^3 - \frac{14}{3} \right) - \left(\frac{1}{3} \times 1^3 - \frac{14}{1} \right) = (9 - \frac{14}{3}) - (\frac{1}{3} - 14)$
 $= 9 - \frac{14}{3} - \frac{1}{3} + 14 = 9 - 5 + 14 = 18 //$

b) $\int_1^2 x^4 + 3 - \frac{2}{5x^2} dx = \int_1^2 x^4 + 3 - \frac{2}{5}x^{-2} dx = \left[\frac{1}{5}x^5 + 3x + \frac{2}{5}x^{-1} \right]_1^2$
 $= \left[\frac{1}{5}x^5 + 3x + \frac{2}{5x} \right]_1^2 = \left(\frac{32}{5} + 6 + \frac{1}{5} \right) - \left(\frac{1}{5} + 3 + \frac{2}{5} \right)$
 $= \frac{32}{5} + 6 + \frac{1}{5} - \frac{1}{5} - 3 - \frac{2}{5} = \frac{32}{5} + 3 - \frac{2}{5} = \frac{32-2}{5} + 3 = \frac{30}{5} + 3 = 6 + 3 = 9 //$

c) $\int_1^5 2x - \frac{15}{x^2} dx = \int_1^5 2x - 15x^{-2} dx = \left[x^2 + 15x^{-1} \right]_1^5 = \left[x^2 + \frac{15}{x} \right]_1^5$
 $= (25 + 3) - (1 + 15) = 28 - 16 = 12 //$

d) $\int_1^4 \frac{x^3 + 2\sqrt{x}}{x} dx = \int_1^4 \frac{x^3}{x} + \frac{2\sqrt{x}}{x} dx = \int_1^4 x^2 + 2x^{-\frac{1}{2}} dx = \left[\frac{1}{3}x^3 + 4x^{\frac{1}{2}} \right]_1^4$
 $= \left[\frac{1}{3}x^3 + 4\sqrt{x} \right]_1^4 = \left(\frac{64}{3} + 8 \right) - \left(\frac{1}{3} + 4 \right) = \frac{64}{3} + 8 - \frac{1}{3} - 4 = \frac{64-1}{3} + 4 = \frac{63}{3} + 4 = 21 + 4 = 25 //$

e) $\int_1^4 \sqrt{x} (5x - 3) dx = \int_1^4 x^{\frac{1}{2}} (5x - 3) dx = \int_1^4 5x^{\frac{3}{2}} - 3x^{\frac{1}{2}} dx$
 $= \left[\frac{10}{5}x^{\frac{5}{2}} - \frac{6}{\frac{3}{2}}x^{\frac{3}{2}} \right]_1^4 = \left[2x^{\frac{5}{2}} - 4x^{\frac{3}{2}} \right]_1^4$
 $= (2 \times 4^{\frac{5}{2}} - 4 \times 4^{\frac{3}{2}}) - (2 \times 1^{\frac{5}{2}} - 4 \times 1^{\frac{3}{2}}) = 2 \times 32 - 2 \times 8$
 $= 64 - 16 = 48 //$

Question 9

Evaluate the following integrals.

$$\text{a) } \int_1^2 \frac{2x^5 + 3}{x^2} dx \quad \boxed{9}$$

$$\text{b) } \int_1^3 \frac{2x^5 - 21}{x^3} dx \quad \boxed{8}$$

$$\text{c) } \int_1^9 (1 + 3\sqrt{x})^2 dx \quad \boxed{472}$$

$$\text{d) } \int_0^6 \left(x^{\frac{1}{2}} + x^{\frac{3}{2}} \right)^2 dx \quad \boxed{486}$$

$$\text{e) } \int_0^4 \left(x^{\frac{1}{2}} - 3 \right)^2 dx \quad \boxed{12}$$

$$\begin{aligned} \text{a) } \int_1^2 \frac{2x^5 + 3}{x^2} dx &= \int_1^2 \frac{2x^5}{x^2} + \frac{3}{x^2} dx = \int_1^2 2x^3 + 3x^{-2} dx = \left[\frac{2x^4}{4} - \frac{3}{x} \right]_1^2 = \left[\frac{1}{2}x^4 - \frac{3}{x} \right]_1^2 = \left(8 - \frac{3}{2} \right) - \left(\frac{1}{2} - 3 \right) \\ &= 8 - \frac{3}{2} - \frac{1}{2} + 3 = 11 - 2 = 9 \\ \text{b) } \int_1^3 \frac{2x^5 - 21}{x^3} dx &= \int_1^3 \frac{2x^5}{x^3} - \frac{21}{x^3} dx = \int_1^3 2x^2 - 21x^{-3} dx = \left[\frac{2}{3}x^3 + \frac{21}{2}x^{-2} \right]_1^3 = \left[\frac{2}{3}x^3 + \frac{21}{2x^2} \right]_1^3 = \left(18 + \frac{21}{18} \right) - \left(\frac{2}{3} + \frac{21}{2} \right) \\ &= 18 + \frac{7}{6} - \frac{2}{3} - \frac{21}{2} = \frac{36 \times 4 + 7 - 6 \times 2 - 63 \times 2}{6} = \frac{144 + 7 - 12 - 126}{6} = \frac{115 - 12}{6} = \frac{103}{6} = 17 \frac{1}{6} \\ \text{c) } \int_1^9 (1 + 3\sqrt{x})^2 dx &= \int_1^9 (1 + 6\sqrt{x} + 9x) dx = \int_1^9 (1 + 6x^{\frac{1}{2}} + 9x) dx = \left[x + \frac{6}{\frac{3}{2}}x^{\frac{3}{2}} + \frac{9}{2}x^2 \right]_1^9 = \left[x + 4x^{\frac{3}{2}} + \frac{9}{2}x^2 \right]_1^9 \\ &= \left(9 + 108 + \frac{729}{2} \right) - \left(1 + 4 + \frac{9}{2} \right) = 117 + \frac{729}{2} - 5 - \frac{9}{2} = 112 + 360 = 472 \\ \text{d) } \int_0^6 (x^{\frac{1}{2}} + x^{\frac{3}{2}})^2 dx &= \int_0^6 (x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}})(x^{\frac{3}{2}}) + (x^{\frac{3}{2}})^2 dx = \int_0^6 (x + 2x^2 + x^3) dx = \left[\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 \right]_0^6 \\ &= \left(18 + 144 + 36 \right) - (0) = 198 \\ \text{e) } \int_0^4 (x^{\frac{1}{2}} - 3)^2 dx &= \int_0^4 (x^{\frac{1}{2}})^2 - 2(x^{\frac{1}{2}})(3) + 3^2 dx = \int_0^4 (x - 6x^{\frac{1}{2}} + 9) dx = \left[\frac{1}{2}x^2 - \frac{6}{\frac{3}{2}}x^{\frac{3}{2}} + 9x \right]_0^4 = \left[\frac{1}{2}x^2 - 4x^{\frac{3}{2}} + 9x \right]_0^4 \\ &= \left(8 - 32 + 36 \right) - (0) = 12 \end{aligned}$$

