THE OPERATION OF DIFFERENTIATION
Question 1
Evaluate the following.

a) \( \frac{d}{dx}(5x^6) = 30x^5 \)

b) \( \frac{d}{dx}(2x^{\frac{3}{2}}) = 3x^{\frac{1}{2}} \)

c) \( \frac{d}{dx}(6x^4 - x^3) = 24x^3 - 3x^2 \)

d) \( \frac{d}{dx}(3x^2 + 5x + 1) = 6x + 5 \)

e) \( \frac{d}{dx}(4x^{\frac{1}{2}} - 2x - 7) = 2x^{\frac{1}{2}} - 2 \)
Question 2
Evaluate the following.

a) \( \frac{d}{dx}(4x^3) \)

\[ \frac{d}{dx}(4x^3) = 12x^2 \]

b) \( \frac{d}{dx}(7x^5) \)

\[ \frac{d}{dx}(7x^5) = 35x^4 \]

c) \( \frac{d}{dx}(4x^2 + 3x^4) \)

\[ \frac{d}{dx}(4x^2 + 3x^4) = 8x + 12x^3 \]

d) \( \frac{d}{dx}(x^2 + 7x + 5) \)

\[ \frac{d}{dx}(x^2 + 7x + 5) = 2x + 7 \]

e) \( \frac{d}{dx}\left(8x^{\frac{1}{2}} + 2x^{-2}\right) \)

\[ \frac{d}{dx}\left(8x^{\frac{1}{2}} + 2x^{-2}\right) = 4x^{-\frac{1}{2}} - 4x^{-3} \]
Question 3
Differentiate the following expressions with respect to $x$

a) $y = x^2 - 4x^6$
\[
\frac{dy}{dx} = 2x - 24x^5
\]

b) $y = 5x^3 - 6x^2$
\[
\frac{dy}{dx} = 15x^2 - 9x^2
\]

c) $y = 9x^{-3} + 7x^{-2}$
\[
\frac{dy}{dx} = -27x^{-4} - 14x^{-3}
\]

d) $y = 5 - 5x^{-1}$
\[
\frac{dy}{dx} = 5x^{-2}
\]

e) $y = 7x + \sqrt{x}$
\[
\frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}
\]
Question 4
Differentiate the following expressions with respect to $x$

a) $y = x^6 - 7x^2$
\[
\frac{dy}{dx} = 6x^5 - 14x
\]

b) $y = 1 - 6x^2$
\[
\frac{dy}{dx} = -12x
\]

c) $y = 2x + 8x^{-2}$
\[
\frac{dy}{dx} = 2 + 16x^{-3}
\]

d) $y = (2x - 1)(4x + 3)$
\[
\frac{dy}{dx} = 16x + 2
\]

e) $y = 4x^3 (2 - 3x)$
\[
\frac{dy}{dx} = 24x^2 - 48x^3
\]
Question 5
Find \( f'(x) \) for each of the following functions.

a) \( f(x) = 4x^3 - 9x + 2 \)
\[ f'(x) = 12x^2 - 9 \]

b) \( f(x) = 6x^{\frac{1}{2}} + 2x \)
\[ f'(x) = -3x^{\frac{1}{2}} + 2 \]

c) \( f(x) = x^4 + 2x^\frac{3}{2} \)
\[ f'(x) = 4x^3 + 5x^{\frac{3}{2}} \]

d) \( f(x) = \frac{1}{2}x^2 - 4x^{-\frac{3}{2}} \)
\[ f'(x) = x + 6x^{-\frac{5}{2}} \]

e) \( f(x) = \frac{1}{2}x^3 + 5x \)
\[ f'(x) = \frac{1}{6}x^\frac{5}{2} + 5 \]
Question 6
Differentiate each of the following functions with respect to $x$.

a) $f(x) = 6x^{-rac{3}{2}} + 4x + 1$
   
   $f'(x) = -9x^{-rac{5}{2}} + 4$

b) $g(x) = x^4 - x^{-1}$
   
   $g'(x) = 4x^3 + x^{-2}$

c) $h(x) = 9x^2 - rac{1}{2}x^4$
   
   $h'(x) = 18x - 2x^3$

d) $p(x) = 4x^{rac{1}{2}} - 6x^{rac{3}{4}} + rac{1}{2}x^{-rac{1}{4}}$
   
   $p'(x) = 2x^{-rac{1}{2}} - 2x^{-rac{2}{3}} - rac{1}{8}x^{-rac{5}{4}}$

e) $v(x) = (8x + rac{1}{2})^2$
   
   $v'(x) = 128x + 8$
Carry out the following differentiations.

a) \[ \frac{d}{dt}(4t^2 - 7t + 5) = 8t - 7 \]

b) \[ \frac{d}{dy}\left(\frac{1}{2}y - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y + \frac{1}{3}y^{-\frac{3}{2}} \]

c) \[ \frac{d}{dz}(2z^2 - 3z^{-1} + z) = 4z + 3z^{-2} + 1 \]

d) \[ \frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}} \]

e) \[ \frac{d}{dx}(ax^2 - 3x^2) = 2ax - 6x \]
Question 8

Carry out the following differentiations.

