

Created by T. Madas

VECTOR OPERATORS

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GRADIENT

$$\text{grad } \varphi \equiv \nabla \varphi$$

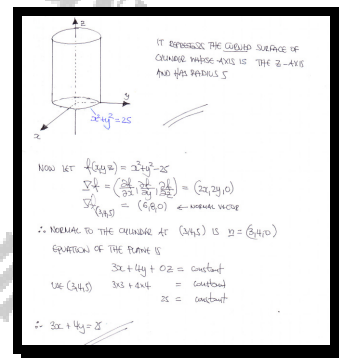
Question 1

A surface S is given by the Cartesian equation

$$x^2 + y^2 = 25.$$

- Draw a sketch of S , and describe it geometrically.
- Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(3, 4, 5)$.

$$3x + 4y = 25$$



Question 2

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}$$

proof

$$\begin{aligned} \nabla \left(\frac{1}{r} \right) &= \nabla \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \left[\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right), \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right), \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \right] \\ &= \left(-\frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{1}{2} \frac{2y}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{1}{2} \frac{2z}{(x^2 + y^2 + z^2)^{3/2}} \right) \\ &= \left(-\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) = -\frac{1}{r^3} (x, y, z) \\ &= -\frac{1}{r^3} \mathbf{r} \end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector find

$$\nabla(\mathbf{a} \cdot \mathbf{r}).$$

$$\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$$

$$\begin{aligned} \nabla(\mathbf{a} \cdot \mathbf{r}) &= \nabla[(a_1, a_2, a_3) \cdot (x, y, z)] = \nabla(a_1x + a_2y + a_3z) \\ &= \left[\frac{\partial}{\partial x}(a_1x), \frac{\partial}{\partial y}(a_2y), \frac{\partial}{\partial z}(a_3z) \right] = (a_1, a_2, a_3) = \mathbf{a} \end{aligned}$$

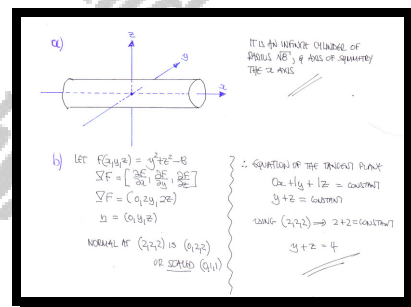
Question 4

A surface S is defined by the Cartesian equation

$$y^2 + z^2 = 8.$$

- Draw a sketch of S , and describe it geometrically.
- Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(2, 2, 2)$.

$$y + z = 4$$



Question 5

The scalar function V is defined as

$$V(x, y, z) = (y+z)^2 + y^2(x+y) + xyz + 1.$$

Determine the value of the directional derivative of V at the point $P(1, -1, 1)$, in the direction $-\mathbf{i} + \mathbf{j} + \mathbf{k}$.

 $\sqrt{3}$

$\bullet V = (y+z)^2 + y^2(x+y) + xyz + 1$
 $\bullet P(1, -1, 1)$
 $\bullet \mathbf{u} = (-1, 1, 1)$

FIRST: $\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)$
 $\nabla V = [y^2 + 2yz, 2y(x+z) + 2yz + 2xy, x(2y) + 2xy]$
 $\nabla V \Big|_{(1,-1,1)} = [0, -2+3+1, 1]$
 $\nabla V \Big|_{(1,-1,1)} = (0, 2, 1)$

NEXT: $\mathbf{u} = (-1, 1, 1)$
 $|\mathbf{u}| = \sqrt{3}$
 $\hat{\mathbf{u}} = \frac{1}{\sqrt{3}}(-1, 1, 1)$

SO DIRECTIONAL DERIVATIVE AT THE REQUIRED POINT AND DIRECTION
 $\nabla V \cdot \hat{\mathbf{u}} = (0, 2, 1) \cdot \frac{1}{\sqrt{3}}(-1, 1, 1) = \frac{0-2+1}{\sqrt{3}} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

Question 6

The scalar function ϕ is defined as

$$\phi(x, y, z) = e^{x-y} \sin z.$$

Determine the value of the directional derivative of ϕ at the point $P(1, 1, 0)$, in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

 $\frac{1}{\sqrt{3}}$

WE REQUIRE $\nabla \phi \cdot \hat{\mathbf{u}}$ EVALUATED AT $(1, 1, 0)$

$\bullet \phi(x, y, z) = e^{x-y} \sin z$
 $\bullet \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = [e^{x-y}, -e^{x-y}, e^{x-y} \cos z]$
 $\bullet \mathbf{u} = (1, 1, 1) \Rightarrow \hat{\mathbf{u}} = \frac{1}{\sqrt{3}}(1, 1, 1)$
 $\bullet \nabla \phi \cdot \hat{\mathbf{u}} = e^{x-y} [\cos z - \sin z] \cdot \frac{1}{\sqrt{3}} [1, 1, 1]$
 $= \frac{1}{\sqrt{3}} e^{x-y} [\cos z - \sin z]$
 $\bullet \nabla \phi \cdot \hat{\mathbf{u}} \Big|_{(1,1,0)} = \frac{1}{\sqrt{3}} e^{1-1} \cos 0 = \frac{1}{\sqrt{3}} \times 1 \times 1 = \frac{1}{\sqrt{3}}$

Question 7

The point $P(1,2,3)$ lies on the surface with Cartesian equation

$$2z^2 = 6x^2 + 3y^2.$$

The scalar function u is defined as

$$u(x, y, z) = x^2 yz + x^2 y.$$

Determine the value of the directional derivative of u at the point P in the direction to the normal at P .

$$\boxed{6\sqrt{3}}$$

Handwritten solution for Question 7:

Given surface: $2z^2 = 6x^2 + 3y^2$
 Let $f(x,y,z) = 2z^2 - 6x^2 - 3y^2$
 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$
 $\nabla f = (-12x, -6y, 4z)$
 At point $P(1,2,3)$:
 $\nabla f|_P = (-12, -12, 12)$
 Direction vector $\hat{n} = \frac{1}{\sqrt{3}}(1, 1, -1)$

Directional derivative at $P(1,2,3)$:
 $\nabla u \cdot \hat{n} = (6, 4, 2) \cdot \frac{1}{\sqrt{3}}(1, 1, -1)$
 $= \frac{6+4-2}{\sqrt{3}} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$

Question 8

The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3.$$

The two surfaces intersect each other at an angle θ , at the point P .

Given further that P lies in the first octant, determine the exact value of $\cos \theta$.

$$\cos \theta = \frac{8}{3\sqrt{21}}$$

Handwritten solution for Question 8:

Given: $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$.

Substituting $z = x^2 + y^2 - 3$ into $x^2 + y^2 + z^2 = 9$:

$$x^2 + y^2 + (x^2 + y^2 - 3)^2 = 9$$

$$x^2 + y^2 + x^4 + y^4 - 6x^2 - 6y^2 + 9 = 9$$

$$x^4 + y^4 - 5x^2 - 5y^2 = 0$$

$$x^2(x^2 - 5) + y^2(y^2 - 5) = 0$$

Since $x, y \in \mathbb{R}$, $x^2 \geq 0$ and $y^2 \geq 0$, we have $x^2 - 5 = 0$ and $y^2 - 5 = 0$.

$$x^2 = 5 \Rightarrow x = \pm\sqrt{5}$$

$$y^2 = 5 \Rightarrow y = \pm\sqrt{5}$$

Since P is in the first octant, $x = 1$ and $y = 2$.

Substituting $x = 1$ and $y = 2$ into $z = x^2 + y^2 - 3$:

$$z = 1^2 + 2^2 - 3 = 0$$

So $P(1, 2, 0)$.

Let $f(x, y, z) = x^2 + y^2 + z^2 - 9$ and $g(x, y, z) = x^2 + y^2 - z - 3$.

Gradients at $P(1, 2, 0)$:

$$\nabla f = (2x, 2y, 2z) = (2, 4, 0)$$

$$\nabla g = (2x, 2y, -1) = (2, 4, -1)$$

At P :

$$\nabla f = (2, 4, 0)$$

$$\nabla g = (2, 4, -1)$$

For the angle θ between the surfaces:

$$\cos \theta = \frac{|\nabla f \cdot \nabla g|}{|\nabla f| |\nabla g|}$$

$$= \frac{|(2, 4, 0) \cdot (2, 4, -1)|}{\sqrt{2^2 + 4^2 + 0^2} \sqrt{2^2 + 4^2 + (-1)^2}}$$

$$= \frac{|4 + 16 + 0|}{\sqrt{20} \sqrt{21}}$$

$$= \frac{20}{\sqrt{420}}$$

$$= \frac{20}{2\sqrt{105}}$$

$$= \frac{10}{\sqrt{105}}$$

$$= \frac{8}{3\sqrt{21}}$$

Question 9

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $\phi(r) = \ln r$, show that

$$\nabla \phi(r) = \frac{\mathbf{r}}{r^2}.$$

proof

Handwritten proof for Question 9:

$$\begin{aligned} \phi(x, y, z) &= \ln |\mathbf{r}| \quad \mathbf{r} = (x, y, z) \\ \phi(x, y, z) &= \ln(x^2 + y^2 + z^2)^{\frac{1}{2}} \\ \phi(x, y, z) &= \frac{1}{2} \ln(x^2 + y^2 + z^2) \\ \nabla \phi &= \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right] = \left[\frac{1}{2} \frac{2x}{x^2 + y^2 + z^2}, \frac{1}{2} \frac{2y}{x^2 + y^2 + z^2}, \frac{1}{2} \frac{2z}{x^2 + y^2 + z^2} \right] \\ &= \frac{1}{x^2 + y^2 + z^2} [x, y, z] = \frac{\mathbf{r}}{r^2} \end{aligned}$$

Question 10

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla r^3 \equiv 3r\mathbf{r}.$$

proof

Handwritten proof for Question 10:

$$\begin{aligned} \nabla(r^3) &= \nabla((x^2 + y^2 + z^2)^{\frac{3}{2}}) \\ &= \left[\frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{\frac{3}{2}}, \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{\frac{3}{2}}, \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{\frac{3}{2}} \right] \\ &\quad \downarrow \\ &= \left[\frac{3}{2} (x^2 + y^2 + z^2)^{\frac{1}{2}} (2x), \frac{3}{2} (x^2 + y^2 + z^2)^{\frac{1}{2}} (2y), \frac{3}{2} (x^2 + y^2 + z^2)^{\frac{1}{2}} (2z) \right] \\ &\quad \text{AND BY CYCLIC SIMILARITY ...} \\ &= \left[3x(x^2 + y^2 + z^2)^{\frac{1}{2}}, 3y(x^2 + y^2 + z^2)^{\frac{1}{2}}, 3z(x^2 + y^2 + z^2)^{\frac{1}{2}} \right] \\ &= 3(x, y, z) (x^2 + y^2 + z^2)^{\frac{1}{2}} \\ &= 3|\mathbf{r}| \mathbf{r} \end{aligned}$$