Question 10

Evaluate the following integrals.

a) $\int_{\frac{1}{2}}^1 \frac{4-x}{2x^3} dx$

$$\boxed{\frac{5}{2}}$$

b) $\int_1^5 3\sqrt{x} - \frac{1}{\sqrt{x}} dx$

$$\boxed{8\sqrt{5}}$$

c) $\int_0^{36} (2+\sqrt{x})^2 dx$

$$\boxed{1368}$$

d) $\int_0^1 \frac{15(2x+1)^2}{2\sqrt{x}} dx$

$$\boxed{47}$$

e) $\int_1^2 \left(x^{\frac{3}{2}} - 8x^{-\frac{3}{2}} \right)^2 dx$

$$\boxed{\frac{47}{4}}$$

a) $\int_{\frac{1}{2}}^1 \frac{4-x}{2x^3} dx = \int_{\frac{1}{2}}^1 \frac{4}{2x^3} - \frac{x}{2x^3} dx = \int_{\frac{1}{2}}^1 2x^{-3} - \frac{1}{2}x^{-2} dx = \left[-x^{-2} + \frac{1}{2}x^{-1} \right]_{\frac{1}{2}}^1$
 $= \left[\frac{1}{2x} - \frac{1}{2x^2} \right]_{\frac{1}{2}}^1 = \left(\frac{1}{2} - 1 \right) - \left(1 - 4 \right) = \frac{1}{2} - 1 - 1 + 4 = \frac{5}{2}$

b) $\int_1^5 3\sqrt{x} - \frac{1}{\sqrt{x}} dx = \int_1^5 3x^{\frac{1}{2}} - x^{-\frac{1}{2}} dx = \left[2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} \right]_1^5$
 $= (2 \times 5^{\frac{3}{2}} - 2 \times 5^{\frac{1}{2}}) - (2 - 2) = 2 \times 5\sqrt{5} - 2\sqrt{5} = 10\sqrt{5} - 2\sqrt{5} = 8\sqrt{5}$

c) $\int_0^{36} (2+\sqrt{x})^2 dx = \int_0^{36} 4 + 4\sqrt{x} + x dx = \int_0^{36} 4 + 4x^{\frac{1}{2}} + x dx$
 $= \left[4x + \frac{8}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^{36} = (144 + 576 + 648) - 0 = 1368$

d) $\int_0^1 \frac{15(2x+1)^2}{2\sqrt{x}} dx = \int_0^1 \frac{15(4x^2+4x+1)}{2\sqrt{x}} dx = \int_0^1 \frac{60x^2+60x+15}{2\sqrt{x}} dx$
 $= \int_0^1 30x^{\frac{3}{2}} + 30x^{\frac{1}{2}} + \frac{15}{2}x^{-\frac{1}{2}} dx = \left[12x^{\frac{5}{2}} + 20x^{\frac{3}{2}} + 15x^{\frac{1}{2}} \right]_0^1$
 $= (12 + 20 + 15) - (0 + 0 + 0) = 47$

e) $\int_1^2 (x^{\frac{3}{2}} - 8x^{-\frac{3}{2}})^2 dx = \int_1^2 x^3 - 16 + 64x^{-3} dx = \left[\frac{1}{4}x^4 - 16x - 32x^{-2} \right]_1^2$
 $= \left[\frac{1}{4}x^4 - 16x - \frac{32}{x^2} \right]_1^2 = (4 - 32 - 8) - \left(\frac{1}{4} - 16 - 32 \right)$
 $= 4 - 32 - 8 - \frac{1}{4} + 16 + 32 = 12 - \frac{1}{4} = \frac{47}{4}$

Created by T. Madas

VARIOUS INTEGRATION QUESTIONS

Created by T. Madas

Question 1

$$y = 2x^2 - \frac{6}{x^3} + 8x^3, \quad x \neq 0.$$

Find an expression for

$$\int y \, dx.$$

$$\frac{2}{3}x^3 + 3x^{-2} + 2x^4 + C$$

$$\int y \, dx = \int 2x^2 - 6x^{-3} + 8x^3 \, dx = \frac{2}{3}x^3 - \frac{6}{-2}x^{-2} + \frac{8}{4}x^4 + C$$

$$= \frac{2}{3}x^3 + 3x^{-2} + 2x^4 + C$$

Question 2

$$y = 3x^2 - 6\sqrt{x} - \frac{1}{x^2} + 4, \quad x > 0.$$

Find an expression for

$$\int y \, dx.$$

$$4x + x^3 - 4x^{\frac{3}{2}} + x^{-1} + C$$

$$\int y \, dx = \int 3x^2 - 6x^{\frac{1}{2}} - \frac{1}{x^2} + 4 \, dx = \frac{3}{3}x^3 - \frac{6}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{1}{-1}x^{-1} + 4x + C$$

$$= x^3 - 4x^{\frac{3}{2}} + x^{-1} + 4x + C$$

Question 3

$$f(x) = 6x + 9\sqrt{x} - \frac{4}{x^2}, \quad x > 0.$$

Find an expression for

$$\int f(x) \, dx.$$

$$3x^2 + 6x^{\frac{3}{2}} + 4x^{-1} + C$$

$$\int f(x) \, dx = \int 6x + 9x^{\frac{1}{2}} - 4x^{-2} \, dx = \frac{6x^2}{2} + \frac{9x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{4x^{-2+1}}{-2+1} + C = 3x^2 + 6x^{\frac{3}{2}} + 4x^{-1} + C$$

Question 4

The point $P(1,3)$ lies on the curve with equation $y = f(x)$, whose gradient function is given by

$$f'(x) = 6x^2 - 4x, \quad x \in \mathbb{R}.$$

Find an equation for $f(x)$.

$$f(x) = 2x^3 - 2x^2 + 3$$

$$\begin{aligned} f(x) &= \int 6x^2 - 4x \, dx \\ f(x) &= 2x^3 - 2x^2 + C \\ \bullet \text{ when } x=1, \quad y=3 \\ 3 &= 2 - 2 + C \\ C &= 3 \end{aligned} \quad \text{Hence } f(x) = 2x^3 - 2x^2 + 3$$