a) \[ \frac{d}{dy}(4y^3 + 6y + 2) \]
   \[ \frac{d}{dy}(4y^3 + 6y + 2) = 12y^2 + 6 \]

b) \[ \frac{d}{dt}(7t^2 - 4t^{1/2}) \]
   \[ \frac{d}{dt}(7t^2 - 4t^{1/2}) = 14t - 2t^{-1/2} \]

c) \[ \frac{d}{dx}(ax^2 + bx + c) \]
   \[ \frac{d}{dx}(ax^2 + bx + c) = 2ax + b \]

d) \[ \frac{d}{dz}\left(\frac{1}{4z^2} - \frac{1}{z}\right) \]
   \[ \frac{d}{dz}\left(\frac{1}{4z^2} - \frac{1}{z}\right) = -\frac{1}{2z} + \frac{1}{z^2} \]

e) \[ \frac{d}{dw}\left(\frac{1}{4w^3} + \frac{k}{w^2}\right) \]
   \[ \frac{d}{dw}\left(\frac{1}{4w^3} + \frac{k}{w^2}\right) = \frac{1}{5w^{5/2}} - \frac{2k}{w^3} \]
Question 9

a) If \( A = \pi x^2 - 20x \), find the rate of change of \( A \) with respect to \( x \).

b) If \( V = x - 2\pi x^3 \), find the rate of change of \( V \) with respect to \( x \).

c) If \( P = at^2 - bt \), find the rate of change of \( P \) with respect to \( t \).

d) If \( W = 6kh^2 - h \), find the rate of change of \( W \) with respect to \( h \).

e) If \( N = (at + b)^2 \), find the rate of change of \( N \) with respect to \( t \).

\[
\begin{align*}
\frac{dA}{dx} &= 2\pi x - 20, \\
\frac{dV}{dx} &= 1 - 6\pi x^2, \\
\frac{dP}{dt} &= 2at - b, \\
\frac{dW}{dh} &= 3kh - \frac{1}{2} - 1, \\
\frac{dN}{dt} &= 2a^2t + 2ab
\end{align*}
\]
DIFFERENTIATING INDICES
Question 1

Differentiate the following expressions with respect to $x$.

a) $y = 4\sqrt{x} - 3\sqrt{x}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{3}{2}}$$

b) $y = 2\sqrt{x} - 4\sqrt{x^3}$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 6x^{\frac{3}{2}}$$

c) $y = \frac{1}{2\sqrt{x}} + \frac{4}{x^2}$

$$\frac{dy}{dx} = -\frac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}$$

d) $y = x\sqrt{x} - \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-3}$$

e) $y = 4\sqrt{x} + \frac{1}{4\sqrt{x}}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}$$
Question 2

Find $f'(x)$ for each of the following functions.

a) $f(x) = \frac{2}{x^3} + 5x^{\frac{2}{3}}$

\[ f'(x) = -6x^{-4} + \frac{10}{3}x^{-\frac{1}{3}} \]

b) $f(x) = 8x^{\frac{3}{4}} - \frac{2}{x^4}$

\[ f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5} \]

c) $f(x) = 2x - \frac{3}{x^2} + 4\sqrt{x} + 2$

\[ f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}} \]

d) $f(x) = 3\sqrt{x}^2 - \frac{3}{2x^3}$

\[ f'(x) = \frac{3}{2}x^\frac{1}{2} + \frac{9}{2}x^{-4} \]

e) $f(x) = \sqrt{x^2} - \frac{1}{2x^2}$

\[ f'(x) = \frac{3}{2}x^2 + x^{-3} \]
Question 3
Differentiate the following expressions with respect to $x$

a) $\frac{dy}{dx} = \frac{8}{3}x^{-3} - 12x^{-4}$

\[ y = \frac{4}{x^3} - \frac{4}{3x^2} \]

b) $\frac{dy}{dx} = 30\frac{x^{-2}}{2} - \frac{3}{2}x^{-3}$

\[ y = -\frac{3}{4x^2} + \frac{12}{x^2\sqrt{x}} \]

c) $\frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{-\frac{3}{2}} - \frac{1}{6}x^{-\frac{5}{2}}$

\[ y = \frac{1}{3x} + \frac{2x^3 + 1}{3\sqrt{x}} \]

d) $\frac{dy}{dx} = 21x^\frac{1}{2} - 5x^\frac{3}{2}$

\[ y = 2\sqrt{x}(7x - x^2) \]

e) $\frac{dy}{dx} = 6x^\frac{1}{3} + 4$

\[ y = (3 + 2\sqrt{x})^2 \]
Question 4
Evaluate the following.

a) \( \frac{d}{dx} \left( 6x^{\frac{4}{3}} - 2x^{\frac{5}{2}} \right) \)

b) \( \frac{d}{dx} \left( \frac{1}{x} - \frac{1}{\sqrt{x}} \right) \)

c) \( \frac{d}{dx} \left( \sqrt[3]{x} - \frac{27}{x} \right) \)

d) \( \frac{d}{dx} \left( \frac{3\sqrt{x} - 2}{x^{\frac{3}{2}}} \right) \)

e) \( \frac{d}{dx} \left[ \frac{1}{3\sqrt{x}} \left( \frac{2}{x} - 3 \right) \right] \)
Question 5
Evaluate the following.