Question 11

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $\psi(r) = \frac{1}{r}$, show that

$$\nabla \psi(r) = -\frac{\mathbf{r}}{r^3}.$$

proof

$$\begin{aligned}\psi(x, y, z) &= \frac{1}{|\mathbf{r}|} \quad \text{as } \mathbf{r} = (x, y, z) \\ \psi(x, y, z) &= \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ \nabla \psi &= \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right) = \left[\frac{-\frac{1}{2}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot 2x, \frac{-\frac{1}{2}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot 2y, \frac{-\frac{1}{2}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot 2z \right] \\ &= \frac{-1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} [x, y, z] = \frac{-1}{r^3} \mathbf{r} = -\frac{\mathbf{r}}{r^3}\end{aligned}$$

Question 12

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show clearly that

$$\nabla(r^n) = nr^{n-2}\mathbf{r}.$$

proof

$$\begin{aligned}\nabla(r^n) &= \nabla[(x^2 + y^2 + z^2)^{\frac{n}{2}}] = \nabla[(x^2 + y^2 + z^2)^{\frac{n}{2}}] \\ &= \left[\frac{\partial}{\partial x}[(x^2 + y^2 + z^2)^{\frac{n}{2}}], \frac{\partial}{\partial y}[(x^2 + y^2 + z^2)^{\frac{n}{2}}], \frac{\partial}{\partial z}[(x^2 + y^2 + z^2)^{\frac{n}{2}}] \right] \\ &= \left[\frac{n}{2} \times 2x (x^2 + y^2 + z^2)^{\frac{n}{2}-1}, \frac{n}{2} \times 2y (x^2 + y^2 + z^2)^{\frac{n}{2}-1}, \frac{n}{2} \times 2z (x^2 + y^2 + z^2)^{\frac{n}{2}-1} \right] \\ &= n(x^2 + y^2 + z^2)^{\frac{n}{2}-1} [x, y, z] = n(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \mathbf{r} \\ &= n \frac{r^{n-2}}{r} \mathbf{r} = nr^{n-2} \mathbf{r}\end{aligned}$$

Question 13

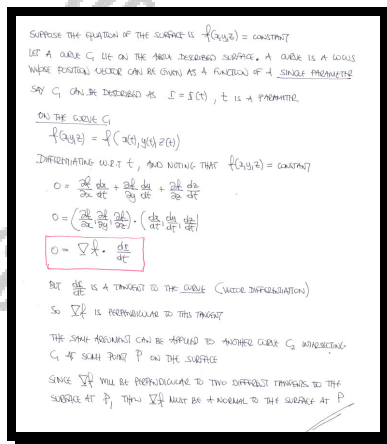
The surface S has Cartesian equation

$$f(x, y, z) = \text{constant},$$

where f is a smooth function.

Given that $\nabla f \neq \mathbf{0}$, show that ∇f is a normal to S .

proof



Question 14

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla f(r) = \frac{\mathbf{r}}{r} f'(r).$$

, proof

MANIPULATE AS BEFORE

$$\nabla f(r) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = \left[\frac{\partial f}{\partial r} \frac{\partial r}{\partial x}, \frac{\partial f}{\partial r} \frac{\partial r}{\partial y}, \frac{\partial f}{\partial r} \frac{\partial r}{\partial z} \right]$$

$$= \frac{\partial f}{\partial r} \left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right]$$

NOW WE HAVE

$$\Rightarrow f = f(x, y, z)$$

$$\Rightarrow r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x) = \frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{x}{r}$$

AND SIMILARLY $\frac{\partial r}{\partial y} = \frac{y}{r}$ AND $\frac{\partial r}{\partial z} = \frac{z}{r}$ ALL THREE IS CLEAR SIMILARITY

RETURNING TO THE MAIN LINE WE OBTAIN

$$\dots = \frac{\partial f}{\partial r} \left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right] = f'(r) \times \frac{1}{r} (x, y, z) = \frac{f'(r)}{r} \mathbf{r}$$

As required

NOTE THE INITIAL MANIPULATION CAN BE THOUGHT OF A STANDARD CHAIN RULE

$$\nabla f(r) = f'(r) \nabla r = f'(r) \left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right] = \dots \text{AS ABOVE}$$

Question 15

The smooth functions $F(x, y, z)$ and $G(x, y, z)$ are given.

Show that

$$\nabla \left[\frac{F}{G} \right] = \frac{G(\nabla F) - F(\nabla G)}{G^2}.$$

proof

Let $F = F(x, y, z)$
 $G = G(x, y, z)$

● CONSIDER THE \hat{i} COMPONENT OF THE GRADIENT
 $\nabla \left[\frac{F}{G} \right] = \frac{\partial}{\partial x} \left[\frac{F}{G} \right] = \frac{G \frac{\partial F}{\partial x} - F \frac{\partial G}{\partial x}}{G^2}$

● AND SIMILARLY THE \hat{j} & \hat{k}

● $\therefore \nabla \left[\frac{F}{G} \right] = \left[\frac{G \frac{\partial F}{\partial x} - F \frac{\partial G}{\partial x}}{G^2}, \frac{G \frac{\partial F}{\partial y} - F \frac{\partial G}{\partial y}}{G^2}, \frac{G \frac{\partial F}{\partial z} - F \frac{\partial G}{\partial z}}{G^2} \right]$

$= \frac{1}{G^2} \left[G \frac{\partial F}{\partial x} - F \frac{\partial G}{\partial x}, G \frac{\partial F}{\partial y} - F \frac{\partial G}{\partial y}, G \frac{\partial F}{\partial z} - F \frac{\partial G}{\partial z} \right]$

$= \frac{1}{G^2} \left[\left(G \frac{\partial F}{\partial x}, G \frac{\partial F}{\partial y}, G \frac{\partial F}{\partial z} \right) - \left(F \frac{\partial G}{\partial x}, F \frac{\partial G}{\partial y}, F \frac{\partial G}{\partial z} \right) \right]$

$= \frac{1}{G^2} \left[G \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] - F \left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right] \right]$

$= \frac{1}{G^2} [G \nabla F - F \nabla G]$

Question 16

$$\Psi(x, y, z) = (x^2 + y^2 + z^2) e^{-\sqrt{x^2 + y^2 + z^2}}$$

Show that

$$\nabla [\Psi(x, y, z)] = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \left(2 - \sqrt{x^2 + y^2 + z^2} \right) e^{-\sqrt{x^2 + y^2 + z^2}}.$$

proof

$\Psi(x, y, z) = (x^2 + y^2 + z^2) e^{-\sqrt{x^2 + y^2 + z^2}}$

Let $\vec{r} = (x, y, z)$
 $r = |\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} \Rightarrow \Psi(r) = r^2 e^{-r}$

NOW FIND THE GRADIENT OF GRADIENT
 $\nabla \left[\frac{1}{r} \right] = \hat{r} \left(-\frac{1}{r^2} \right)$

$\nabla [\Psi(r)] = \Psi'(r) \frac{\vec{r}}{r} = \left[2re^{-r} - r^2 e^{-r} \right] \frac{\vec{r}}{r}$

$= \left[2e^{-r} - r e^{-r} \right] \vec{r} = (2 - r) e^{-r} \vec{r}$

$\therefore \nabla \Psi(\vec{r}) = \left[2 - \sqrt{x^2 + y^2 + z^2} \right] e^{-\sqrt{x^2 + y^2 + z^2}} [x, y, z]$

Question 17

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show that

$$\nabla \left(9r^2 + \frac{4}{r} - 12\sqrt{r} \right) = 2 \left(3 - 2r^{-\frac{3}{2}} \right) \left(3 + r^{-\frac{3}{2}} \right) \mathbf{r}.$$

proof

$\nabla \left[9r^2 + \frac{4}{r} - 12\sqrt{r} \right] = \nabla \left[\frac{4}{r} \right]$ where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$
 By the chain rule
 $\nabla \left[\frac{4}{r} \right] = \frac{4}{r^2} \nabla r = \frac{4}{r^2} \left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right]$
 $= \frac{4}{r^2} \left[\frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right]$
 $= \frac{4}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} [x, y, z] = \frac{4}{r^3} \mathbf{r}$
 This $\nabla \left[9r^2 + \frac{4}{r} - 12\sqrt{r} \right] = \nabla \left[\frac{4}{r} \right]$
 $= 2[3r - 2r^{-\frac{3}{2}}] \left[3 + r^{-\frac{3}{2}} \right] \frac{\mathbf{r}}{r}$
 $= 2[3 - 2r^{-\frac{3}{2}}] \left[3 + r^{-\frac{3}{2}} \right] \mathbf{r}$

Question 18

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

The smooth function $f(r)$ satisfies

$$\nabla [f(r)] = 10r^3 \mathbf{r}.$$

Determine a simplified expression for $f(r)$.

$$f(r) = 2r^5 + C$$

Chain rule
 $\nabla [f(r)]$ where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$
 By the chain rule
 $\nabla [f(r)] = f'(r) \nabla r = f'(r) \left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right]$
 $= f'(r) \left[\frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right]$
 $= f'(r) \left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right] = \frac{1}{r} f'(r) [x, y, z] = \frac{f'(r)}{r} \mathbf{r}$
 Now $10r^3 \mathbf{r} = \frac{f'(r)}{r} \mathbf{r} \Rightarrow \dots \nabla (r^5)$
 $\therefore f(r) = 2r^5 + C$

DIVERGENCE

$$\operatorname{div} \mathbf{F} \equiv \nabla \cdot \mathbf{F}$$

Question 1

A Cartesian position vector is denoted by \mathbf{r} .

Determine the value of

$$\nabla \cdot \mathbf{r}.$$

$$\nabla \cdot \mathbf{r} = 3$$

$$\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, y, z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

Question 2

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{a} is a constant vector, find

$$\nabla \cdot (\mathbf{a} \wedge \mathbf{r}).$$

$$\nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = 0$$

$$\begin{aligned} \nabla \cdot (\mathbf{a} \wedge \mathbf{r}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \\ &= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{a} is a constant vector, show that

$$\nabla \cdot [(\mathbf{a} \cdot \mathbf{r})\mathbf{r}] = 4\mathbf{a} \cdot \mathbf{r}.$$

proof

Handwritten proof for Question 3:

$$\begin{aligned} \nabla \cdot (\mathbf{a} \cdot \mathbf{r})\mathbf{r} &= (\mathbf{a} \cdot \nabla) \nabla \cdot \mathbf{r} + \nabla (\mathbf{a} \cdot \mathbf{r}) \cdot \mathbf{r} \\ \nabla \cdot \mathbf{r} &= 3 \\ &= [a_1, a_2, a_3] \cdot [x, y, z] = a_1x + a_2y + a_3z \\ &= (a_1x + a_2y + a_3z) \times 3 + (a_1, a_2, a_3) \cdot (x, y, z) \\ &= 3(a_1x + a_2y + a_3z) + (a_1x + a_2y + a_3z) \\ &= 4(a_1x + a_2y + a_3z) \\ &= 4\mathbf{a} \cdot \mathbf{r} \end{aligned}$$