Question 5

The point $P(3, -1)$ lies on the curve with equation $y = f(x)$, whose gradient function is given by

$$f'(x) = 1 - x^2, \quad x \in \mathbb{R}.$$

Find an equation for $f(x)$.

$$f(x) = 5 + x - \frac{1}{3}x^3$$

Handwritten solution for Question 5:

$$f(x) = \int (1 - x^2) dx$$

$$f(x) = x - \frac{1}{3}x^3 + C$$

• When $x = 3$, $y = -1$

$$-1 = 3 - \frac{1}{3}(3)^3 + C$$

$$-1 = 3 - 9 + C$$

$$5 = C$$

$\therefore f(x) = 5 + x - \frac{1}{3}x^3$

Question 6

By showing clear workings, find the value of

$$\int_1^9 6\sqrt{x} - \frac{6}{\sqrt{x}} dx.$$

80

Handwritten solution for Question 6:

$$\int_1^9 \left(6\sqrt{x} - \frac{6}{\sqrt{x}} \right) dx = \int_1^9 \left(6x^{\frac{1}{2}} - 6x^{-\frac{1}{2}} \right) dx = \left[4x^{\frac{3}{2}} - 12x^{\frac{1}{2}} \right]_1^9$$

$$= (108 - 36) - (4 - 12) = 72 + 8 = 80$$

Question 7

The curve C with equation $y = f(x)$ has gradient function

$$\frac{dy}{dx} = 9x^2 + \frac{7}{x^2}, \quad x \neq 0.$$

The point $A(-1, -1)$ lies on C .

Find an equation for C .

$$y = 3x^3 - \frac{7}{x} - 5$$

Handwritten working for Question 7:

$$y = \int (9x^2 + \frac{7}{x^2}) dx = \int 9x^2 + 7x^{-2} dx$$

$$y = 3x^3 - 7x^{-1} + C$$

$$y = 3x^3 - \frac{7}{x} + C$$

• Use $x = -1, y = -1$

$$-1 = 3(-1)^3 - \frac{7}{-1} + C$$

$$-1 = -3 + 7 + C$$

$$C = -5$$

Final $y = 3x^3 - \frac{7}{x} - 5$

Question 8

$$y = x(6x - 5\sqrt{x}), \quad x \geq 0.$$

By showing all steps in the workings, find an expression for

$$\int y \, dx.$$

$$2x^3 - 2x^{\frac{5}{2}} + C$$

Handwritten working for Question 8:

$$\int y \, dx = \int x(6x - 5\sqrt{x}) \, dx = \int (6x^2 - 5x^{\frac{3}{2}}) \, dx$$

$$= \int 6x^2 - 5x^{\frac{3}{2}} \, dx = \frac{6}{3}x^3 - \frac{5}{\frac{3}{2}+1}x^{\frac{3}{2}+1} + C$$

$$= 2x^3 - 2x^{\frac{5}{2}} + C$$

Question 9

The point $P(4,9)$ lies on the curve with equation $y = f(x)$, whose gradient function is given by

$$f'(x) = 1 + \frac{2}{\sqrt{x}}, \quad x > 0.$$

Find an equation for $f(x)$.

$$f(x) = x + 4\sqrt{x} - 3$$

$\text{If } f'(x) = 1 + \frac{2}{\sqrt{x}} = 1 + 2x^{-\frac{1}{2}}$ where $a=4, \quad b=9 \leftarrow f(4)$
 $\text{Then } f(x) = \int (1 + 2x^{-\frac{1}{2}}) dx$ $\therefore 9 = a + b + C$
 $\Rightarrow f(x) = x + 4x^{\frac{1}{2}} + C$ $C = -5$
 $\Rightarrow f(x) = x + 4\sqrt{x} - 3$ $\therefore f(x) = x + 4\sqrt{x} - 3$

Question 10

$$\frac{dy}{dx} = 4 + \frac{1}{x^2}, \quad x \neq 0.$$

Given that $y = 5$ when $x = 1$, express y in terms of x .

$$y = 4x - \frac{1}{x} + 2$$

$\text{If } \frac{dy}{dx} = 4 + \frac{1}{x^2} = 4 + x^{-2}$ where $x=1, y=5$
 $\text{Then } y = \int (4 + x^{-2}) dx$ $S = 4x - \frac{1}{x} + C$
 $\Rightarrow y = 4x - \frac{1}{x} + C$ $5 = 4 - 1 + C$
 $\Rightarrow y = 4x - \frac{1}{x} + C$ $5 = 3 + C$
 $\Rightarrow y = 4x - \frac{1}{x} + 2$ $C = 2$
 $\therefore y = 4x - \frac{1}{x} + 2$

Question 11

$$f'(x) = (3x-1)^2.$$

Given that $f(3) = 56$, find an expression for $f(x)$.

$$f(x) = 3x^3 - 3x^2 + x - 1$$

Handwritten solution for Question 11:

$$\begin{aligned} \text{if } f'(x) &= (3x-1)^2 \text{ then} & \text{when } x=3, y=56 \leftarrow f(3) \\ \Rightarrow f(x) &= \int (3x-1)^2 dx & 56 = 81 - 27 + 3 + C \\ \Rightarrow f(x) &= \int 9x^2 - 6x + 1 dx & 56 = 57 + C \\ \Rightarrow f(x) &= 3x^3 - 3x^2 + x + C & C = -1 \\ & \therefore f(x) = 3x^3 - 3x^2 + x - 1 \end{aligned}$$

Question 12

The point $P(8,18)$ lies on the curve C , whose gradient function is given by

$$\frac{dy}{dx} = 8\sqrt[3]{x} - 10, \quad x \geq 0.$$

Find an equation for C .

$$y = 6x^{\frac{4}{3}} - 10x + 2$$

Handwritten solution for Question 12:

$$\begin{aligned} \text{If } \frac{dy}{dx} &= 8\sqrt[3]{x} - 10 = 8x^{\frac{1}{3}} - 10 \\ \text{Then } y &= \int (8x^{\frac{1}{3}} - 10) dx & \text{when } x=8, y=18 \\ \Rightarrow y &= \frac{8 \times 3}{4} x^{\frac{4}{3}} - 10x + C & 18 = 6 \times 8^{\frac{4}{3}} - 10 \times 8 + C \\ \Rightarrow y &= 6x^{\frac{4}{3}} - 10x + C & 18 = 96 - 80 + C \\ & \therefore y = 6x^{\frac{4}{3}} - 10x + 2 \end{aligned}$$