a) \[ \frac{d}{dx} \left( \frac{x + x^2}{\sqrt{x}} \right) \]

\[ \frac{1}{2} x^{-\frac{3}{2}} + \frac{3}{2} x^{\frac{1}{2}} \]

b) \[ \frac{d}{dx} \left( \frac{4x + \sqrt{x}}{2x^2} \right) \]

\[ -2x^{-2} - \frac{3}{4} x^{-\frac{3}{2}} \]

c) \[ \frac{d}{dx} \left( \frac{x^2 + 2}{x^3} \right) \]

\[ -x^{-2} - 6x^{-4} \]

d) \[ \frac{d}{dx} \left( \frac{1 - \sqrt{x}}{4x^2} \right) \]

\[ -\frac{3}{4} x^{-4} + \frac{5}{8} x^{-\frac{7}{2}} \]

e) \[ \frac{d}{dx} \left[ \frac{3\sqrt{x} - 2x\sqrt{x}}{3x} \right] \]

\[ 2x^{-\frac{1}{3}} - \frac{1}{3} x^{-\frac{1}{2}} \]
Question 6
Differentiate the following expressions with respect to $x$

a) $y = \frac{4+x}{2x^3}$

\[
\frac{dy}{dx} = -6x^{-4} - x^{-3}
\]

b) $y = \frac{x^2 + 3x}{2\sqrt{x}}$

\[
\frac{dy}{dx} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{\frac{1}{2}}
\]

c) $y = \frac{x + 4\sqrt{x}}{2x^3}$

\[
\frac{dy}{dx} = -5x^{-\frac{5}{2}} - x^{-3}
\]

d) $y = \frac{\sqrt{x}(2x-4)}{3x^2}$

\[
\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{5}{2}} + 2x^{-\frac{3}{2}}
\]

e) $y = \frac{(x+2)(2x-3)}{4x^5}$

\[
\frac{dy}{dx} = -\frac{3}{2}x^{-4} - x^{-5} + \frac{15}{2}x^{-6}
\]
Question 7

Find $f'(x)$ for each of the following functions.

a) $f(x) = x\left(\sqrt{x} + x^{-4}\right)$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4}$$

b) $f(x) = \frac{1}{\sqrt{x}} \left( \frac{2}{x} - \frac{3}{4x^2} \right)$

$$f'(x) = -3x^{-\frac{3}{2}} + \frac{15}{8}x^{-\frac{7}{2}}$$

c) $f(x) = 4x^2 \left( \frac{6}{x^2} - \frac{5}{\sqrt{x}} \right)$

$$f'(x) = 36x^{\frac{1}{2}} - 60x^2$$

d) $f(x) = 2\sqrt{x} \left( \frac{5}{x} + x^2 \right)$

$$f'(x) = -5x^{\frac{3}{2}} + 5x^{\frac{5}{2}}$$

e) $f(x) = \frac{2}{x^2} \left( 7x^3 - 5x^2 \right)$

$$f'(x) = \frac{7}{4}x^{-\frac{1}{2}} + \frac{5}{4}x^{-\frac{3}{2}}$$
Question 8

Differentiate the following expressions with respect to $x$

a) $y = \frac{(2x-1)(3x-2)}{2x^{\frac{1}{2}}}$

\[
\frac{dy}{dx} = \frac{\frac{3}{2}x^{-\frac{1}{2}} + \frac{7}{4}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{5}{2}}}{4x}
\]

b) $y = \frac{(3+2\sqrt{x})^2}{4x}$

\[
\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{9}{4}x^{-2}
\]

c) $y = \frac{4x^3 + \sqrt{x^3}}{4\sqrt{x}}$

\[
\frac{dy}{dx} = \frac{1}{2}x + \frac{5}{2}x^2
\]

d) $y = \frac{(4x + \sqrt{x})(x^2 - 3)}{3\sqrt{x}}$

\[
\frac{dy}{dx} = \frac{2}{3}x + \frac{10}{3}x^3 - 2x^{-\frac{1}{2}}
\]

e) $y = \frac{\left(2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right)\left(6x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}\right)}{3x}$

\[
\frac{dy}{dx} = \frac{4}{3}x^{\frac{3}{2}} + 8x^{-3} + 4
\]
TANGENTS

&

NORMALS
Question 1 (non calculator)

For each of the following curves find an equation of the tangent to the curve at the point whose $x$ coordinate is given.

a) $y = x^2 - 9x + 13$, where $x = 6$
   \[ y = 3x - 23 \]

b) $y = x^4 + x + 1$, where $x = 1$
   \[ y = 5x - 2 \]

c) $y = 2x^2 + 6x + 7$, where $x = -1$
   \[ y = 2x + 5 \]

d) $y = 2x^3 - 4x + 5$, where $x = 1$
   \[ y = 2x + 1 \]

e) $y = 2x^3 - 4x^2 - 3$, where $x = 2$
   \[ y = 8x - 19 \]

f) $y = 3x^3 - 17x^2 + 24x - 9$, where $x = 2$
   \[ y = -8x + 11 \]
For each of the following curves find an equation of the tangent to the curve at the point whose $x$ coordinate is given.

