VOLUMES OF SOLIDS
BY SLICING

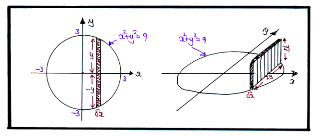
Question 1

A solid's base is bounded by the circle with equation

$$x^2 + y^2 = 9 .$$

Determine the volume of the solid, given that every vertical cross-section of the solid, which is perpendicular to the x axis, is a square.

$\boxed{}$, $\boxed{144}$



LOOKING AT THE FIGURES ABOVE

- $x^2+y^2=9$
  $y^2 = 9-x^2$
  $y = \pm\sqrt{9-x^2}$
- THE LENGTH OF THE SQUARE IS $2y = +2\sqrt{9-x^2}$
- THE AREA OF THE SQUARE IS $4y^2 = 4(9-x^2)$
- THE VOLUME OF THE "INFINITESIMAL SLICE" IS $4(9-x^2)\delta x$

SUMMING ALL "INFINITESIMAL SLICES" OF THICKNESS δx, FROM $x=-3$ TO $x=3$, & TAKE LIMITS

$\Rightarrow V = \int_{-3}^{3} 4(9-x^2)\,dx$ EVEN INTEGRAND

$\Rightarrow V = 8\int_{0}^{3} 9-x^2\,dx$

$\Rightarrow V = 8\left[9x - \frac{1}{3}x^3\right]_0^3$

$\Rightarrow V = 8\left[(27-9)-0\right]$

$\Rightarrow V = 144$

Question 2

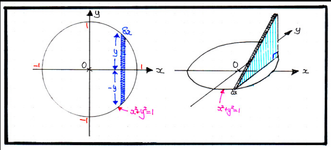
A solid's base is bounded by the circle with equation

$$x^2 + y^2 = 1 .$$

Every vertical cross-section of the solid, perpendicular to the x axis, is a right angled isosceles triangle, with one of its non hypotenuse sides on the base of the solid.

Determine the volume of the solid.

$\boxed{}$, $\boxed{\frac{8}{3}}$



LOOKING AT THE FIGURES ABOVE

- $x^2+y^2=1$
  $y = \pm\sqrt{1-x^2}$
- BOTH THE BASE & HEIGHT OF THE INFINITESIMAL TRIANGLE ARE $2\sqrt{1-x^2}$
- THE VOLUME OF THE INFINITESIMAL TRIANGULAR PRISM IS $\frac{1}{2}\left(2\sqrt{1-x^2}\right)^2 \delta x$

SUMMING ALL THE INFINITESIMAL TRIANGULAR PRISMS FROM $x=-1$ TO $x=1$

$$\Rightarrow V = \int_{-1}^{1} \frac{1}{2}\left(2\sqrt{1-x^2}\right)^2 \, dx = \int_{-1}^{1} 2(1-x^2) \, dx$$

... EVEN INTEGRAND ...

$$= \int_0^1 4(1-x^2) \, dx = \int_0^1 4 - 4x^2 \, dx$$

$$= \left[4x - \frac{4}{3}x^3\right]_0^1 = \left(4 - \frac{4}{3}\right) - (0)$$

$$= \frac{8}{3}$$

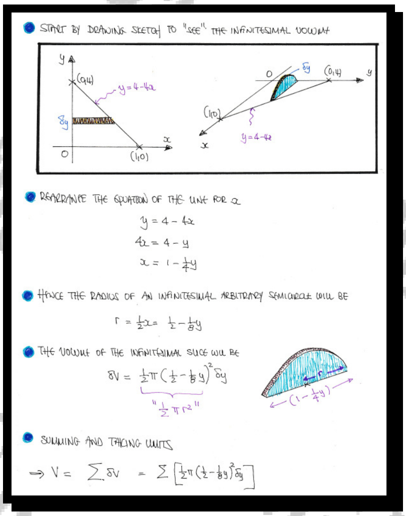
Question 3

A solid's base is bounded by the right angled triangle in the x-y plane whose vertices have coordinates $(0,0)$, $(1,0)$ and $(0,4)$.

Every vertical cross-section of the solid, perpendicular to the y axis, is a semicircle with its diameter lying on the base of the solid.

Use calculus to determine the volume of the above described solid.

$\boxed{}$, $\boxed{\frac{\pi}{6}}$



- START BY DRAWING SKETCH TO "SEE" THE INFINITESIMAL VOLUME

- REARRANGE THE EQUATION OF THE LINE FOR x

$$y = 4 - 4x$$
$$4x = 4 - y$$
$$x = 1 - \frac{1}{4}y$$

- HENCE THE RADIUS OF AN INFINITESIMAL ARBITRARY SEMICIRCLE WILL BE

$$r = \frac{1}{2}x = \frac{1}{2} - \frac{1}{8}y$$

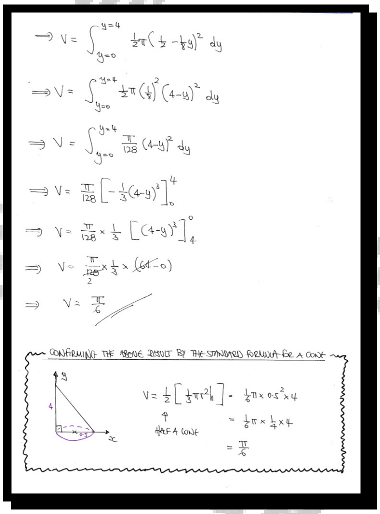
- THE VOLUME OF THE INFINITESIMAL SLICE WILL BE