Question 13

$$f(x) = \frac{5\sqrt{x}(3x^2 - 2)}{x}, \quad x > 0.$$

Show clearly that

$$\int f(x) dx = P\sqrt{x} + Qx^{\frac{5}{2}} + C,$$

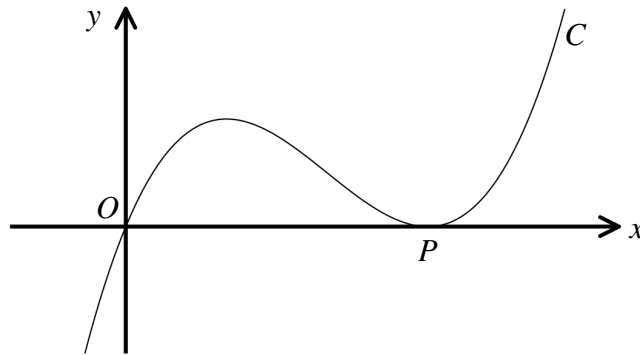
where P and Q are integers to be found, and C is an arbitrary constant.

$$\boxed{P = -20}, \quad \boxed{Q = 6}$$

$$\begin{aligned} \int f(x) dx &= \int \frac{5\sqrt{x}(3x^2-2)}{x} dx = \int \frac{5x^{1/2}(3x^2-2)}{x} dx \\ &= \int \frac{5x^{3/2} - 10x^{1/2}}{1} dx = \int 5x^{3/2} - 10x^{1/2} dx \\ &= \int 5x^{3/2} dx - \int 10x^{1/2} dx = \frac{5 \cdot 2}{5/2} x^{5/2} - \frac{10 \cdot 2}{3/2} x^{3/2} + C \\ &= 6x^{5/2} - 20x^{3/2} + C \end{aligned}$$

If $P = -20$
 $Q = 6$

Question 14



The figure above shows the cubic curve C which meets the coordinates axes at the origin O and at the point P .

The gradient function of C is given by

$$f'(x) = 3x^2 - 8x + 4.$$

- Find an equation for C .
- Determine the coordinates of P .

$$f(x) = x^3 - 4x^2 + 4x, \quad P(2,0)$$

<p>Q If $f'(x) = 3x^2 - 8x + 4$ Then $f(x) = \int 3x^2 - 8x + 4 \, dx$ $f(x) = x^3 - 4x^2 + 4x + C$ But cubic goes through $(0,0)$ $0 = 0 - 0 + 0 + C$ $\therefore C = 0$ $\therefore f(x) = x^3 - 4x^2 + 4x$</p>	<p>(b) $f(x) = x^3 - 4x^2 + 4x$ $f(x) = x(x^2 - 4x + 4)$ $f(x) = x(x-2)^2$ \therefore when $y=0$ $x = 0$ or 2 $\therefore P(2,0)$</p>
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Question 15

The point $P(-1, -1)$ lies on the curve C , whose gradient function is given by

$$\frac{dy}{dx} = \frac{5x^3 - 6}{x^3}, \quad x \neq 0.$$

Find an equation for C .

$$y = 5x + \frac{3}{x^2} + 1$$

Handwritten solution for Question 15:

$$\begin{aligned} \frac{dy}{dx} &= \frac{5x^3 - 6}{x^3} = 5 - \frac{6}{x^3} = 5 - 6x^{-3} \\ \text{Then } y &= \int (5 - 6x^{-3}) dx \\ &= 5x + 3x^{-2} + C \\ &= 5x + \frac{3}{x^2} + C \end{aligned}$$

When $x = -1, y = -1$

$$\begin{aligned} -1 &= 5(-1) + \frac{3}{(-1)^2} + C \\ -1 &= -5 + 3 + C \\ -1 &= -2 + C \\ C &= 1 \end{aligned}$$

$\therefore y = 5x + \frac{3}{x^2} + 1$

Question 16

Show clearly that

$$\int_3^4 3\sqrt{x} - \frac{4}{\sqrt{x}} \, dx = k\sqrt{3},$$

where k is an integer to be found.

$$k = 2$$

Handwritten solution for Question 16:

$$\begin{aligned} \int_3^4 3\sqrt{x} - \frac{4}{\sqrt{x}} \, dx &= \int_3^4 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} \, dx = \left[\frac{3 \times 2}{3} x^{\frac{3}{2}} - \frac{4 \times 2}{-1} x^{\frac{1}{2}} \right]_3^4 \\ &= \left[2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} \right]_3^4 = (2 \times 4^{\frac{3}{2}} - 8 \times 4^{\frac{1}{2}}) - (2 \times 3^{\frac{3}{2}} - 8 \times 3^{\frac{1}{2}}) \\ &= (16 - 16) - (2 \times 3\sqrt{3} - 8 \times \sqrt{3}) \\ &= -(2 \times 3\sqrt{3} - 8\sqrt{3}) = 8\sqrt{3} - 6\sqrt{3} = 2\sqrt{3} \end{aligned}$$

$k = 2$

Question 17

$$f(x) = 2x^2 + 3x + k, \text{ where } k \text{ is a constant.}$$

Find the value of k , given that

$$\int_1^3 f(x) \, dx = \frac{4}{3}.$$

$$k = -14$$

Handwritten solution for Question 17:

$$\int_1^3 (2x^2 + 3x + k) \, dx = \frac{4}{3}$$

$$\Rightarrow \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 + kx \right]_1^3 = \frac{4}{3}$$

$$\Rightarrow \left(\frac{2}{3} \times 27 + \frac{3}{2} \times 9 + 3k \right) - \left(\frac{2}{3} \times 1 + \frac{3}{2} \times 1 + k \right) = \frac{4}{3}$$

$$\Rightarrow 18 + \frac{27}{2} + 3k - \frac{2}{3} - \frac{3}{2} - k = \frac{4}{3}$$

$$\Rightarrow 2k + 18 + \frac{27}{2} - \frac{3}{2} - \frac{2}{3} = \frac{4}{3}$$

$$\Rightarrow 2k + 18 + 12 = 2 \Rightarrow 2k = -28 \Rightarrow k = -14$$

Question 18

The cubic equation C passes through the origin O and its gradient function is

$$\frac{dy}{dx} = 6x^2 - 6x - 20.$$

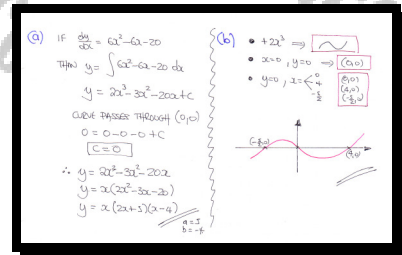
- a) Show clearly that the equation of C can be written as

$$y = x(2x + a)(x + b),$$

where a and b are constants.