a) $f(x) = x^3 - 4x^2 + 2x - 1$, where $x = 2$

\[ y = -2x - 1 \]

b) $f(x) = 3x^3 + x^2 - 8x - 5$, where $x = 1$

\[ y = 3x - 12 \]

c) $f(x) = 2x^3 - 5x^2 + 2x - 1$, where $x = 2$

\[ y = 6x - 13 \]

d) $f(x) = x^3 - x^2 - 3x - 2$, where $x = 1$

\[ y = -2x - 3 \]

e) $f(x) = 2x^3 + x^2 - 2x - 2$, where $x = 1$

\[ y = 6x - 7 \]
Question 3  \hspace{1cm} \text{(non calculator)}

For each of the following curves find an equation of the tangent to the curve at the point whose \( x \) coordinate is given.

\begin{enumerate}
\item \( y = x^2 - \frac{3}{x} - \frac{1}{2} \), where \( x = -2 \) \hspace{1cm} 13x + 4y + 6 = 0
\item \( y = x^3 - 6x + \frac{8}{x} + 1 \), where \( x = 2 \) \hspace{1cm} y = 4x - 7
\item \( y = 4x^2 + \frac{5}{x} - 1 \), where \( x = 1 \) \hspace{1cm} y = 3x + 5
\item \( y = 2\sqrt{x} - \frac{6}{\sqrt{x}} \), where \( x = 4 \) \hspace{1cm} 7x - 8y - 20 = 0
\item \( y = 3x^\frac{3}{2} - \frac{32}{x} \), where \( x = 4 \) \hspace{1cm} y = 11x - 28
\end{enumerate}
Question 4 (non calculator)
For each of the following curves find an equation of the normal to the curve at the point whose $x$ coordinate is given.

a) \( f(x) = x^3 - 4x^2 + 1 \), where \( x = 2 \)
\[ 4y = x - 30 \]

b) \( f(x) = x^3 - 7x^2 + 11x \), where \( x = 3 \)
\[ 4y = x - 15 \]

c) \( f(x) = 3x^4 - 7x^3 + 5 \) where \( x = 2 \)
\[ 12y + x + 34 = 0 \]

d) \( f(x) = \frac{1}{4}x^5 - 18x + 11 \) where \( x = 2 \)
\[ 2y + x + 32 = 0 \]
Question 5 (non calculator)

For each of the following curves find an equation of the normal to the curve at the point whose $x$ coordinate is given.

a) \( f(x) = 2x^3 - 3x^2 - 10x + 18 \), where \( x = 2 \)

\[ x + 2y = 6 \]

b) \( f(x) = x^3 - 4x^2 + 6x + 1 \), where \( x = 1 \)

\[ x + y = 5 \]

c) \( f(x) = 4x^3 + 2x^2 - 18x - 10 \) where \( x = -2 \)

\[ 22y + x = 42 \]

d) \( f(x) = -2x^3 + 4x^2 - 1 \), where \( x = 2 \)

\[ 8y = x - 10 \]
Question 6 (non calculator)

For each of the following curves find an equation of the normal to the curve at the point whose $x$ coordinate is given.

a) $y = x^2 (x - 6) + \frac{5}{x} - 1$, where $x = 1$

\[ x - 14y - 15 = 0 \]

b) $y = 2x^{\frac{3}{2}} - \frac{16}{x}$, where $x = 4$

\[ x + 7y = 88 \]

c) $y = 4x^2 + x^{-\frac{3}{2}}$, where $x = 1$

\[ 2x + 13y = 67 \]

d) $y = 2x^2 - 4x^{\frac{3}{2}} - \frac{8}{x} - 1$, where $x = 4$

\[ 2x + 9y + 19 = 0 \]
STATIONARY POINTS
Question 1  (non calculator)

For each of the following cubic equations find the coordinates of their stationary points and determine their nature.

a)  \( y = x^3 - 3x^2 - 9x + 3 \)

b)  \( y = x^3 + 12x^2 + 45x + 50 \)

c)  \( y = 2x^3 - 6x^2 + 12 \)

d)  \( y = 25 - 24x + 9x^2 - x^3 \)

\[
\begin{align*}
\text{min}(3,-24), \text{max}(-1,8), & \quad \text{min}(-3,-4), \text{max}(-5,0) \\
\text{min}(2,4), \text{max}(0,12), & \quad \text{min}(2,5), \text{max}(4,9)
\end{align*}
\]
Question 2
For each of the following equations find the coordinates of their stationary points and determine their nature.

a) \( y = x + \frac{4}{x}, \quad x \neq 0 \)

b) \( y = x^2 + \frac{16}{x}, \quad x \neq 0 \)

c) \( y = x - 4\sqrt{x}, \quad x > 0 \)

d) \( y = 4x^2 + \frac{1}{x}, \quad x \neq 0 \)

\[
\begin{align*}
\min (2, 4), & \max (-2, -4), & \min (2, 12), & \min (4, -4), & \min \left(\frac{1}{2}, 3\right)
\end{align*}
\]
Question 3
For each of the following equations find the coordinates of their stationary points and determine their nature.