Question 4

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{a} is a constant vector, show that

$$\nabla \cdot [\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{a})] = 2\mathbf{a} \cdot \mathbf{r}.$$

proof

Handwritten proof for Question 4:

$$\begin{aligned} \nabla \cdot [\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{a})] &= (\mathbf{r} \wedge \nabla) \cdot \mathbf{r} - \mathbf{r} \cdot \nabla (\mathbf{r} \wedge \mathbf{a}) \\ \nabla \cdot (\mathbf{r} \wedge \mathbf{a}) &= \mathbf{r} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{r} \\ \nabla \cdot \mathbf{r} &= 3 \\ &= -\mathbf{r} \cdot \nabla (\mathbf{r} \wedge \mathbf{a}) = \dots \\ &= -\mathbf{r} \cdot (-2\mathbf{a}) \\ &= 2\mathbf{r} \cdot \mathbf{a} \\ &= 2\mathbf{a} \cdot \mathbf{r} \end{aligned}$$

Additional calculations shown in the image:

$$\begin{aligned} \mathbf{r} \wedge \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = (a_3y - a_2z, a_1z - a_3x, a_2x - a_1y) \\ \nabla (\mathbf{r} \wedge \mathbf{a}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix} \\ &= \begin{pmatrix} -a_1 - a_2 - a_3 & -a_2 - a_3 - a_1 & -a_3 - a_1 - a_2 \end{pmatrix} \\ &= (-2a_1, -2a_2, -2a_3) \\ &= -2\mathbf{a} \end{aligned}$$

Question 5

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3} \right) = 0.$$

proof

$\mathbf{r} = (x, y, z)$
 $|\mathbf{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$
 $\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right)$
 $= \frac{\partial}{\partial x} \left[\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] + \frac{\partial}{\partial y} \left[\frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] + \frac{\partial}{\partial z} \left[\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right]$
 \downarrow
 $\frac{(x^2 + y^2 + z^2)^{\frac{3}{2}}(1 - 3x^2) - 3x^2(x^2 + y^2 + z^2)^{\frac{3}{2}}}{(x^2 + y^2 + z^2)^3}$
 $= \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}}[(x^2 + y^2 + z^2) - 3x^2]}{(x^2 + y^2 + z^2)^3} = \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$
 As the expression is symmetrical
 $\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{2x^2 - y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{2x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = 0$
Alternative
 $\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} = \nabla \cdot \left[\frac{\mathbf{r}}{|\mathbf{r}|^3} \right] = (\nabla \cdot \mathbf{r}) \frac{1}{|\mathbf{r}|^3} + \mathbf{r} \cdot \nabla \left(\frac{1}{|\mathbf{r}|^3} \right)$
 $= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, y, z) \cdot \frac{1}{|\mathbf{r}|^3} + \mathbf{r} \cdot \left[-\frac{3}{|\mathbf{r}|^4} \nabla |\mathbf{r}| \right]$
 $= 3 \times \frac{1}{|\mathbf{r}|^3} - \frac{3}{|\mathbf{r}|^4} \mathbf{r} \cdot \nabla |\mathbf{r}|$
 $= \frac{3}{|\mathbf{r}|^3} - \frac{3}{|\mathbf{r}|^4} \mathbf{r} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$
 $= \frac{3}{|\mathbf{r}|^3} - \frac{3|\mathbf{r}|^2}{|\mathbf{r}|^5} = 0$
 Note: $\nabla |\mathbf{r}| = \frac{\mathbf{r}}{|\mathbf{r}|}$

Question 6

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{m} is a constant vector, show that

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{|\mathbf{r}|^3} \right) = 0.$$

, proof

FIRSTLY DEFINE THE COMPONENTS OF SOME VECTORS

- $\mathbf{m} = (m_1, m_2, m_3)$ CONSTANT VECTOR
- $\mathbf{r} = (x, y, z)$ CARTESIAN POSITION VECTOR

$\mathbf{m} \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ m_1 & m_2 & m_3 \\ x & y & z \end{vmatrix} = (m_2 z - m_3 y, m_3 x - m_1 z, m_1 y - m_2 x)$

PUTTING ALL THE RESULTS TOGETHER

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{r^3} \right) = \nabla \cdot \left[\frac{m_2 z - m_3 y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{m_3 x - m_1 z}{(x^2 + y^2 + z^2)^{3/2}}, \frac{m_1 y - m_2 x}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

APPLY THE DIVERGENCE CONCEPT

$$\begin{aligned} &= \frac{\partial}{\partial x} \left[\frac{m_2 z - m_3 y}{(x^2 + y^2 + z^2)^{3/2}} \right] + \frac{\partial}{\partial y} \left[\frac{m_3 x - m_1 z}{(x^2 + y^2 + z^2)^{3/2}} \right] + \frac{\partial}{\partial z} \left[\frac{m_1 y - m_2 x}{(x^2 + y^2 + z^2)^{3/2}} \right] \\ &= \frac{(m_2 z - m_3 y)(-\frac{3}{2})(2x)}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(m_3 x - m_1 z)(-\frac{3}{2})(2y)}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(m_1 y - m_2 x)(-\frac{3}{2})(2z)}{(x^2 + y^2 + z^2)^{5/2}} \\ &= \frac{3(x^2 + y^2 + z^2)^{-5/2} \left[-x(m_2 z - m_3 y) - y(m_3 x - m_1 z) - z(m_1 y - m_2 x) \right]}{(x^2 + y^2 + z^2)^{5/2}} \\ &= \frac{3(x^2 + y^2 + z^2)^{-5/2} \left[-m_2 x z + m_3 x y - m_3 x y + m_1 y z - m_1 y z + m_2 x z \right]}{(x^2 + y^2 + z^2)^{5/2}} \\ &= 0 \end{aligned}$$

As required

Question 7

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \right) = \frac{2}{|\mathbf{r}|}.$$

proof

• The unit vector of a position vector $\mathbf{r} = (x, y, z)$ is

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

• Finding the divergence of $\hat{\mathbf{r}}$ from first principles

$$\begin{aligned} \nabla \cdot \hat{\mathbf{r}} &= \nabla \cdot \left[\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right] \\ &= \left[\frac{\partial}{\partial x} \frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{\partial}{\partial y} \frac{y}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right] \\ &= \frac{\partial}{\partial x} \left[\frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right] + \frac{\partial}{\partial y} \left[\frac{y}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right] + \frac{\partial}{\partial z} \left[\frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \right] \\ &= \frac{(x^2 + y^2 + z^2)^{-\frac{1}{2}} - \frac{x^2(x^2 + y^2 + z^2)^{-\frac{3}{2}}}{2}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{(x^2 + y^2 + z^2)^{-\frac{1}{2}} - \frac{y^2(x^2 + y^2 + z^2)^{-\frac{3}{2}}}{2}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \\ &\quad + \frac{(x^2 + y^2 + z^2)^{-\frac{1}{2}} - \frac{z^2(x^2 + y^2 + z^2)^{-\frac{3}{2}}}{2}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \\ &= \frac{(x^2 + y^2 + z^2)^{-\frac{1}{2}}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{(x^2 + y^2 + z^2)^{-\frac{1}{2}}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{(x^2 + y^2 + z^2)^{-\frac{1}{2}}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \\ &= \frac{2(x^2 + y^2 + z^2)^{-\frac{1}{2}}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{2}{(x^2 + y^2 + z^2)} = \frac{2}{|\mathbf{r}|} \end{aligned}$$

Question 8

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla \cdot (\mathbf{r} |\mathbf{r}|^3) = 6 |\mathbf{r}|^3.$$

proof

$$\begin{aligned}\nabla \cdot (\vec{r}^4) &= \left(\frac{\partial}{\partial x} \cdot \frac{1}{3} \left(\frac{\partial}{\partial x} \right) \cdot \left[(x^2+y^2+z^2)^{\frac{3}{2}} (x, y, z) \right] \right. \\ &= \frac{1}{2\alpha} \left[2\alpha x y^2 + 2x^2 y \right] + \frac{1}{2\alpha} \left[2\alpha y x^2 + 2x^2 y \right] + \frac{1}{2\alpha} \left[2\alpha (x^2+y^2+z^2)^{\frac{3}{2}} \right] \\ &\quad \left(x^2+y^2+z^2 \right)^{\frac{3}{2}} + 2\alpha \cdot \frac{1}{2} \left(x^2+y^2+z^2 \right)^{\frac{3}{2}} \cdot 2\alpha \\ &= \left(x^2+y^2+z^2 \right)^{\frac{3}{2}} \left[2x y^2 + 2x^2 y + 2\alpha \right] \\ &= \left(x^2+y^2+z^2 \right)^{\frac{3}{2}} \left(2x^2+y^2+z^2 \right) \\ &= r \left(2r^2+y^2+z^2 \right)\end{aligned}$$

By Outer Symmetry we deduce

$$\begin{aligned}\nabla \cdot (\vec{r}^4) &= r \left(4x^2+y^2+z^2 \right) + r \left(x^2+4y^2+z^2 \right) + r \left(x^2+y^2+4z^2 \right) \\ &= r \left[6x^2+6y^2+6z^2 \right] \\ &= 6r \left(x^2+y^2+z^2 \right) \\ &= 6r \left(r^2 \right) \\ &= 6r^3\end{aligned}$$

Question 9

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show that

$$\nabla \cdot (r^n \mathbf{r}) = (n+3)r^n.$$

, proof

• START "EXPANDING" BY THE VECTOR CALCULUS IDENTITY

$$\nabla \cdot (\frac{1}{r} \mathbf{A}) = \frac{1}{r} \nabla \cdot \mathbf{A} + \nabla \frac{1}{r} \cdot \mathbf{A}$$

$$\Rightarrow \nabla \cdot (r^n \mathbf{r}) = r^n \nabla \cdot \mathbf{r} + \nabla r^n \cdot \mathbf{r}$$

$$= r^n \left[\frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right] \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) + \left[\frac{\partial}{\partial x} r^n \frac{\partial}{\partial x} + \frac{\partial}{\partial y} r^n \frac{\partial}{\partial y} + \frac{\partial}{\partial z} r^n \frac{\partial}{\partial z} \right] \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= r^n \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] + \left[2 \frac{\partial}{\partial x} r^n + 2 \frac{\partial}{\partial y} r^n + 2 \frac{\partial}{\partial z} r^n \right] \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

• NOW $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

$$r^n = (x^2 + y^2 + z^2)^{\frac{n}{2}}$$

$$\frac{\partial}{\partial x} r^n = \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{\frac{n}{2}} = \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} \cdot 2x = nx (x^2 + y^2 + z^2)^{\frac{n}{2} - 1}$$

AND SIMILARLY THE REST AS THESE EXPRESSIONS ARE SYMMETRICAL

• TRYING OF RATHER WE GET

$$\Rightarrow \nabla \cdot (r^n \mathbf{r}) = 3r^n + nx^2 (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} + ny^2 (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} + nz^2 (x^2 + y^2 + z^2)^{\frac{n}{2} - 1}$$

$$= 3r^n + n(x^2 + y^2 + z^2)^{\frac{n}{2} - 1} [x^2 + y^2 + z^2]$$

$$= 3r^n + n(x^2 + y^2 + z^2)^{\frac{n}{2}}$$

$$= 3r^n + nr^n$$

$$= (n+3)r^n \quad \text{as required}$$

Question 10

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla \cdot \left[|\mathbf{r}| \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \right] = \frac{3}{|\mathbf{r}|^4}.$$

proof

[illegible]