- b) Sketch the graph of C , indicating clearly the coordinates of the points where the graph meets the coordinate axes.

$$a = 5, \quad b = -4$$



Question 19

The gradient of every point on the curve C , with equation $y = f(x)$, satisfies

$$f'(x) = 3x^2 - 4x + k,$$

where k is a constant.

The points $P(0, -3)$ and $Q(2, 7)$ both lie on C .

Find an equation for C .

$$y = x^3 - 2x^2 + 5x - 3$$

Handwritten solution for Question 19:

$$f'(x) = 3x^2 - 4x + k$$

Integrate:

$$f(x) = \int (3x^2 - 4x + k) dx$$

$$f(x) = x^3 - 2x^2 + kx + C$$

Use point $P(0, -3)$:

$$-3 = 0^3 - 2(0)^2 + k(0) + C$$

$$\Rightarrow C = -3$$

Use point $Q(2, 7)$:

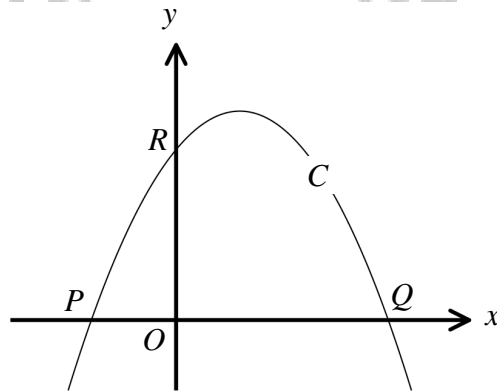
$$7 = 2^3 - 2(2)^2 + k(2) - 3$$

$$10 = 2k$$

$$k = 5$$

$\therefore f(x) = x^3 - 2x^2 + 5x - 3$

Question 20



The figure above shows the curve C which meets the coordinates axes at the points P , Q and R .

Given the gradient function of C is given by

$$f'(x) = 3 - 4x,$$

and that $f(1) = 2f(2)$, determine the coordinates of P , Q and R .

$$P(-1,0), P\left(\frac{5}{2},0\right), R(0,5)$$

$$\begin{aligned} f'(x) &= 3 - 4x \\ f(x) &= \int 3 - 4x \, dx = 3x - 2x^2 + k \\ \therefore f(x) &= -2x^2 + 3x + k \\ f(1) &= 2f(2) \\ -2 + 3 + k &= 2(-8 + 6 + k) \\ 1 + k &= -4 + 2k \\ 5 &= k \\ \therefore f(x) &= -2x^2 + 3x + 5 \\ \text{when } x=0, f(0) &= 5 \therefore R(0,5) \\ \text{when } y=0, 0 &= -2x^2 + 3x + 5 \\ 2x^2 - 3x - 5 &= 0 \\ (2x-5)(x+1) &= 0 \end{aligned}$$

$$\therefore x = \frac{5}{2} \text{ or } -1 \quad \begin{matrix} P(-1,0) \\ Q\left(\frac{5}{2},0\right) \end{matrix}$$

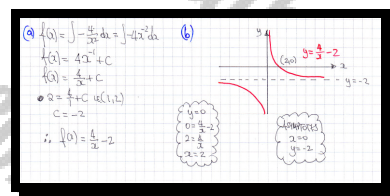
Question 21

The curve C with equation $y = f(x)$ satisfies

$$f'(x) = -\frac{4}{x^2}, \quad x \neq 0.$$

- a) Given that $f(1) = 2$, find an expression for $f(x)$.
- b) Sketch the graph of $f(x)$, indicating clearly the asymptotes of the curve and the coordinates of any points where the curve crosses the coordinate axes.

$$f(x) = \frac{4}{x} - 2, \quad (2, 0)$$



Question 22

$$f(x) = \left(x^{\frac{1}{2}} - 4\right)\left(x^{-\frac{1}{2}} - 3\right), \quad x > 0.$$

Show clearly that

$$\int f(x) dx = P\sqrt{x} + Qx + Rx^{\frac{3}{2}} + C,$$

where P , Q and R are integers to be found, and C is an arbitrary constant.

$$P = -8, \quad Q = 13, \quad R = -2$$

$$\begin{aligned} \int f(x) dx &= \int (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 3) dx = \int (x^0 - 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} + 12) dx \\ &= \int (13 - 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}) dx = 13x - \frac{3}{\frac{3}{2}} x^{\frac{3}{2}} - \frac{4}{-\frac{1}{2}} x^{\frac{1}{2}} + C \\ &= 13x - 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + C \end{aligned}$$

$\therefore P = -8$
 $Q = 13$
 $R = -2$

Question 23

$$f'(x) = 5 - \frac{8}{x^2}, \quad x \neq 0.$$

Find the value of $f(4)$, given that $2f(1) = 4 + f(2)$.

$$f(4) = 14$$

$f'(x) = 5 - \frac{8}{x^2}$
 $f(x) = \int 5 - \frac{8}{x^2} dx = \int 5 - 8x^{-2} dx = 5x - \frac{8}{-1} x^{-1} + c$
 $\therefore f(x) = 5x + \frac{8}{x} + c$
 $2f(1) = 4 + f(2)$
 $2\left(5 + 8 + c\right) = 4 + \left(10 + 4 + c\right)$
 $26 + 2c = 14 + c$
 $c = -12$
 $\therefore f(x) = 5x + \frac{8}{x} - 12$
 $\therefore f(4) = 20 + 2 - 12$
 $f(4) = 10$

Question 24

$$f(x) = \frac{(3x^2 - 2)^2}{x^2}, \quad x \neq 0.$$

Show clearly that

$$\int_1^2 f(x) dx = 11.$$

proof

$f(x) = \frac{(3x^2 - 2)^2}{x^2} = \frac{9x^4 - 12x^2 + 4}{x^2} = \frac{9x^4}{x^2} - \frac{12x^2}{x^2} + \frac{4}{x^2}$
 $= 9x^2 - 12 + 4x^{-2}$
 $\therefore \int_1^2 f(x) dx = \int_1^2 (9x^2 - 12 + 4x^{-2}) dx = \left[3x^3 - 12x - \frac{4x^{-1}}{1} \right]_1^2$
 $= (3 \times 2^3 - 12 \times 2 - \frac{4}{2}) - (3 \times 1^3 - 12 \times 1 - \frac{4}{1})$
 $= (24 - 24 - 2) - (3 - 12 - 4) = -2 - (-13) = 11$