a) \( y = 12\sqrt{x} - x^{\frac{3}{2}}, \quad x > 0 \)

b) \( y = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}, \quad x > 0 \)

c) \( y = 6x^{\frac{1}{2}} - 4x - 2, \quad x > 0 \)

d) \( y = x^2 - 14x^2 + 100, \quad x > 0 \)

\[
\begin{align*}
\text{max} (4,16), & \quad \text{min} (2, -4\sqrt{2}), & \quad \text{max} \left( \frac{9}{16}, \frac{1}{4} \right), & \quad \text{min} (4,4)
\end{align*}
\]
Question 4

For each of the following equations find the coordinates of their stationary points and determine their nature.

a) \( y = x^3 - 16x^2 + 60, \ x > 0 \)

b) \( y = 5x^2 - 6x^{\frac{5}{3}} + 10, \ x > 0 \)

c) \( y = 6x^\frac{4}{3} - x^2 - 20, \ x > 0 \)

d) \( y = 5x^2 - 2x^{\frac{5}{3}} - 10, \ x > 0 \)

\[ \min(4, -4), \min(1, 9), \max(8, 12), \max(4, 6) \]
Question 5

For each of the following equations find the coordinates of their stationary points and determine their nature.

a) \[ y = \frac{1}{x} - \frac{1}{\sqrt{x}}, \quad x > 0 \]

b) \[ y = \frac{3\sqrt{x} - 2}{x^2}, \quad x > 0 \]

c) \[ y = \frac{3}{x^2} + \frac{27}{x}, \quad x > 0 \]

d) \[ y = \frac{1}{3\sqrt{x}} \left( \frac{2}{x} - 3 \right), \quad x > 0 \]

\[ \min \left( 4, -\frac{1}{4} \right), \quad \max \left( 1, 1 \right), \quad \min (27, 4), \quad \min \left( 2, -\frac{\sqrt{2}}{3} \right) \]
INCREASING and DECREASING FUNCTIONS
Question 1
For each of the following equations find the range of the values of \( x \), for which \( y \) is increasing or decreasing.

a) \( y = 2x^3 - 3x^2 - 12x + 2 \), increasing

b) \( y = x^3 - 6x^2 + 12 \), decreasing

c) \( y = x^3 - 3x + 8 \), increasing

d) \( y = 1 - 3x^2 - x^3 \), decreasing

\[ x < -1 \text{ or } x > 2, \quad 0 < x < 4, \quad x < -1 \text{ or } x > 1, \quad x < -2 \text{ or } x > 0 \]
Question 2

Find the range of the values of $x$, for which $f(x)$ is increasing or decreasing.

a) $f(x) = x^3 - 3x^2 - 9x + 10$, increasing

b) $f(x) = -x^3 + 9x^2 - 15x - 13$, increasing

c) $f(x) = 4x^3 - 3x^2 - 6x$, decreasing

d) $f(x) = 4x^3 - 3x$, decreasing

$x < -1$ or $x > 3$, $1 < x < 5$, $-\frac{1}{2} < x < \frac{1}{2}$
DIFFERENTIATION PRACTICE IN CONTEXT
Question 1

The curve $C$ has equation

$$f(x) = 3x^2 - 8x + 2.$$ 

a) Find the gradient at the point on $C$, where $x = -1$.

The point $A$ lies on $C$ and the gradient at that point is 4.

b) Find the coordinates of $A$.

$(-14, A(2, -2))$
Question 2
The curve $C$ has equation

$$y = x^3 - 11x + 1.$$  

a) Find the gradient at the point on $C$, where $x = 3$.

The point $P$ lies on $C$ and the gradient at that point is 1.

b) Find the possible coordinates of $P$.

$$P(2,-13) \text{ or } P(-2,15)$$
Question 3

The curve $C$ has equation

$$y = 2x^2 - 4x - 1.$$ 

a) Find the gradient at the point on $C$, where $x = 2$.

The point $P$ lies on $C$ and the gradient at that point is 2.

b) Find the coordinates of $P$.

$$(4, P\left(\frac{3}{2}, -\frac{5}{2}\right))$$
Question 4

The curve $C$ has equation

$$f(x) = x + \frac{1}{x}, \quad x \neq 0.$$ 

a) Find the gradient at the point on $C$, where $x = \frac{1}{2}$.

The point $A$ lies on $C$ and the gradient at that point is $\frac{3}{4}$.

b) Find the possible coordinates of $A$.