CURL

$$\text{curl} \mathbf{F} \equiv \nabla \wedge \mathbf{F}$$

Question 1

A Cartesian vector is denoted by

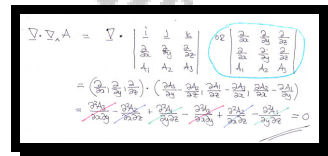
$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

where $A_i = f(x, y, z)$, $i = 1, 2, 3$

Given that A_i are differentiable functions, show that

$$\nabla \cdot \nabla \wedge \mathbf{A} = 0.$$

proof



Handwritten proof for Question 1:

$$\begin{aligned} \nabla \cdot \nabla \wedge \mathbf{A} &= \nabla \cdot \left(\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \right) \\ &= \left(\frac{\partial}{\partial x} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial x} \right) + \left(\frac{\partial}{\partial y} \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial y} \right) + \left(\frac{\partial}{\partial z} \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial z} \right) \\ &= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z^2} = 0 \end{aligned}$$

Question 2

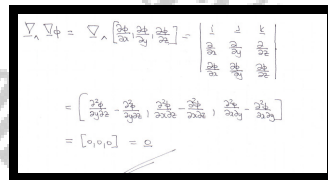
A function φ is denoted by

$$\varphi = \varphi(x, y, z).$$

Given that φ is differentiable show that

$$\nabla \wedge \nabla \varphi = \mathbf{0}.$$

proof



Handwritten proof for Question 2:

$$\begin{aligned} \nabla \wedge \nabla \varphi &= \nabla \wedge \left(\begin{bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{\partial^2 \varphi}{\partial y \partial x} - \frac{\partial^2 \varphi}{\partial x \partial y} \\ \frac{\partial^2 \varphi}{\partial z \partial y} - \frac{\partial^2 \varphi}{\partial y \partial z} \\ \frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \varphi}{\partial z \partial x} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Determine the value of

$$\nabla \wedge \mathbf{r}.$$

$$\nabla \wedge \mathbf{r} = \mathbf{0}$$

$$\nabla \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \begin{pmatrix} \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Question 4

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, find

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{r}).$$

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{r}) = 2\mathbf{a}$$

$$\begin{aligned} \nabla \wedge (\mathbf{a} \wedge \mathbf{r}) &= \nabla \wedge \left[\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \right] = \nabla \wedge \begin{bmatrix} a_1 z - a_3 y \\ a_3 x - a_1 z \\ a_2 x - a_3 y \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3 x - a_1 z & a_2 x - a_3 y & a_1 z - a_3 y \end{bmatrix} \\ &= \begin{bmatrix} a_1 - (-a_1) & a_2 - (-a_2) & a_3 - (-a_3) \end{bmatrix} = \begin{bmatrix} 2a_1 & 2a_2 & 2a_3 \end{bmatrix} \\ &= 2(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) = 2\mathbf{a} \end{aligned}$$

Question 5

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Determine

$$\nabla \wedge (\mathbf{r} + y^2 \mathbf{k}).$$

$$\nabla \wedge (\mathbf{r} + y^2 \mathbf{k}) = 2y \mathbf{i}$$

Handwritten solution for Question 5:

Let $\mathbf{r} = (x, y, z)$

Then $\nabla \wedge (\mathbf{r} + y^2 \mathbf{k}) = \nabla \wedge (x, y, z) + \nabla \wedge (0, y^2, 0)$

For the first term, $\nabla \wedge (x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0, 0, 0)$

For the second term, $\nabla \wedge (0, y^2, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & y^2 & 0 \end{vmatrix} = (0, 0, 0) + (2y, 0, 0) = 2y \mathbf{i}$

Question 6

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, show that

$$\nabla \wedge (r^2 \mathbf{a}) \equiv 2\mathbf{r} \wedge \mathbf{a}.$$

proof

Handwritten solution for Question 6:

$\nabla \wedge (r^2 \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)a_1 & (x^2+y^2+z^2)a_2 & (x^2+y^2+z^2)a_3 \end{vmatrix}$

$= [2y_1y_2 - 2a_1a_2, 2a_1a_2 - 2a_2a_1, 2a_2a_3 - 2a_3a_2]$

$= 2 [a_1a_2 - a_2a_1, a_1a_2 - a_2a_1, a_2a_3 - a_3a_2]$

$= 2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = 2 (x, y, z) \wedge (a_1, a_2, a_3)$

$= 2 \mathbf{r} \wedge \mathbf{a}$

As required

Question 7

A vector function \mathbf{A} is defined as

$$\mathbf{A} = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}.$$

Given that the standard Cartesian position vector is denoted by \mathbf{r} , show that

$$\nabla \cdot (\mathbf{A} \wedge \mathbf{r}) \equiv \mathbf{r} \cdot \nabla \wedge \mathbf{A}.$$

proof

$$\begin{aligned} \nabla \cdot (\underline{\hat{A}}, \underline{\hat{c}}) &= \text{divergence} \quad \underline{\hat{A}} = (A_1, A_2, A_3) \quad \text{where } A_i = A_i(x,y,z) \\ &= (x,y,z) \\ \nabla \cdot (\underline{\hat{A}}, \underline{\hat{c}}) &= \nabla \cdot \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{bmatrix} = \nabla \cdot [A_{1x} - A_{1y}, A_{2x} - A_{2y}, A_{3x} - A_{3y}] \\ &= \frac{\partial}{\partial x} [A_1 - A_2] + \frac{\partial}{\partial y} [A_2 - A_1] + \frac{\partial}{\partial z} [A_{1z} - A_{2z}] \\ &= \frac{\partial}{\partial x} \frac{\partial}{\partial x} A_1 - \frac{\partial}{\partial x} \frac{\partial}{\partial x} A_2 + \frac{\partial}{\partial y} \frac{\partial}{\partial y} A_2 - \frac{\partial}{\partial y} \frac{\partial}{\partial y} A_1 + \frac{\partial}{\partial z} \frac{\partial}{\partial z} A_1 - \frac{\partial}{\partial z} \frac{\partial}{\partial z} A_2 \\ &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} A_1 - \frac{\partial}{\partial x} A_2 \right) + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} A_2 - \frac{\partial}{\partial y} A_1 \right) + \frac{\partial}{\partial z} \left(\frac{\partial}{\partial z} A_1 - \frac{\partial}{\partial z} A_2 \right) \\ &= (x,y,z) \cdot \left[\frac{\partial}{\partial x} \frac{\partial}{\partial x} A_1 - \frac{\partial}{\partial x} \frac{\partial}{\partial x} A_2 + \frac{\partial}{\partial y} \frac{\partial}{\partial y} A_2 - \frac{\partial}{\partial y} \frac{\partial}{\partial y} A_1 + \frac{\partial}{\partial z} \frac{\partial}{\partial z} A_1 - \frac{\partial}{\partial z} \frac{\partial}{\partial z} A_2 \right] \\ &= \underline{\hat{c}} \cdot \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{bmatrix} = \underline{\hat{c}} \cdot \nabla \cdot \underline{\hat{A}} \end{aligned}$$

Question 8

The smooth vector functions, **A** and **B**, are both irrotational.

Show that $\mathbf{A} \wedge \mathbf{B}$ is solenoidal.

proof

• $\text{INTEGRAL} \Rightarrow \nabla_A A = 0$
 $\nabla_A B = 0$

• $\text{SCALAR} \Rightarrow \text{INVARIA} = 0$

Now $\nabla_A (A_a B^a) = \dots$ IDENTITY
 $= B^a (\underbrace{\nabla_A a}_0) - \underbrace{A_a (\nabla_A B^a)}_{0}$
 $= 0$

∴ SCALARIAL INVAR

Question 9

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the value of

$$\nabla \wedge \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right).$$

0

$$\begin{aligned} \nabla \wedge \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right) &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y} \left(\frac{z}{x^2+y^2+z^2} \right) - \frac{\partial}{\partial z} \left(\frac{y}{x^2+y^2+z^2} \right) \right] \mathbf{i} \\ &\quad + \left[\frac{\partial}{\partial z} \left(\frac{x}{x^2+y^2+z^2} \right) - \frac{\partial}{\partial x} \left(\frac{z}{x^2+y^2+z^2} \right) \right] \mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x} \left(\frac{y}{x^2+y^2+z^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2+z^2} \right) \right] \mathbf{k} \\ &= \left[\frac{\partial}{\partial y} \left[\frac{z(x^2+y^2+z^2)^{-1}} \right] - \frac{\partial}{\partial z} \left[\frac{y(x^2+y^2+z^2)^{-1}} \right] \right] \mathbf{i} \\ &\quad + \left[\frac{\partial}{\partial z} \left[\frac{x(x^2+y^2+z^2)^{-1}} \right] - \frac{\partial}{\partial x} \left[\frac{z(x^2+y^2+z^2)^{-1}} \right] \right] \mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x} \left[\frac{y(x^2+y^2+z^2)^{-1}} \right] - \frac{\partial}{\partial y} \left[\frac{x(x^2+y^2+z^2)^{-1}} \right] \right] \mathbf{k} \\ &= \left[-2yz(x^2+y^2+z^2)^{-2} - \left[-2yz(x^2+y^2+z^2)^{-2} \right] \right] \mathbf{i} \\ &\quad + \left[-2zx(x^2+y^2+z^2)^{-2} - \left[-2zx(x^2+y^2+z^2)^{-2} \right] \right] \mathbf{j} \\ &\quad + \left[-2xy(x^2+y^2+z^2)^{-2} - \left[-2xy(x^2+y^2+z^2)^{-2} \right] \right] \mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Question 10

If $\varphi = \varphi(x, y, z)$ is a smooth function, prove that.

$$\nabla \wedge (\varphi \nabla \varphi) = \mathbf{0}.$$

proof

$$\begin{aligned} \nabla \wedge (\varphi \nabla \varphi) &= \nabla \wedge (\varphi \Delta) \quad \text{where } \Delta = \nabla \varphi \\ &= \varphi (\nabla \wedge \Delta) + (\nabla \varphi \wedge \Delta) \\ &= \varphi [\nabla \wedge \nabla \varphi] + \nabla \varphi \wedge \nabla \varphi \\ &= \mathbf{0} \end{aligned}$$

$$\nabla \wedge [\varphi \nabla \varphi] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi \frac{\partial \varphi}{\partial x} & \varphi \frac{\partial \varphi}{\partial y} & \varphi \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i} \left[\frac{\partial}{\partial y} \left(\varphi \frac{\partial \varphi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\varphi \frac{\partial \varphi}{\partial y} \right) \right] + \mathbf{j} \left[\frac{\partial}{\partial z} \left(\varphi \frac{\partial \varphi}{\partial x} \right) - \frac{\partial}{\partial x} \left(\varphi \frac{\partial \varphi}{\partial z} \right) \right]$$

$$+ \mathbf{k} \left[\frac{\partial}{\partial x} \left(\varphi \frac{\partial \varphi}{\partial y} \right) - \frac{\partial}{\partial y} \left(\varphi \frac{\partial \varphi}{\partial x} \right) \right]$$

$$= \mathbf{i} \left[\varphi \frac{\partial^2 \varphi}{\partial y \partial z} + \frac{\partial \varphi}{\partial y} \frac{\partial \varphi}{\partial z} - \varphi \frac{\partial^2 \varphi}{\partial z \partial y} - \frac{\partial \varphi}{\partial z} \frac{\partial \varphi}{\partial y} \right]$$

$$+ \mathbf{j} \left[\varphi \frac{\partial^2 \varphi}{\partial z \partial x} + \frac{\partial \varphi}{\partial z} \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial z} \right]$$

$$+ \mathbf{k} \left[\varphi \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} - \varphi \frac{\partial^2 \varphi}{\partial y \partial x} - \frac{\partial \varphi}{\partial y} \frac{\partial \varphi}{\partial x} \right]$$

$$= \mathbf{0i} + \mathbf{0j} + \mathbf{0k}$$

Question 11

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Given that \mathbf{a} is a constant vector, find the value of

$$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}).$$

0

$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}) = \mathbf{0}$ As \mathbf{r} is a vector of AT MOST 1st order derivatives, so 2nd partial derivatives will be zero. EXCEPT IT TO ZERO