Question 25

$$y = \frac{x^{\frac{1}{2}}(3x^2 + 1)}{x^2}, \quad x > 0.$$

Show clearly that

$$\int_1^4 y \, dx = 15.$$

proof

$$\begin{aligned} y &= \frac{x^{\frac{1}{2}}(3x^2 + 1)}{x^2} = \frac{3x^{\frac{5}{2}} + x^{\frac{1}{2}}}{x^2} = 3x^{\frac{1}{2}} + x^{-\frac{3}{2}} \\ \therefore \int_1^4 y \, dx &= \int_1^4 (3x^{\frac{1}{2}} + x^{-\frac{3}{2}}) \, dx = \left[2x^{\frac{3}{2}} - 2x^{-\frac{1}{2}} \right]_1^4 \\ &= (2 \times 4^{\frac{3}{2}} - 2 \times 4^{-\frac{1}{2}}) - (2 \times 1^{\frac{3}{2}} - 2 \times 1^{-\frac{1}{2}}) \\ &= (16 - 1) - (2 - 2) = 15 \end{aligned}$$

Question 26

Find the exact value of

$$\int_1^3 3\sqrt{x} - \frac{4}{\sqrt{x}} \, dx,$$

giving the answer in the form $p + q\sqrt{3}$, where p and q are integers.

$$6 - 2\sqrt{3}$$

$$\begin{aligned} \int_1^3 3\sqrt{x} - \frac{4}{\sqrt{x}} \, dx &= \int_1^3 (3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}) \, dx = \left[2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} \right]_1^3 \\ &= (2 \times 3^{\frac{3}{2}} - 8 \times 3^{\frac{1}{2}}) - (2 \times 1^{\frac{3}{2}} - 8 \times 1^{\frac{1}{2}}) \\ &= (2 \times 3\sqrt{3} - 8\sqrt{3}) - (2 - 8) \\ &= 6\sqrt{3} - 8\sqrt{3} - 2 + 8 \\ &= 6 - 2\sqrt{3} \end{aligned}$$

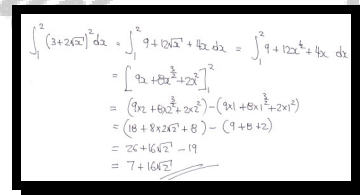
Question 27

Find the exact value of

$$\int_1^2 (3+2\sqrt{x})^2 dx,$$

giving the answer in the form $a+b\sqrt{2}$, where a and b are integers.

$$\boxed{7+16\sqrt{2}}$$



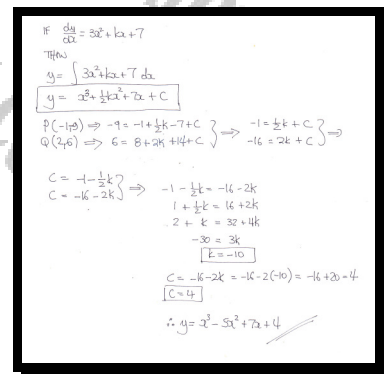
$$\begin{aligned} \int_1^2 (3+2\sqrt{x})^2 dx &= \int_1^2 9 + 12\sqrt{x} + 4x dx = \int_1^2 9 + 12x^{\frac{1}{2}} + 4x dx \\ &= \left[9x + 8x^{\frac{3}{2}} + 2x^2 \right]_1^2 \\ &= (18 + 16\sqrt{2} + 4) - (9 + 8 + 2) \\ &= 24 + 16\sqrt{2} - 19 \\ &= 7 + 16\sqrt{2} \end{aligned}$$

Question 28A cubic curve passes through the points $P(-1, -9)$ and $Q(2, 6)$ and its gradient function is given by

$$\frac{dy}{dx} = 3x^2 + kx + 7, \text{ where } k \text{ is a constant.}$$

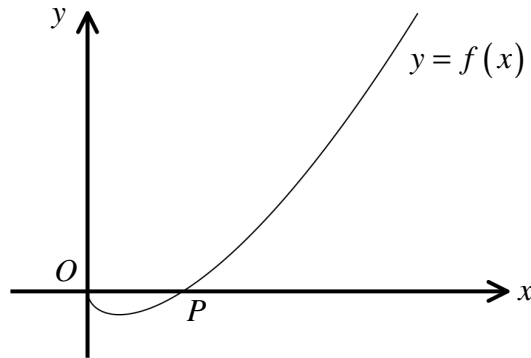
Find an equation for this cubic curve.

$$\boxed{y = x^3 - 5x^2 + 7x + 4}$$



$$\begin{aligned} \text{If } \frac{dy}{dx} &= 3x^2 + kx + 7 \\ \text{Then } y &= \int 3x^2 + kx + 7 dx \\ y &= x^3 + \frac{1}{2}kx^2 + 7x + C \\ P(-1, -9) &\Rightarrow -9 = -1 + \frac{1}{2}k - 7 + C \\ Q(2, 6) &\Rightarrow 6 = 8 + 2k + 14 + C \Rightarrow \begin{cases} -1 + \frac{1}{2}k + C = -9 \\ -16 + 2k + C = 6 \end{cases} \Rightarrow \begin{cases} -1 + \frac{1}{2}k + C = -9 \\ -16 + 2k + C = 6 \end{cases} \\ C = -1 - \frac{1}{2}k &\Rightarrow \begin{cases} -1 + \frac{1}{2}k - 1 - \frac{1}{2}k = -16 - 2k \\ -2 + k = -16 - 2k \\ 3k = -14 \\ k = -\frac{14}{3} \end{cases} \\ C = -16 - 2k &\Rightarrow \begin{cases} -16 - 2(-\frac{14}{3}) = -16 + \frac{28}{3} = -\frac{14}{3} \\ C = -\frac{14}{3} \end{cases} \\ \therefore y &= x^3 - 5x^2 + 7x + 4 \end{aligned}$$

Question 29



The figure above shows a curve with equation $y = f(x)$ which meets the x axis at the origin O and at the point P .