\(-3, \ A\left(2, \frac{5}{2}\right) \text{ or } A\left(-2, -\frac{5}{2}\right)\)
Question 5
The curve $C$ has equation
\[ y = x^3 - x^2 - 5x + 2. \]

Find the $x$ coordinates of the points on $C$ with gradient 3.

\[ x = -\frac{4}{3}, 2 \]

Question 6
The curve $C$ has equation
\[ y = x^3 - 6x^2 - 3x + 25. \]

Find an equation of the tangent to $C$ at the point where $x = 2$.

\[ y = 5x - 7 \]
The curve $C$ has equation

$$y = -x^2(x+1), \ x \in \mathbb{R}.$$  

The curve meets the coordinate axes at the origin $O$ and at the point $A$.

a) Sketch the graph of $C$, indicating clearly the coordinates of $A$.

b) Show that the straight line with equation

$$x + y + 1 = 0,$$

is a tangent to $C$ at $A$. 

$A(-1,0)$
Question 8

The curve \( C \) has equation

\[ y = \frac{6}{x^2} + \frac{5x}{4} - 4, \quad x \neq 0. \]

\( dy \)

\( dx \)

a) Find an expression for \( \frac{dy}{dx} \).

b) Determine an equation of the normal to the curve at the point where \( x = 2 \).

\[ \frac{dy}{dx} = \frac{5}{4x^3}, \quad y = 4x - 8 \]
Question 9

The curve \( C \) has equation

\[
f(x) = 4x\sqrt{x} - \frac{25x^2}{16}, \ x \geq 0.
\]

a) Find a simplified expression for \( f'(x) \).

b) Determine an equation of the tangent to \( C \) at the point where \( x = 4 \), giving the answer in the form \( ax + by = c \), where \( a \), \( b \) and \( c \) are integers.
Question 10

A curve has the following equation

\[ f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}}, \quad x > 0. \]

a) Express \( f(x) \) in the form \( Ax^\frac{3}{2} + Bx^{\frac{1}{2}} + Cx^{-\frac{1}{2}} \), where \( A \), \( B \) and \( C \) are constants to be found.

b) Show that the tangent to the curve at the point where \( x = 1 \) is parallel to the line with equation

\[ 2y = 13x + 2. \]

\[ A = 2, \quad B = 1, \quad C = -6 \]
Question 11
A cubic curve has equation
\[ f(x) = 2x^3 - 7x^2 + 6x + 1. \]

The point \( P(2,1) \) lies on the curve.

a) Find an equation of the tangent to the curve at \( P \).

The point \( Q \) lies on the curve so that the tangent to the curve at \( Q \) is parallel to the tangent to the curve at \( P \).

b) Determine the \( x \) coordinate of \( Q \).

\[ y = 2x - 3, \quad x_Q = \frac{1}{3} \]
Question 12

The curve $C$ has equation

$$y = 2x^3 - 9x^2 + 12x - 10.$$ 

a) Find the coordinates of the two points on the curve where the gradient is zero.

The point $P$ lies on $C$ and its $x$ coordinate is $-1$.

b) Determine the gradient of $C$ at the point $P$.

The point $Q$ lies on $C$ so that the gradient at $Q$ is the same as the gradient at $P$.

c) Find the coordinates of $Q$.

$$(1, -5), (2, -6), \frac{36}{36}, Q(4, 22)$$
Question 13

The curve $C$ has equation
\[ y = ax^3 + bx^2 - 10, \]
where $a$ and $b$ are constants.

The point $A(2, 2)$ lies on $C$.

Given that the gradient at $A$ is $4$, determine the value of $a$ and the value of $b$.

\[ a = -2, \quad b = 7 \]
Question 14

The curve $C$ has equation

$$y = x^3 - 4x^2 + 6x - 3.$$ 

The point $P(2,1)$ lies on $C$ and the straight line $L_1$ is the tangent to $C$ at $P$.

a) Find an equation of $L_1$.

The straight line $L_2$ is a tangent to $C$ at the point $Q$.

b) Given that $L_2$ is parallel to $L_1$, determine …

i. … the exact coordinates of $Q$.

ii. … an equation of $L_2$. 

$$27y = 54x - 49$$

Created by T. Madas
Question 15

A curve $C$ and a straight line $L$ have respective equations

\[ y = 2x^2 - 6x + 5 \quad \text{and} \quad 2y + x = 4. \]

a) Find the coordinates of the points of intersection between $C$ and $L$.

b) Show that $L$ is normal to $C$.

The tangent to $C$ at the point $P$ is parallel to $L$.

c) Determine the $x$ coordinate of $P$.

\[ (2, 1), \left(\frac{3}{4}, \frac{13}{8}\right), \quad x_P = \frac{11}{8} \]
Question 16

The curve $C$ has equation

$$y = 2x^3 - 6x^2 + 3x + 5.$$

The point $P(2,3)$ lies on $C$ and the straight line $L_1$ is the tangent to $C$ at $P$.

(a) Find an equation of $L_1$.

The straight lines $L_2$ and $L_3$ are parallel to $L_1$, and they are the respective normals to $C$ at the points $Q$ and $R$.

(b) Determine the $x$ coordinate of $Q$ and the $x$ coordinate of $R$.

$$y = 3x - 3, \quad x = \frac{1}{3}, \frac{5}{3}$$

\[\begin{array}{c}
\text{Diagram showing curves and tangent.}
\end{array} \]
Question 17

The figure above shows the curve with equation

\[ y = \frac{1}{4}(x^2 - 12x + 35) \, . \]

The curve crosses the \( x \) axis at the points \( P(x_1, 0) \) and \( Q(x_2, 0) \), where \( x_2 > x_1 \).