OR

$$\mathbf{r} \wedge \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{bmatrix} a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{bmatrix}$$

$$\nabla \wedge (\mathbf{r} \wedge \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$$= \begin{bmatrix} -a_1 - a_1 - a_2 - a_2 - a_3 - a_3 \\ -2a_1 - 2a_2 - 2a_3 \\ -2a_1 - 2a_2 - 2a_3 \end{bmatrix}$$

$$\nabla \wedge (\nabla \wedge (\mathbf{r} \wedge \mathbf{a})) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2a_1 - 2a_2 - 2a_3 \\ -2a_1 - 2a_2 - 2a_3 \\ -2a_1 - 2a_2 - 2a_3 \end{vmatrix} = \mathbf{0}$$

Question 12

The irrotational vector field \mathbf{F} is given by

$$\mathbf{F} = (8x + 8y + az)\mathbf{i} + (bx + 8y - 4z)\mathbf{j} + (-4x + cy + 2z)\mathbf{k},$$

where a , b and c are scalar constants.

Determine a smooth scalar function $\phi(x, y, z)$ such that

$$\nabla \phi = \mathbf{F}.$$

$$\phi(x, y, z) = (2x + 2y - x)^2 + \text{constant} = 4x^2 + 4y^2 + z^2 - 4xz - 4yz + 8xy + \text{constant}$$

Handwritten solution for Question 12:

$$\nabla \phi = \mathbf{F} = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix} = \begin{bmatrix} 8x + 8y + az \\ bx + 8y - 4z \\ -4x + cy + 2z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

If irrotational, $\nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \begin{cases} a = -4 \\ b = 8 \\ c = -4 \end{cases} \Rightarrow \nabla \phi = \mathbf{0}$

$\phi = \int \frac{\partial \phi}{\partial x} dx + \int \frac{\partial \phi}{\partial y} dy + \int \frac{\partial \phi}{\partial z} dz$

$$\frac{\partial \phi}{\partial x} = 8x + 8y - 4z \Rightarrow \phi(x, y, z) = 4x^2 + 8xy - 4xz + G_1(y, z)$$

$$\frac{\partial \phi}{\partial y} = 8x + 8y - 4z \Rightarrow \phi(x, y, z) = 4x^2 + 8xy - 4xz + G_2(z)$$

$$\frac{\partial \phi}{\partial z} = -4x - 4y + 2z \Rightarrow \phi(x, y, z) = 4x^2 + 8xy - 4xz + G_3(z)$$

$\therefore \phi(x, y, z) = 4x^2 + 8xy + z^2 - 4xz - 4yz + C$

Question 13

- a) Define the vector calculus operators grad, div and curl.

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

- b) Determine the vector

$$\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + \nabla (\sin e^{xyz})]$$

$$\boxed{}, \quad \nabla \wedge [\mathbf{r} \wedge \mathbf{i} + \nabla (\sin e^{xyz})] = -2\mathbf{i}$$

a) GRADIENT OF A SCALAR FUNCTION $\phi = \phi(x,y,z)$
 $\nabla \phi = \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right]$
DIVIDENCE OF A SMOOTH VECTOR FIELD $\mathbf{F} = (F_1, F_2, F_3)$
 $\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
CURL OF A SMOOTH VECTOR FIELD $\mathbf{F} = (F_1, F_2, F_3)$
 $\nabla \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

b) MANIPULATE AS USUAL
 $\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + \nabla (\sin e^{xyz})] = \nabla \wedge (\mathbf{r} \wedge \mathbf{i}) + \nabla \wedge \nabla (\sin e^{xyz})$
 $= \nabla \wedge [\mathbf{r} \wedge \mathbf{i}] + \nabla \wedge \nabla (\sin e^{xyz})$
 $\text{SINCE } \nabla \wedge \nabla \phi = 0$
 $= \nabla \wedge \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix}$
 $= \nabla \wedge (y\mathbf{j} - z\mathbf{k})$
 $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & y & -z \end{vmatrix}$
 $= (-1, 0, 0)$
 $= -\mathbf{i}$

Question 14

The smooth functions f and \mathbf{A} are defined as

$$f = f(x, y, z) \quad \text{and} \quad \mathbf{A} = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}.$$

- a) Define the vector calculus operators grad, div and curl with reference to f and \mathbf{A} .

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

- b) Determine the vector**

$$\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + (x + y)\mathbf{k}].$$

- c) Show that

$$\nabla \cdot \nabla \left(\frac{1}{r^2} \right) = \frac{2}{r^4}.$$

$$\boxed{}, \boxed{-\mathbf{i} - \mathbf{j}}$$

4) THE GRADIENT OF THE SCALAR FIELD $f = (x+y+z)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

THE DIVERGENCE OF THE VECTOR FIELD $A = [A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)]$

$$\nabla \cdot A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

THE CURL OF THE VECTOR FIELD $A = [A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)]$

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

6) STATE BY USING $\nabla \cdot (A+B) = \nabla \cdot A + \nabla \cdot B$

$$\nabla \cdot [x\hat{i} + (x+y)\hat{j}] = \nabla \cdot (x\hat{i}) + \nabla \cdot ((x+y)\hat{j})$$

$$= \nabla \cdot \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \nabla \cdot \begin{bmatrix} 0 \\ x+y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \nabla \cdot [0\hat{i}, x+y\hat{j}] + [1, 1, 0]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1, 1, 0)$$

$$= (-1, -1, 0) + (1, 1, 0)$$

$$= (-1, -1, 0)$$

$$= -\hat{i} - \hat{j}$$

4. SWITCH INTO CARBON COMPONENTS

$$\begin{aligned} \Sigma \cdot \nabla \left(\frac{1}{r} \right) &= \nabla^2 \left(\frac{1}{r} \right) = \nabla^2 \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right] \\ &= \frac{\partial^2}{\partial x^2} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right] + \frac{\partial^2}{\partial y^2} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right] + \frac{\partial^2}{\partial z^2} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right] \end{aligned}$$

THE EXPRESSION HAS ONLY SUMMATION OVER ONE OF THE DIMENSIONS

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left[\frac{1}{\sqrt{x^2+y^2+z^2}} \right] &= \frac{\partial}{\partial x} \left[-\frac{x}{(x^2+y^2+z^2)^{3/2}} \right] = \frac{\partial}{\partial x} \left[-\frac{2x}{(x^2+y^2+z^2)^{3/2}} \right] \\ &= \frac{(3x^2+y^2+z^2-2x) \cdot 2(x^2+y^2+z^2)^{5/2}}{(x^2+y^2+z^2)^6} \\ &= -\frac{2(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}} \\ &= -\frac{2(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{3x^2 - 2x^2 - 2y^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}} \\ &= \frac{3x^2 - 2y^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}} \end{aligned}$$

AND BY SUMMATION THE OTHER TWO COMPONENTS

$$\begin{aligned} \Sigma \cdot \nabla \left(\frac{1}{r} \right) &= \frac{3x^2 - 2y^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3y^2 - 2x^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3z^2 - 2x^2 - 2y^2}{(x^2+y^2+z^2)^{5/2}} \\ &= \frac{3x^2 + 3y^2 + 3z^2 - 2x^2 - 2y^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} \\ &= \frac{2}{r^3} \end{aligned}$$

IS ZERO

Question 15

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, show that

$$\nabla \wedge [(\mathbf{a} \cdot \mathbf{r})\mathbf{r}] = \mathbf{a} \wedge \mathbf{r}.$$

proof

$$\begin{aligned} \nabla \cdot [(\mathbf{a} \cdot \mathbf{r})\mathbf{r}] &= (\mathbf{a} \cdot \nabla) \nabla \cdot \mathbf{r} + \nabla(\mathbf{a} \cdot \mathbf{r}) \cdot \mathbf{r} = \dots \\ \nabla \cdot (\phi \mathbf{A}) &= \phi \nabla \cdot \mathbf{A} + \nabla \phi \cdot \mathbf{A} \\ \nabla \cdot \mathbf{r} &= \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3 \\ \therefore &= \nabla(\mathbf{a} \cdot \mathbf{r}) \cdot \mathbf{r} \\ &= \nabla(a_1x + a_2y + a_3z) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= \nabla(a_1x + a_2y + a_3z) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= \begin{bmatrix} \frac{\partial}{\partial x}(a_1x + a_2y + a_3z) & \frac{\partial}{\partial y}(a_1x + a_2y + a_3z) & \frac{\partial}{\partial z}(a_1x + a_2y + a_3z) \end{bmatrix} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= (a_1, a_2, a_3) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= \mathbf{a} \cdot \mathbf{r} \end{aligned}$$

Question 16

The smooth functions $\mathbf{A} = \mathbf{A}(x, y, z)$, $\mathbf{B} = \mathbf{B}(x, y, z)$ and $\phi = \phi(x, y, z)$ are defined as

$$\mathbf{A} = yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k}, \quad \mathbf{B} = x^2\mathbf{i} + yz\mathbf{j} - xy\mathbf{k} \quad \text{and} \quad \phi = xyz.$$

Find, in simplified form, expressions for

a) $(\mathbf{A} \cdot \nabla)\phi$.

b) $(\mathbf{B} \cdot \nabla)\mathbf{A}$.

c) $(\mathbf{A} \wedge \nabla)\phi$.