The gradient function of the curve is given by

$$f'(x) = \frac{12x-1}{\sqrt{x}}, \quad x > 0.$$

- Find an equation of the curve.
- Determine the coordinates of P .

$$f(x) = 8x^{\frac{3}{2}} - 2\sqrt{x}, \quad P\left(\frac{1}{4}, 0\right)$$

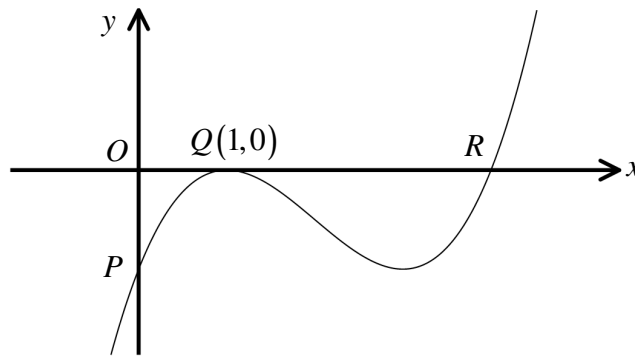
(a) At P, $z = -1$
 $f'(z) = \frac{8(z-1)^2}{(z-3)^3} = \frac{-8}{(-2)^3} = -1$
 ∴ tangent
 $y - y_1 = m(x - x_1)$
 $y - 0 = -1(x + 1)$
 $y = -x - 1$

(b) $f(z) = \frac{8z^3 - 1}{z^2}$
 $f'(z) = \frac{8z^2 \cdot \frac{1}{z^2} - 1 \cdot \frac{2}{z^3}}{z^4}$
 $f'(z) = \frac{8z - 2}{z^3}$
 $\therefore f'(z) = 8 - 2z^3$
 $\text{at } f'(z) = 8 - \frac{2}{3^3}$

(c) If $f'(z) = 8z - 2z^3$
 $f(z) = \int 8z - 2z^3 dz$
 $f(z) = 4z^2 + \frac{1}{2}z^4 + C$
 $f'(z) = 8z + \frac{1}{2}z^3$
 Apply condition, $(-1, 0)$
 $0 = 4(-1)^2 + \frac{1}{2} + C$
 $0 = 4 + \frac{1}{2} + C$
 $C = -4\frac{1}{2}$
 $\therefore f(z) = 4z^2 + \frac{1}{2}z^3 - 4\frac{1}{2}$

(d) $y = 0$ $4x^2 + \frac{1}{2}x^3 - 4\frac{1}{2} = 0$
 $4x^2 + \frac{1}{2}x^3 - 4\frac{1}{2} = 0$
 $4x^2 - 3x + 1 = 0$
 $(x+1)(4x^2 - 3x + 1) = 0$
 Point P By Newton's
 $\frac{f(x)}{f'(x)} = -4$
 $\frac{(x+1)(4x^2 - 3x + 1)}{(2x+1)(2x-1)} = 0$
 $\therefore (x+1) = 0$

Question 30



The figure above shows the graph of a cubic curve, which touches the x axis at the point $Q(1,0)$.

- a) Determine an equation for the cubic curve, given its gradient is given by

$$\frac{dy}{dx} = 3x^2 - 12x + 9.$$

The cubic curve crosses the x axis and the y axis at the points R and P , respectively.

- b) Determine the coordinates ...

i. ... of the point P .

ii. ... of the point R .

$$y = x^3 - 6x^2 + 9x - 4, \quad P(0, -4), \quad R(4, 0)$$

(a) IF $\frac{dy}{dx} = 3x^2 - 12x + 9$
 Then $y = \int 3x^2 - 12x + 9 \, dx$
 $y = x^3 - 6x^2 + 9x + C$
 Apply condition $Q(1,0)$
 $0 = 1 - 6 + 9 + C$
 $C = -4$
 $\therefore y = x^3 - 6x^2 + 9x - 4$

(b) (i) When $x = 0$
 $y = -4$
 $\therefore P(0, -4)$
 (ii) $y = x^3 - 6x^2 + 9x - 4$
 $y = (x-1)^2(x-4)$
 Touches AT $Q(1,0)$
 $\therefore R(4,0)$

Question 31

$$\sqrt{y} = 2\sqrt[3]{x} - 3, \quad x > 0.$$

Show clearly that

$$\int_1^8 y \, dx = \frac{12}{5}.$$

proof

Handwritten solution for Question 31:

$$\begin{aligned} \sqrt{y} &= 2\sqrt[3]{x} - 3 \\ \sqrt{y} &= 2x^{\frac{1}{3}} - 3 \\ y &= (2x^{\frac{1}{3}} - 3)^2 \\ y &= 4x^{\frac{2}{3}} - 12x^{\frac{1}{3}} + 9 \end{aligned}$$

$$\int_1^8 y \, dx = \int_1^8 (4x^{\frac{2}{3}} - 12x^{\frac{1}{3}} + 9) \, dx$$

$$= \left[\frac{12}{5}x^{\frac{5}{3}} - 12x^{\frac{4}{3}} + 9x \right]_1^8$$

$$= \left(\frac{12}{5} \cdot 8^{\frac{5}{3}} - 12 \cdot 8^{\frac{4}{3}} + 9 \cdot 8 \right) - \left(\frac{12}{5} - 12 + 9 \right)$$

$$= \frac{12}{5} \cdot 64 - 12 \cdot 16 + 72 - \left(\frac{12}{5} - 3 \right)$$

$$= \frac{768}{5} - 192 + 72 - \frac{12}{5} + 3$$

$$= \frac{768 - 960 + 360 - 12 + 15}{5}$$

$$= \frac{12}{5}$$

Question 32

$$y = 6 + 6\sqrt{x} + 5x, \quad x \geq 0.$$

Show clearly that

$$\int (y^2 - x^2) \, dx = 36x + Px^{\frac{3}{2}} + 48x^2\sqrt{x} + Qx^{\frac{5}{2}} + Rx^3 + C,$$

where P , Q and R are constants to be found, and C is an arbitrary constant.