The tangent to the curve at \( Q \) is the straight line \( L_1 \).

\[ \text{a) Find an equation of } L_1. \]

The tangent to the curve at the point \( R \) is denoted by \( L_2 \). It is further given that \( L_2 \) meets \( L_1 \) at right angles, at the point \( S \).

\[ \text{b) Find an equation of } L_2. \]

\[ \text{c) Determine the exact coordinates of } S. \]
The point $P(1,0)$ lies on the curve $C$ with equation

$$y = x^3 - x, \ x \in \mathbb{R}.$$

**a)** Find an equation of the tangent to $C$ at $P$, giving the answer in the form $y = mx + c$, where $m$ and $c$ are constants.

The tangent to $C$ at $P$ meets $C$ again at the point $Q$.

**b)** Determine the coordinates of $Q$.

$$y = 2x - 2, \ Q(-2,-6)$$
Question 19

A curve $C$ with equation

$$y = 4x^3 + 7x^2 + x + 11, \quad x \in \mathbb{R}.$$ 

The point $P$ lies on $C$, where $x = -1$.

a) Find an equation of the tangent to $C$ at $P$.

b) Determine the $x$ coordinate of $Q$.

\[
y = 12 - x, \quad x_Q = \frac{1}{4}
\]
The figure above shows the curve \( C \) with equation \( y = 2x^2 - x + 3 \).

\( C \) crosses the \( y \) axis at the point \( P \). The normal to \( C \) at \( P \) is the straight line \( L_1 \).

a) Find an equation of \( L_1 \).

\( L_1 \) meets the curve again at the point \( Q \).

b) Determine the coordinates of \( Q \).

The tangent to \( C \) at \( Q \) is the straight line \( L_2 \).

\( L_2 \) meets the \( y \) axis at the point \( R \).

c) Show that the area of the triangle \( PQR \) is one square unit.

\[ y = x + 3, \quad Q(1,4) \]
The figure above shows the curve $C$ with equation

$$y = 2x^3 + 3x^2 - 11x - 6.$$ 

The curve crosses the $x$ axis at the points $P$, $Q$ and $R(2,0)$.

The tangent to $C$ at $R$ is the straight line $L_1$.

a) Find an equation of $L_1$.

The normal to $C$ at $P$ is the straight line $L_2$.

The straight lines $L_1$ and $L_2$ meet at the point $S$.

b) Show that $\angle PSR = 90^\circ$.
Find the coordinates of the stationary point of the curve and determine whether it is a local maximum, a local minimum or a point of inflexion.

\[
y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16, \quad x \in \mathbb{R}, \quad x \geq 0.
\]

\textbf{local minimum at } (16, -2800)
Question 23

A curve has equation

\[ y = x^2 - 6x \sqrt[3]{x} + 2, \quad x \in \mathbb{R}, \quad x \geq 0. \]

Find the coordinates of the stationary points of the curve and classify them as local maxima, local minima or a points of inflexion.

- Local minimum at \( (8, -30) \)
- Local maximum at \( (0, 2) \)
Question 24

A curve has equation

\[ y = x \left( x^2 - 128 \right), \quad x \in \mathbb{R}, \quad x > 0. \]

The curve has a single stationary point with coordinates \( \left( 2^\alpha, -2^\beta \right) \), where \( \alpha \) and \( \beta \) are positive integers.

Find the value of \( \beta \) and justify that the stationary point is a local minimum.

\[ \beta = 12 \]
Question 25

The point $P$, whose $x$ coordinate is $\frac{1}{4}$, lies on the curve with equation

$$y = \frac{k + 4\sqrt{x}}{7x}, \ x \in \mathbb{R}, \ x > 0,$$

where $k$ is a non-zero constant.

a) Determine, in terms of $k$, the gradient of the curve at $P$.

The tangent to the curve at $P$ is parallel to the straight line with equation

$$44x + 7y - 5 = 0.$$

b) Find an equation of the tangent to the curve at $P$.

$$\frac{dy}{dx} \bigg|_{x=\frac{1}{4}} = \frac{4-16k}{7}, \quad 44x + 7y = 25$$
Question 26

The figure above shows the curve \( C \) with equation

\[
y = \frac{x^2}{2} - \frac{4}{x}, \quad x \neq 0.
\]

The curve crosses the \( x \) axis at the point \( P \).

The straight line \( L \) is the normal to \( C \) at \( P \).

a) Find …

i. … the coordinates of \( P \).

ii. … an equation of \( L \).

b) Show that \( L \) does not meet \( C \) again.

\[
P(2,0), \quad x + 3y = 2
\]
Question 27

The curve $C$ has equation

$$y = (x-1)(x^2 + 4x + 5), \ x \in \mathbb{R}.$$ 

a) Show that $C$ meets the $x$ axis at only one point.

The point $A$, where $x = -1$, lies on $C$.

b) Find an equation of the normal to $C$ at $A$.