$$(\mathbf{A} \cdot \nabla)\phi = y^2z^2 - x^3yz + x^2yz^2,$$

$$(\mathbf{B} \cdot \nabla)\mathbf{A} = (yz^2 - xy^2)\mathbf{i} + (-2x^3y - x^2yz)\mathbf{j} + (x^2z^2 - 2x^2yz)\mathbf{k},$$

$$(\mathbf{A} \wedge \nabla)\phi = (-x^3y^2 - x^2z^3)\mathbf{i} + (xyz^3 - xy^2z)\mathbf{j} + (xyz^2 + x^2y^2z)\mathbf{k}$$

$$\begin{aligned} \mathbf{A} &= (yz, -x^2y, xz^2) \quad \mathbf{B} = (x^2, yz, -xy) \quad \phi = xyz \\ \text{a) } (\mathbf{A} \cdot \nabla)\phi &= (yz, -x^2y, xz^2) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (xyz) \\ &= yz \frac{\partial}{\partial x} (xyz) - x^2y \frac{\partial}{\partial y} (xyz) + xz^2 \frac{\partial}{\partial z} (xyz) \\ &= yz \frac{\partial}{\partial x} (xyz) - x^2y \frac{\partial}{\partial y} (xyz) + xz^2 \frac{\partial}{\partial z} (xyz) \\ &= yz^2 - x^3yz + x^2yz^2 \\ &= yz^2 - x^3yz + x^2yz^2 \\ \text{b) } (\mathbf{B} \cdot \nabla)\mathbf{A} &= (x^2, yz, -xy) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (yz, -x^2y, xz^2) \\ &= \left[x^2 \frac{\partial}{\partial x} (yz) + yz \frac{\partial}{\partial y} (yz) - xy \frac{\partial}{\partial z} (yz) \right] \mathbf{i} \\ &\quad + \left[x^2 \frac{\partial}{\partial x} (-x^2y) + yz \frac{\partial}{\partial y} (-x^2y) - xy \frac{\partial}{\partial z} (-x^2y) \right] \mathbf{j} \\ &\quad + \left[x^2 \frac{\partial}{\partial x} (xz^2) + yz \frac{\partial}{\partial y} (xz^2) - xy \frac{\partial}{\partial z} (xz^2) \right] \mathbf{k} \\ &= [yz^2 - x^3yz + x^2yz^2] \mathbf{i} + [-2x^3y - x^2yz] \mathbf{j} + [x^2z^2 - 2x^2yz] \mathbf{k} \\ \text{c) } (\mathbf{A} \wedge \nabla)\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} (xyz) \\ &= \begin{vmatrix} yz \frac{\partial}{\partial x} - x^2y \frac{\partial}{\partial y} + xz^2 \frac{\partial}{\partial z} & yz \frac{\partial}{\partial x} - x^2y \frac{\partial}{\partial y} + xz^2 \frac{\partial}{\partial z} & yz \frac{\partial}{\partial x} - x^2y \frac{\partial}{\partial y} + xz^2 \frac{\partial}{\partial z} \end{vmatrix} (xyz) \\ &= \begin{vmatrix} -x^3y^2 - x^2z^3 & xyz^3 - xy^2z & xyz^2 + x^2y^2z \end{vmatrix} \\ &= (-x^3y^2 - x^2z^3)\mathbf{i} + (xyz^3 - xy^2z)\mathbf{j} + (xyz^2 + x^2y^2z)\mathbf{k} \end{aligned}$$

Question 17

a) Define the vector calculus operators grad, div and curl.

b) Given that $\varphi(x, y, z)$ and $\psi(x, y, z)$ are smooth functions, show that

$$\nabla \wedge [\varphi \nabla \psi] \equiv \nabla \varphi \wedge \nabla \psi.$$

c) Evaluate

$$\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left((x+y+z)^3 \mathbf{i} + (4x^3 - yz) \mathbf{j} + (xyz) \mathbf{k} \right) \right] \right] \right].$$

d) Use the vector function

$$\mathbf{A}(x, y, z) = (e^{x+y}) \mathbf{i} + (x \sin z) \mathbf{j} + (4\sqrt{x}) \mathbf{k}$$

to verify the validity of the identity

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

$$\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left((x+y+z)^3 \mathbf{i} + (4x^3 - yz) \mathbf{j} + (xyz) \mathbf{k} \right) \right] \right] \right] = \mathbf{0}$$

The image shows two pages of handwritten mathematical work. The left page contains the definition of the curl operator and the evaluation of the vector function \mathbf{A} for part (c). The right page contains the verification of the identity $\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ for part (d).

Left Page:

a) If $\varphi(x, y, z)$ is a smooth scalar field, then $\text{grad } \varphi = \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$.

If $\mathbf{F} = [F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)]$ is a smooth vector field, then $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

Then $\text{curl } \mathbf{F} = \nabla \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$.

b) $\nabla \wedge [\varphi \nabla \psi] = \nabla \wedge \left[\varphi \left(\frac{\partial \psi}{\partial x} \mathbf{i} + \frac{\partial \psi}{\partial y} \mathbf{j} + \frac{\partial \psi}{\partial z} \mathbf{k} \right) \right] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi \frac{\partial \psi}{\partial x} & \varphi \frac{\partial \psi}{\partial y} & \varphi \frac{\partial \psi}{\partial z} \end{vmatrix}$.

Using the product rule for differentiation, we get:

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi \frac{\partial \psi}{\partial x} & \varphi \frac{\partial \psi}{\partial y} & \varphi \frac{\partial \psi}{\partial z} \end{vmatrix} = \nabla \varphi \wedge \nabla \psi.$$

Right Page:

d) $\nabla \wedge [\nabla \wedge \mathbf{A}] = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

Let $\mathbf{A} = (e^{x+y}) \mathbf{i} + (x \sin z) \mathbf{j} + (4\sqrt{x}) \mathbf{k}$.

First, calculate $\nabla \cdot \mathbf{A}$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} (e^{x+y}) + \frac{\partial}{\partial y} (e^{x+y}) + \frac{\partial}{\partial z} (x \sin z) = e^{x+y} + e^{x+y} + x \cos z = 2e^{x+y} + x \cos z.$$

Next, calculate $\nabla^2 \mathbf{A}$:

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) = \nabla (2e^{x+y} + x \cos z) = \left(\frac{\partial}{\partial x} (2e^{x+y} + x \cos z), \frac{\partial}{\partial y} (2e^{x+y} + x \cos z), \frac{\partial}{\partial z} (2e^{x+y} + x \cos z) \right) = (2e^{x+y} + \cos z, 2e^{x+y}, -x \sin z) \mathbf{i} + (2e^{x+y}, 2e^{x+y} + \cos z, -x \sin z) \mathbf{j} + (2e^{x+y}, -x \sin z, -x \sin z) \mathbf{k}.$$

Finally, calculate $\nabla \wedge [\nabla \wedge \mathbf{A}]$:

$$\nabla \wedge [\nabla \wedge \mathbf{A}] = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = (2e^{x+y} + x \cos z) \mathbf{i} + (2e^{x+y} + x \cos z) \mathbf{j} + (2e^{x+y} + x \cos z) \mathbf{k} - (2e^{x+y} + \cos z) \mathbf{i} - (2e^{x+y}, 2e^{x+y} + \cos z, -x \sin z) \mathbf{j} - (2e^{x+y}, -x \sin z, -x \sin z) \mathbf{k} = \mathbf{0}.$$

Question 19

Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ is a twice differentiable vector function, show that

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

proof

$$\begin{aligned} \nabla_r (\nabla_r A) &= \nabla (\nabla \cdot A) - \nabla^2 A \\ \nabla_r (\nabla_r A) &= \nabla_r \left[\frac{i}{\partial_1} \frac{j}{\partial_2} \frac{k}{\partial_3} \right] = \nabla_r \left[\frac{\partial_1}{\partial_1} \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} \frac{\partial_1}{\partial_1} \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} \right] \\ &= \begin{bmatrix} \frac{i}{\partial_1} & \frac{j}{\partial_2} & \frac{k}{\partial_3} \\ \frac{\partial_1}{\partial_1} & \frac{\partial_1}{\partial_2} & \frac{\partial_1}{\partial_3} \\ \frac{\partial_2}{\partial_1} & \frac{\partial_2}{\partial_2} & \frac{\partial_2}{\partial_3} \\ \frac{\partial_3}{\partial_1} & \frac{\partial_3}{\partial_2} & \frac{\partial_3}{\partial_3} \end{bmatrix} \\ &= \left(\frac{\partial_1^2}{\partial_1 \partial_1} \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} - \frac{\partial_1^2}{\partial_1 \partial_2} \frac{\partial_1}{\partial_3} \right) i \\ &\quad + \left(\frac{\partial_1^2}{\partial_2 \partial_1} \frac{\partial_1}{\partial_3} - \frac{\partial_1^2}{\partial_2 \partial_2} \frac{\partial_1}{\partial_3} \right) j \\ &\quad + \left(\frac{\partial_1^2}{\partial_3 \partial_1} \frac{\partial_1}{\partial_3} - \frac{\partial_1^2}{\partial_3 \partial_2} \frac{\partial_1}{\partial_3} \right) k \\ &= \left(\frac{\partial_1^2}{\partial_1^2} - \frac{\partial_1^2}{\partial_2^2} \right) A_1 i + \left(\frac{\partial_1^2}{\partial_2^2} - \frac{\partial_1^2}{\partial_3^2} \right) A_2 j + \left(\frac{\partial_1^2}{\partial_3^2} - \frac{\partial_1^2}{\partial_1^2} \right) A_3 k \\ &= -\nabla^2 A + \nabla (\nabla \cdot A) \end{aligned}$$

At Equiv 10

ALTERNATIVE

$$\begin{aligned} \nabla_r (\nabla_r A) &= \frac{\partial_1}{\partial_1} \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} - \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} \\ \text{let } A &= \nabla \\ B &= \nabla \\ C &= A \\ \nabla_r (\nabla_r A) &= \frac{\partial_1}{\partial_1} \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} - \frac{\partial_1}{\partial_2} \frac{\partial_1}{\partial_3} \\ \therefore \nabla_r (\nabla_r A) &= \nabla (\nabla \cdot A) - \nabla^2 A \end{aligned}$$

At Equiv 10

Question 20

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show that $\mathbf{r} f(|\mathbf{r}|)$ is irrotational, where $f(|\mathbf{r}|)$ is differentiable function of $|\mathbf{r}|$.

proof

$\mathbf{F} = r f(r), f(r) \text{ is differentiable}$
 $\nabla \wedge \mathbf{F} = \nabla \wedge (f \mathbf{r}) = \dots$ (identity)
 $\nabla \wedge (f \mathbf{r}) = \nabla f \wedge \mathbf{r} + f (\nabla \wedge \mathbf{r})$
 $= \nabla f \wedge \mathbf{r} + f (\nabla \wedge \mathbf{r})$
 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{1}{r} f'(r), \frac{y}{r^2} f'(r), \frac{z}{r^2} f'(r) \right)$
 $\mathbf{r} = (x, y, z)$
 $\nabla f \wedge \mathbf{r} = \begin{vmatrix} \frac{1}{r} f'(r) & \frac{y}{r^2} f'(r) & \frac{z}{r^2} f'(r) \\ x & y & z \end{vmatrix}$
 $= \left(\frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \right) \wedge (x, y, z)$
 $\dots = \left(\frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \right) \wedge (x, y, z)$
 $= \frac{f'(r)}{r^2} \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \right) \wedge (x, y, z)$
 $= 0$
 $\therefore \text{irrotational}$

Question 21

The vector function \mathbf{E} satisfies

$$\mathbf{E} = \frac{\mathbf{r}}{|\mathbf{r}|^2}, \quad |\mathbf{r}| \neq 0,$$

where $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Show that \mathbf{E} is irrotational, and find a smooth scalar function $\phi(|\mathbf{r}|)$, with $\phi(k) = 0$,

so that $\mathbf{E} = -\nabla\phi(|\mathbf{r}|)$.