$$\delta V = \underbrace{\frac{1}{2}\pi\left(\frac{1}{2} - \frac{1}{8}y\right)^2}_{\text{"}\frac{1}{2}\pi r^2\text{"}}\delta y$$

r
$(1 - \frac{1}{4}y)$

- SUMMING AND TAKING LIMITS

$$\Rightarrow V = \sum \delta V = \sum\left[\frac{1}{2}\pi\left(\frac{1}{2} - \frac{1}{8}y\right)^2\delta y\right]$$



$$\Rightarrow V = \int_{y=0}^{y=4} \frac{1}{2}\pi\left(\frac{1}{2} - \frac{1}{8}y\right)^2 dy$$

$$\Rightarrow V = \int_{y=0}^{y=4} \frac{1}{2}\pi\left(\frac{1}{8}\right)^2(4-y)^2 dy$$

$$\Rightarrow V = \int_{y=0}^{y=4} \frac{\pi}{128}(4-y)^2 dy$$

$$\Rightarrow V = \frac{\pi}{128}\left[-\frac{1}{3}(4-y)^3\right]_0^4$$

$$\Rightarrow V = \frac{\pi}{128} \times \frac{1}{3}\left[(4-y)^3\right]_4^0$$

$$\Rightarrow V = \frac{\pi}{\cancel{128}_{2}} \times \frac{1}{3} \times (\cancel{64} - 0)$$

$$\Rightarrow V = \frac{\pi}{6}$$

CONFIRMING THE ABOVE RESULT BY THE STANDARD FORMULA FOR A CONE

$$V = \frac{1}{2}\left[\frac{1}{3}\pi r^2 h\right] = \frac{1}{6}\pi \times 0.5^2 \times 4$$

HALF A CONE

$$= \frac{1}{6}\pi \times \frac{1}{4} \times 4$$

$$= \frac{\pi}{6}$$

Question 4

The finite region R is bounded by the curve with equation

$$y = x^2 - 1,$$

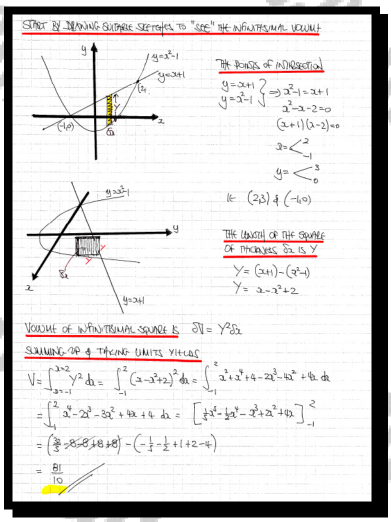
and the straight line with equation

$$y = x + 1$$

A solid's base is bounded by R and every vertical cross-section of the solid, perpendicular to the x axis, is a square with one of sides on the base of the solid.

Determine the volume of the solid.

$\boxed{\text{M.M.C}}$, $\boxed{\frac{81}{10}}$



Question 5

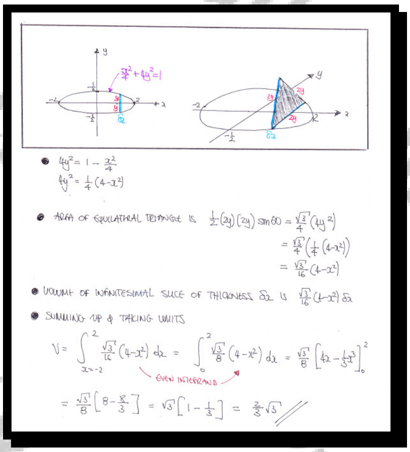
A solid's base is bounded by the ellipse with equation

$$\frac{1}{4}x^2 + 4y^2 = 1.$$

Every vertical cross-section of the solid, perpendicular to the x axis, is an equilateral triangle, with one of its sides on the base of the solid.

Determine the volume of the solid.

$$\boxed{\frac{2}{3}\sqrt{3}}$$



Question 6

An ellipse E has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b are positive constants.

Use integral calculus to show that the volume of a right elliptical cone with base E and height h, is given by

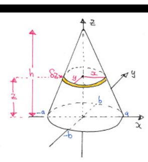
$$\frac{1}{3}\pi abh .$$

You may assume the standard result for the area of an ellipse without proof.

$\boxed{}$, $\boxed{\text{proof}}$

- LOOKING AT THE DIAGRAM OPPOSITE
- THE INFINITESIMAL ELLIPTICAL DISC OF THICKNESS δz HAS VOLUME

$$\delta V = \pi xy\,\delta z$$



- BY SIMILAR TRIANGLES WE HAVE

$$\Rightarrow \frac{x}{a}=\frac{h-z}{h} \quad \& \quad \frac{y}{b}=\frac{h-z}{h}$$

$$\Rightarrow x=\frac{a}{h}(h-z) \quad \& \quad y=\frac{b}{h}(h-z)$$

- SO THE VOLUME IS GIVEN BY

$$\delta V=\pi\left[\frac{a}{h}(h-z)\right]\left[\frac{b}{h}(h-z)\right]\delta z = \frac{\pi ab}{h^2}(h-z)^2\,\delta z$$

$$V=\int_{z=0}^{z=h} 1\,dV = \int_{z=0}^{z=h}\frac{\pi ab}{h^2}(h-z)^2\,dz$$

$$=\frac{\pi ab}{h^2}\left[-\frac{1}{3}(h-z)^3\right]_0^h$$

$$=\frac{\pi ab}{3h^2}\left[(h-z)^3\right]_h^0$$

$$=\frac{\pi ab}{3h^2}\left[h^3-0\right]$$

$$=\frac{1}{3}\pi abh$$