The normal to $C$ at $A$ meets the coordinate axes at the points $P$ and $Q$.

c) Show further that the area of the triangle $OPQ$, where $O$ is the origin, is $12\frac{1}{4}$ square units.

$$2y = x - 7$$
Question 28

A curve has equation

\[ y = x - 8\sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0. \]

The curve meets the coordinate axes at the origin and at the point \( P \).

a) Determine the coordinates of \( P \).

The point \( Q \), where \( x = 4 \), lies on the curve.

b) Find an equation of the normal to curve at \( Q \).

c) Show clearly that the normal to the curve at \( Q \) does not meet the curve again.

\[
P(64, 0), \quad y = x - 16
\]
Question 29

The curve $C$ has equation

$$y = x^3 - 9x^2 + 24x - 19, \quad x \in \mathbb{R}.$$  

a) Show that the tangent to $C$ at the point $P$, where $x = 1$, has gradient 9.

b) Find the coordinates of another point $Q$ on $C$ at which the tangent also has gradient 9.

The normal to $C$ at $Q$ meets the coordinate axes at the points $A$ and $B$.

c) Show further that the approximate area of the triangle $OAB$, where $O$ is the origin, is 11 square units.
Question 30

The point $A(2,1)$ lies on the curve with equation

$$y = \frac{(x-1)(x+2)}{2x}, \quad x \in \mathbb{R}, \ x \neq 0.$$ 

a) Find the gradient of the curve at $A$.

b) Show that the tangent to the curve at $A$ has equation

$$3x - 4y - 2 = 0.$$ 

The tangent to the curve at the point $B$ is parallel to the tangent to the curve at $A$.

c) Determine the coordinates of $B$. 

Gradient at $A = \frac{3}{4}, \quad B(-2,0)$
Question 31

The curve $C$ has equation $y = f(x)$ given by

$$f(x) = 2(x - 2)^3, \ x \in \mathbb{R}.$$ 

a) Sketch the graph of $f(x)$.

b) Find an expression for $f'(x)$.

The point $P(3, 2)$ lies on $C$ and the straight line $l_1$ is the tangent to $C$ at $P$.

c) Find an equation of $l_1$.

The straight line $l_2$ is another tangent at a different point $Q$ on $C$.

d) Given that $l_1$ is parallel to $l_2$ show that an equation of $l_2$ is

$$y = 6x - 8.$$ 

$$f'(x) = 6x^2 - 24x + 24, \ \ y = 6x - 16$$
Question 32

The point \( P(2,9) \) lies on the curve \( C \) with equation

\[
y = x^3 - 3x^2 + 2x + 9, \quad x \in \mathbb{R}, \quad x \geq 1.
\]

a) Find an equation of the tangent to \( C \) at \( P \), giving the answer in the form \( y = mx + c \), where \( m \) and \( c \) are constants.

The point \( Q \) also lies on \( C \) so that the tangent to \( C \) at \( Q \) is perpendicular to the tangent to \( C \) at \( P \).

b) Show that the \( x \) coordinate of \( Q \) is

\[
\frac{6 + \sqrt{6}}{6}.
\]

\[
y = 2x + 5
\]
Question 33

The volume, $V$ cm$^3$, of a soap bubble is modelled by the formula

$$V = (p - qt)^2, \quad t \geq 0,$$

where $p$ and $q$ are positive constants, and $t$ is the time in seconds, measured after a certain instant.

When $t=1$ the volume of a soap bubble is 9 cm$^3$ and at that instant its volume is decreasing at the rate of 6 cm$^3$ per second.

Determine the value of $p$ and the value of $q$.

$$p = 4, \quad q = 1$$
Question 34

A curve \( C \) has equation

\[
y = 2x^3 - 5x^2 + a, \quad x \in \mathbb{R},
\]

where \( a \) is a constant.

The tangent to \( C \) at the point where \( x = 2 \) and the normal to \( C \) at the point where \( x = 1 \), meet at the point \( Q \).

Given that \( Q \) lies on the \( x \) axis, determine in any order …

a) … the value of \( a \).

b) … the coordinates of \( Q \).

\[
a = \frac{8}{3}, \quad Q\left(\frac{2}{3}, 0\right)
\]
Question 35

The curve $C$ has equation

$$y = \frac{x^3(5x\sqrt{x} - 128)}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$ 

a) Determine expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b) Show that the $y$ coordinate of the stationary point of $C$ is $-k\sqrt[3]{4}$, where $k$ is a positive integer.

c) Evaluate $\frac{d^2y}{dx^2}$ at the stationary point of $C$.

Give the answer in terms of $\sqrt[3]{2}$.

d) Find the value of $\frac{d^3y}{dx^3}$ at the point on $C$, where $\frac{d^2y}{dx^2} = 0$.

$$\frac{dy}{dx} = 20x^3 - 320x^{\frac{3}{2}}$$,  
$$\frac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}$$,  
$$\frac{d^3y}{dx^3} = 120x - 240x^{\frac{1}{2}}$$.

$$k = 3072, \quad 960\sqrt[3]{2}, \quad 360$$