$$P = 48, \quad Q = 24, \quad R = 8$$

Handwritten solution for Question 32:

$$\begin{aligned} y^2 &= (6 + 6\sqrt{x} + 5x)^2 = (6 + 6x^{\frac{1}{2}} + 5x)^2 \\ &= 36 + 36x^{\frac{1}{2}} + 30x + 36x^{\frac{3}{2}} + 36x + 25x^2 \\ &= 36 + 72x^{\frac{1}{2}} + 96x + 36x^{\frac{3}{2}} + 25x^2 \end{aligned}$$

$$\therefore \int (y^2 - x^2) \, dx = \int (36 + 72x^{\frac{1}{2}} + 96x + 36x^{\frac{3}{2}} + 25x^2 - x^2) \, dx$$

$$= \int (36 + 72x^{\frac{1}{2}} + 96x + 36x^{\frac{3}{2}} + 24x^2) \, dx$$

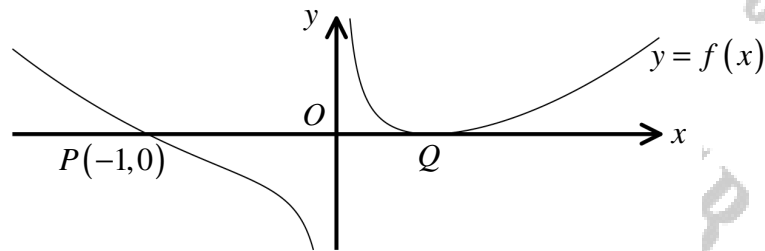
$$= 36x + 72 \cdot \frac{2}{3}x^{\frac{3}{2}} + \frac{96}{2}x^2 + 36 \cdot \frac{2}{5}x^{\frac{5}{2}} + \frac{24}{3}x^3 + C$$

$$= 36x + 48x^{\frac{3}{2}} + 48x^2 + 24x^{\frac{5}{2}} + 8x^3 + C$$

Annotations in the handwritten solution:

- $\frac{72}{3} = 24$ (for the $x^{\frac{3}{2}}$ term)
- $\frac{96}{2} = 48$ (for the x^2 term)
- $\frac{72}{5} = 14.4$ (for the $x^{\frac{5}{2}}$ term)
- $\frac{24}{3} = 8$ (for the x^3 term)

Question 33



The figure above shows a curve with equation $y = f(x)$.

The curve meets the x axis at the points $P(-1, 0)$ and Q , and its gradient function is given by

$$f'(x) = \frac{8x^3 - 1}{x^2}, \quad x \neq 0.$$

a) Find an equation of the tangent to the curve at P .

b) Find an expression for $f''(x)$.

c) Determine ...

i. ... an equation of the curve.

ii. ... the coordinates of Q .

$$\boxed{y = -9x - 9}, \quad \boxed{f''(x) = 8 + 2x^{-3}}, \quad \boxed{y = 4x^2 + \frac{1}{x} - 3}, \quad \boxed{Q\left(\frac{1}{2}, 0\right)}$$

(a) If $f'(x) = \frac{8x^3 - 1}{x^2}$ then

$$f'(x) = \frac{8x^3 - 1}{x^2} = \frac{8x^3}{x^2} - \frac{1}{x^2} = 8x - x^{-2}$$

$$f(x) = \int (8x - x^{-2}) dx = 4x^2 + \frac{1}{x} + C$$

At $P(-1, 0)$, $y = 0$ when $x = -1$

$$0 = 4(-1)^2 + \frac{1}{-1} + C = 4 - 1 + C = 3 + C$$

$$\therefore C = -3$$

$$\therefore f(x) = 4x^2 + \frac{1}{x} - 3$$

(b) $y = 0$

$$0 = 4x^2 + \frac{1}{x} - 3$$

$$0 = \frac{4x^3 + 1 - 3x}{x}$$

$$0 = 4x^3 - 3x + 1$$

$$0 = (4x^2 + 1)(x - \frac{1}{2})$$

$$\therefore x = \frac{1}{2}$$

$$\therefore P\left(\frac{1}{2}, 0\right)$$

Question 34

$$y = \frac{1}{\sqrt{x}} + 5\sqrt{x}, \quad x > 0.$$

Show clearly that

$$\int_1^4 y^3 \, dx = 1931.$$

proof

Handwritten solution for the integral of y^3 from $x=1$ to $x=4$:

$$\begin{aligned}
 y &= \frac{1}{\sqrt{x}} + 5\sqrt{x} = x^{-\frac{1}{2}} + 5x^{\frac{1}{2}} \\
 y^2 &= (x^{-\frac{1}{2}} + 5x^{\frac{1}{2}})^2 \\
 y^3 &= (x^{-\frac{1}{2}} + 5x^{\frac{1}{2}})(x^{-\frac{1}{2}} + 5x^{\frac{1}{2}})^2 \\
 y^3 &= (x^{-\frac{1}{2}} + 5x^{\frac{1}{2}})(x^{-1} + 10x^{\frac{1}{2}} + 25x) \\
 y^3 &= x^{-\frac{1}{2}} + 10x^{\frac{1}{2}} + 25x^{\frac{3}{2}} + 5x^{\frac{1}{2}} + 25x^{\frac{3}{2}} + 125x^{\frac{5}{2}} \\
 y^3 &= x^{-\frac{1}{2}} + 15x^{\frac{1}{2}} + 75x^{\frac{3}{2}} + 125x^{\frac{5}{2}} \\
 \int_1^4 y^3 \, dx &= \int_1^4 (x^{-\frac{1}{2}} + 15x^{\frac{1}{2}} + 75x^{\frac{3}{2}} + 125x^{\frac{5}{2}}) \, dx \\
 &= \left[2x^{\frac{1}{2}} + 30x^{\frac{3}{2}} + 50x^{\frac{5}{2}} + 25x^{\frac{7}{2}} \right]_1^4 \\
 &= (-1 + 60 + 400 + 1600) - (-2 + 30 + 50 + 50) \\
 &= 2059 - 128 \\
 &= 1931
 \end{aligned}$$

Question 35

A quadratic curve C passes through the points $P(a, b)$ and $Q(2a, 2b)$, where a and b are constants.

The gradient at any given point on C is given by

$$\frac{dy}{dx} = 2x - 6.$$

Find an equation for C , in terms of a .

$$y = x^2 - 6x + 2a^2$$

Handwritten solution for Question 35:

$$\begin{aligned} \frac{dy}{dx} &= 2x - 6 \\ y &= \int (2x - 6) dx \\ y &= x^2 - 6x + c \\ P(a, b) &\Rightarrow b = a^2 - 6a + c \\ Q(2a, 2b) &\Rightarrow 2b = 4a^2 - 12a + c \\ \left. \begin{aligned} 2b &= 2a^2 - 12a + 2c \\ 2b &= 4a^2 - 12a + c \end{aligned} \right\} &\Rightarrow \begin{aligned} &\Rightarrow 2a^2 - 12a + 2c = 4a^2 - 12a + c \\ &\Rightarrow c = 2a^2 \\ \therefore y &= x^2 - 6x + 2a^2 \end{aligned} \end{aligned}$$