$$\phi(|\mathbf{r}|) = \ln\left(\frac{k}{|\mathbf{r}|}\right)$$

Handwritten solution for Question 21:

$$\mathbf{E} = \frac{\mathbf{r}}{r^2} = \frac{(x, y, z)}{x^2 + y^2 + z^2} = \left[\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right]$$

• $\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2 + z^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{x^2 + y^2 + z^2} \right)$

$$= \frac{1}{x^2 + y^2 + z^2} - \frac{2x^2}{(x^2 + y^2 + z^2)^2} + \frac{1}{x^2 + y^2 + z^2} - \frac{2y^2}{(x^2 + y^2 + z^2)^2} + \frac{1}{x^2 + y^2 + z^2} - \frac{2z^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{3}{x^2 + y^2 + z^2} - \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^2} = \frac{3}{x^2 + y^2 + z^2} - \frac{2}{x^2 + y^2 + z^2} = \frac{1}{x^2 + y^2 + z^2}$$

• $\mathbf{E} = -\nabla\phi$

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \left(-\frac{x}{x^2 + y^2 + z^2}, -\frac{y}{x^2 + y^2 + z^2}, -\frac{z}{x^2 + y^2 + z^2} \right)$$

• $\frac{\partial \phi}{\partial x} = -\frac{x}{x^2 + y^2 + z^2}$, $\frac{\partial \phi}{\partial y} = -\frac{y}{x^2 + y^2 + z^2}$, $\frac{\partial \phi}{\partial z} = -\frac{z}{x^2 + y^2 + z^2}$

$$\begin{cases} \phi = -\frac{1}{2} \ln(x^2 + y^2 + z^2) + f(y, z) \\ \phi = -\frac{1}{2} \ln(x^2 + y^2 + z^2) + g(x, z) \\ \phi = -\frac{1}{2} \ln(x^2 + y^2 + z^2) + h(x, y) \end{cases} \Rightarrow f(y, z) = g(x, z) = h(x, y) = \text{constant}$$

• $\therefore \phi(x, y, z) = \ln\left(\frac{1}{x^2 + y^2 + z^2}\right) + \text{constant} = \ln\left(\frac{1}{r^2}\right) + \text{constant}$

$$\phi(r) = \ln\left(\frac{1}{r}\right) + \text{constant}$$

$\therefore \phi(k) = 0 \Rightarrow 0 = \ln\left(\frac{1}{k}\right) + \text{constant}$

$$\text{constant} = -\ln k$$

$\therefore \phi(r) = \ln\left(\frac{1}{r}\right) - \ln k = \ln\left(\frac{1}{rk}\right)$

Question 22

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, show that

$$\nabla \wedge \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \frac{3\mathbf{a} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{a}}{r^3}.$$

proof

[illegible]

$$\begin{aligned}
 &= \frac{1}{(\sqrt{2x^2+2})^2} \left[4x^2y + a_2 2x - a_1 (\sqrt{2x^2+2})^2, a_1 2xy + a_2 2x - a_1 (\sqrt{2x^2+2})^2, a_1 2x^2 + a_2 2x - a_1 (\sqrt{2x^2+2})^2 \right] \\
 &\textcircled{1} \quad \Delta \left(\sqrt{\frac{x}{y}}, \frac{y}{x} \right) = \Delta \left(\frac{\sqrt{\frac{x}{y}}}{\sqrt{\frac{x}{y}}}, \frac{\frac{y}{x}}{\sqrt{\frac{x}{y}}} \right) = \frac{\sqrt{\frac{x}{y}}}{(\sqrt{\frac{x}{y}})^2} - \frac{\frac{y}{x}}{(\sqrt{\frac{x}{y}})^2} \quad \leftarrow \text{wrong not needed} \\
 &= \Delta \left(\frac{\sqrt{2x^2+2}}{(\sqrt{2x^2+2})^2} + \frac{x^2y^2-2x^2}{(\sqrt{2x^2+2})^2} + \frac{y^2x^2-2x^2}{(\sqrt{2x^2+2})^2} \right) \\
 &= 0 \\
 &\textcircled{2} \quad \nabla_x \left(g, \frac{1}{f} \right) = \left(\frac{g_1(3xy)}{(\sqrt{2x^2+2})^2} - a_1 (\sqrt{2x^2+2}), \frac{a_1 2xy + a_2 2x + 3ay^2 - a_1 (\sqrt{2x^2+2})^2}{(\sqrt{2x^2+2})^2}, a_1 2x^2 + a_2 2x + a_1 y^2 - a_1 (\sqrt{2x^2+2})^2 \right) \\
 &= \left[\frac{(a_1 x + 3ay + 3ay^2)}{(\sqrt{2x^2+2})^2} - \frac{a_1}{(\sqrt{2x^2+2})^2}, \frac{(a_1 x + 3ay + 3ay^2)}{(\sqrt{2x^2+2})^2} - \frac{a_1}{(\sqrt{2x^2+2})^2}, \frac{(a_1 x + 3ay + 3ay^2)}{(\sqrt{2x^2+2})^2} - \frac{a_1}{(\sqrt{2x^2+2})^2} \right] \\
 &= \left[\frac{3a_1}{\sqrt{2x^2+2}} - \frac{a_1}{\sqrt{2x^2+2}}, \frac{3a_1}{\sqrt{2x^2+2}} - \frac{a_1}{\sqrt{2x^2+2}}, \frac{3a_1}{\sqrt{2x^2+2}} - \frac{a_1}{\sqrt{2x^2+2}} \right] \\
 &= \left[\frac{3a_1 - a_1}{\sqrt{2x^2+2}}, \frac{3a_1 - a_1}{\sqrt{2x^2+2}}, \frac{3a_1 - a_1}{\sqrt{2x^2+2}} \right] = \left[\frac{2a_1}{\sqrt{2x^2+2}}, \frac{2a_1}{\sqrt{2x^2+2}}, \frac{2a_1}{\sqrt{2x^2+2}} \right] \\
 &= \frac{2a_1}{\sqrt{2x^2+2}} (1, 1, 1) = \frac{1}{\sqrt{2}} (a_1, a_1, a_1) \\
 &= \frac{3a_1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{3a_1}{2}
 \end{aligned}$$

$$\Sigma \left(\frac{a}{b}, \frac{c}{d} \right) = \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

ACTIVITE

$$\frac{a}{b} = (a_1, a_2, a_3) \Rightarrow \frac{a}{b} \cdot \frac{c}{d} = \left| \begin{array}{ccc} \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{array} \right| = \begin{pmatrix} \frac{a_2 c_3 - a_3 c_2}{(a_1 c_1)^2 c_3} & \frac{a_3 c_1 - a_1 c_3}{(a_1 c_1)^2 c_3} & \frac{a_1 c_2 - a_2 c_1}{(a_1 c_1)^2 c_3} \end{pmatrix}$$

$$\left| \begin{array}{ccc} \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{array} \right| = \dots \text{WONNE (COMPARAISON DE MATRICES)}$$

1)
$$\frac{2}{3} \left(\frac{a_2 c_3 - a_3 c_2}{(a_1 c_1)^2 c_3} \right) - \frac{2}{3} \left(\frac{a_3 c_1 - a_1 c_3}{(a_1 c_1)^2 c_3} \right) = \frac{(a_2 c_3 - a_3 c_2)(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} - \frac{(a_3 c_1 - a_1 c_3)(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2}$$

$$= \frac{a_2(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} - \frac{a_3(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} + \frac{a_1(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} - \frac{a_1(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} = \frac{a_2(a_1 c_1)^2 c_3 - a_3(a_1 c_1)^2 c_3 + a_1(a_1 c_1)^2 c_3 - a_1(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2}$$

$$= \frac{a_2(a_1 c_1)^2 c_3 - a_3(a_1 c_1)^2 c_3 + a_1(a_1 c_1)^2 c_3 - a_1(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2}$$

2)
$$\frac{2}{3} \left(\frac{a_3 c_1 - a_1 c_3}{(a_1 c_1)^2 c_3} \right) - \frac{2}{3} \left(\frac{a_1 c_2 - a_2 c_1}{(a_1 c_1)^2 c_3} \right) = \frac{(a_3 c_1 - a_1 c_3)(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} - \frac{(a_1 c_2 - a_2 c_1)(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2}$$

$$= \frac{a_3(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} - \frac{a_1(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} - \frac{a_1(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2} + \frac{a_2(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2}$$

$$= \frac{a_3(a_1 c_1)^2 c_3 - a_1(a_1 c_1)^2 c_3 - a_1(a_1 c_1)^2 c_3 + a_2(a_1 c_1)^2 c_3}{(a_1 c_1)^4 c_3^2}$$

[illegible]

Question 23

$$\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) \equiv (\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} - (\nabla \cdot \mathbf{A}) \mathbf{B}.$$

- a) Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ are smooth vector functions, use index summation notation to prove the validity of the above vector identity.
- b) Verify the validity of the vector identity if

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \mathbf{B} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

both sides yield $2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$

q) FIND THE 1^{st} COMPONENT OF $A \cdot B$

$$(A \cdot B)_k = \varepsilon_{ijk} A_i B_j$$

• NEXT THE 4^{th} COMPONENT OF $\Sigma_i (A \cdot B)_i$

$$\begin{aligned} [\Sigma_i (A \cdot B)_i]_4 &= \varepsilon_{i4k} \frac{\partial}{\partial x_i} [\varepsilon_{ijk} A_i B_j] \\ &= \varepsilon_{i4k} \varepsilon_{ijk} \frac{\partial}{\partial x_i} [A_i B_j] \\ &= -\varepsilon_{i4k} \varepsilon_{ijk} \frac{\partial}{\partial x_i} [A_i B_j] \\ &= - \begin{vmatrix} \partial_i & \partial_j & \partial_k \\ \partial_i & \partial_j & \partial_k \end{vmatrix} \varepsilon_{ijk} [A_i B_j] \\ &= (\partial_i \partial_j - \partial_j \partial_i) \frac{\partial}{\partial x_i} [A_i B_j] \\ &= \partial_i \partial_j \frac{\partial}{\partial x_i} [A_i B_j] - \partial_j \partial_i \frac{\partial}{\partial x_i} [A_i B_j] \end{aligned}$$

... USING THE SYMMETRY PROPERTY OF ε ...

$$= \frac{\partial^2}{\partial x_i^2} [A_i B_j] - \frac{\partial^2}{\partial x_j^2} [A_i B_j]$$

... BY PRODUCT RULE ...

$$= \frac{\partial^2}{\partial x_i^2} B_j + \frac{\partial^2}{\partial x_j^2} A_i - \frac{\partial^2}{\partial x_i^2} B_j - \frac{\partial^2}{\partial x_j^2} A_i$$

... PRODUCT OF IDENTICAL TENSORS VANISHES ...

$$\begin{aligned} &= \partial_i \partial_j A_i + A_i \frac{\partial^2}{\partial x_i^2} - \partial_j \frac{\partial^2}{\partial x_i^2} A_i - \frac{\partial^2}{\partial x_j^2} B_j \\ &= (\partial_i \partial_j A + A (\partial_i \partial_j) - (\partial_j \partial_i A + (\partial_j \partial_i) B))_k \end{aligned}$$

• THIS

$$\Sigma_i (A \cdot B)_i = (\partial_i \partial_j A + A (\partial_i \partial_j) - (\partial_j \partial_i A + (\partial_j \partial_i) B))_k$$

[illegible]

LAPLACIAN

$$\nabla \cdot \nabla \varphi \equiv \nabla^2 \varphi$$

Question 1

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Determine the value of

$$\operatorname{div} \left[\operatorname{grad} \left(\frac{1}{r} \right) \right], \quad r \neq 0.$$

$$\operatorname{div} \left[\operatorname{grad} \left(\frac{1}{r} \right) \right] = 0$$

$$\nabla \cdot \nabla \left(\frac{1}{r} \right) = \nabla^2 \left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right) = \nabla^2 \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right]$$

$$= \frac{\partial^2}{\partial x^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] + \frac{\partial^2}{\partial y^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] + \frac{\partial^2}{\partial z^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right]$$
 Look at one of the components

$$\frac{\partial^2}{\partial x^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] = \frac{\partial}{\partial x} \left[-\frac{1}{2} (x^2+y^2+z^2)^{-\frac{3}{2}} (2x) \right]$$

$$= \frac{\partial}{\partial x} \left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right]$$

$$= -\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - x \cdot \frac{3}{2} (x^2+y^2+z^2)^{\frac{1}{2}} (2x)}{(x^2+y^2+z^2)^3}$$

$$= -\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2 (x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= -\frac{(x^2+y^2+z^2)^{\frac{1}{2}} (x^2+y^2+z^2 - 3x^2)}{(x^2+y^2+z^2)^3}$$

$$= -\frac{(x^2+y^2+z^2 - 3x^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$
 As expression is symmetric

$$\Rightarrow \nabla \cdot \nabla \left(\frac{1}{r} \right) = \frac{x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\Rightarrow \nabla \cdot \nabla \left(\frac{1}{r} \right) = 0$$

Question 2

A smooth scalar field is denoted by $\varphi = \varphi(x, y, z)$ and a smooth vector field is denoted by $\mathbf{A} = \mathbf{A}(x, y, z)$.

- a) Use the standard definitions of vector operators to show that

$$\nabla \cdot (\varphi \mathbf{A}) = \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A}.$$

- b) Given further that f and g are functions of x , y and z , whose second partial derivatives exist, deduce that

$$\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f.$$

NOTES, proof

a) DEFINE SOME QUANTITIES FIRST
 $\mathbf{A} = (A_1(x,y,z), A_2(x,y,z), A_3(x,y,z))$ $\varphi = \varphi(x,y,z)$

THEN WE HAVE
 $\nabla \cdot (\varphi \mathbf{A}) = \nabla \cdot (\varphi A_1, \varphi A_2, \varphi A_3)$
 $= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (\varphi A_1, \varphi A_2, \varphi A_3)$

BY THE PRODUCT RULE
 $= \frac{\partial}{\partial x}(\varphi A_1) + \frac{\partial}{\partial y}(\varphi A_2) + \frac{\partial}{\partial z}(\varphi A_3)$
 $= \frac{\partial \varphi}{\partial x} A_1 + \varphi \frac{\partial A_1}{\partial x} + \frac{\partial \varphi}{\partial y} A_2 + \varphi \frac{\partial A_2}{\partial y} + \frac{\partial \varphi}{\partial z} A_3 + \varphi \frac{\partial A_3}{\partial z}$
 $= \left[\frac{\partial \varphi}{\partial x} A_1 + \frac{\partial \varphi}{\partial y} A_2 + \frac{\partial \varphi}{\partial z} A_3 \right] + \varphi \left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right]$
 $= \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) \cdot (A_1, A_2, A_3) + \varphi \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right)$
 $= \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A}$ AS REQUIRED

b) PROCEED USING THE RESULT OF PART (a)
 $\nabla \cdot [f \nabla g - g \nabla f] = \nabla \cdot [f \nabla g] - \nabla \cdot [g \nabla f]$
 $= \nabla \cdot (f \nabla g) + f \nabla \cdot \nabla g - \nabla g \cdot \nabla f - g \nabla \cdot \nabla f$
 $= \nabla \cdot (f \nabla g) + f \nabla^2 g - \nabla g \cdot \nabla f - g \nabla^2 f$
 $= \nabla \cdot (f \nabla g) - g \nabla^2 f$ AS REQUIRED

Question 3

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$, $r \neq 0$.

Show that

$$\nabla^2 (\ln r) = \frac{1}{r^2}.$$

proof

$$\begin{aligned} \nabla^2 (\ln r) &= \nabla \cdot \nabla (\ln r) = \nabla \cdot \left(\frac{\partial}{\partial x} \ln r, \frac{\partial}{\partial y} \ln r, \frac{\partial}{\partial z} \ln r \right) = \dots \\ \text{Now } \frac{\partial (\ln r)}{\partial x} &= \frac{\partial}{\partial x} \left[\ln(x^2 + y^2 + z^2)^{\frac{1}{2}} \right] = \frac{\partial}{\partial x} \left[\frac{1}{2} \ln(x^2 + y^2 + z^2) \right] \\ &= \frac{1}{2} \times \frac{1}{x^2 + y^2 + z^2} \times 2x = \frac{x}{x^2 + y^2 + z^2} = \frac{x}{r^2} \\ \text{Since the expression is cyclic symmetric} \\ \frac{\partial}{\partial x} (\ln r) &= \frac{x}{r^2}, \quad \frac{\partial}{\partial y} (\ln r) = \frac{y}{r^2}, \quad \frac{\partial}{\partial z} (\ln r) = \frac{z}{r^2} \\ \therefore \nabla \cdot \left[\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2} \right] &= \left[\frac{\partial}{\partial x} \left(\frac{x}{r^2} \right), \frac{\partial}{\partial y} \left(\frac{y}{r^2} \right), \frac{\partial}{\partial z} \left(\frac{z}{r^2} \right) \right] \\ &= \frac{\partial}{\partial x} \left(\frac{x}{r^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^2} \right) = \dots \\ \frac{\partial}{\partial x} \left(\frac{x}{r^2} \right) &= \frac{\partial}{\partial x} \left[\frac{x}{x^2 + y^2 + z^2} \right] = \frac{(x^2 + y^2 + z^2) \times 1 - x(2x)}{(x^2 + y^2 + z^2)^2} \\ &= \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2} = \frac{y^2 + z^2 - x^2}{r^4} \\ \text{Summing the 'other two' by cyclic symmetry} \\ &= \frac{y^2 + z^2 - x^2}{r^4} + \frac{x^2 + z^2 - y^2}{r^4} + \frac{x^2 + y^2 - z^2}{r^4} \\ &= \frac{x^2 + y^2 + z^2}{r^4} = \frac{r^2}{r^4} = \frac{1}{r^2} \quad \text{As required} \end{aligned}$$

Question 4

It is given that $\phi = \phi(x, y, z)$ and $\psi = \psi(x, y, z)$ are twice differentiable functions.

Show, with a detailed method, that

$$\nabla^2(\phi\psi) = \psi\nabla^2\phi + 2\nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi.$$

proof

The handwritten proof shows the following steps:

$$\begin{aligned} \nabla^2(\phi\psi) &= \frac{\partial^2}{\partial x^2}(\phi\psi) + \frac{\partial^2}{\partial y^2}(\phi\psi) + \frac{\partial^2}{\partial z^2}(\phi\psi) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \psi + \phi \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \psi + \phi \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \psi + \phi \frac{\partial \psi}{\partial z} \right) \\ &= \frac{\partial^2 \phi}{\partial x^2} \psi + \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \phi \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \psi + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} + \phi \frac{\partial^2 \psi}{\partial y^2} \\ &\quad + \frac{\partial^2 \phi}{\partial z^2} \psi + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z} + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z} + \phi \frac{\partial^2 \psi}{\partial z^2} \\ &= \psi \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right] + 2 \left[\frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z} \right] + \phi \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right] \\ &= \psi \nabla^2 \phi + 2 \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi \end{aligned}$$

Question 5

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla^2(|\mathbf{r}|^n) = n(n+1)|\mathbf{r}|^{n-2}.$$

proof

Handwritten mathematical proof for the divergence of a vector field:

$$\begin{aligned} \nabla^2(r^n) &= \nabla^2[(x^2+y^2+z^2)^{\frac{n}{2}}] \\ \bullet \nabla^2(r^n) &= \frac{\partial^2}{\partial x^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] + \frac{\partial^2}{\partial y^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] + \frac{\partial^2}{\partial z^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] \\ \bullet \text{ Now } \frac{\partial}{\partial x}[(x^2+y^2+z^2)^{\frac{n}{2}}] &= \frac{\partial}{\partial x}[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \times 2x] \\ &= \frac{\partial}{\partial x}[nx(x^2+y^2+z^2)^{\frac{n}{2}-1}] \\ &= n(x^2+y^2+z^2)^{\frac{n}{2}-1} + n(x^2+y^2+z^2)^{\frac{n}{2}-2} \times 2x \\ &= n(x^2+y^2+z^2)^{\frac{n}{2}-2} [x^2+y^2+z^2 + 2x^2] \\ &= n(x^2+y^2+z^2)^{\frac{n}{2}-2} [3x^2+y^2+z^2] \\ \bullet \text{ And by cyclic symmetry} \\ \frac{\partial^2}{\partial x^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] &= n(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-4} \times 2x^2 \\ \frac{\partial^2}{\partial x^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] &= n(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-4} \times 2x^2 \\ \bullet \text{ Adding the above} \\ \nabla^2(r^n) &= 3n(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-4} [x^2+y^2+z^2] \\ &= 3n(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-4} \times (x^2+y^2+z^2) \\ &= (3n + n^2 - 2n)(x^2+y^2+z^2)^{\frac{n}{2}-2} \\ &= (n^2 + n)(x^2+y^2+z^2)^{\frac{n}{2}-2} \\ &= n(n+1)r^{n-2} \end{aligned}$$

Question 6

It is given that

$$\phi(x, y, z) = z + \sinh x \sin y.$$

- a) Verify that ϕ is a solution of Laplace's equation

$$\nabla^2 \phi = 0.$$

- b) Hence find a vector field \mathbf{F} , so that

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{F} = \mathbf{0}.$$

- c) Verify that \mathbf{F} found in part (b) is solenoidal and irrotational.

$$\mathbf{F} = (\cosh x \sin y)\mathbf{i} + (\sinh x \cos y)\mathbf{j} + \mathbf{k}$$

a) $\phi(x, y, z) = \sinh x \sin y + z$
 $\frac{\partial \phi}{\partial x} = \cosh x \sin y = \phi(x, y)$
 $\frac{\partial \phi}{\partial y} = \sinh x \cos y = -\phi(x, y)$
 $\frac{\partial \phi}{\partial z} = 1$
 $\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$
 $= \cosh x \sin y - \sinh x \cos y + 0$
 $= 0$

b) $\nabla \phi = \nabla(\phi) = 0$
 $(\text{Div}(\nabla \phi)) = 0 \leftarrow (\text{Divergence})$
 $\text{Also } \nabla \wedge (\nabla \phi) = 0 \leftarrow (\text{Curl})$
 $\therefore \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = [\cosh x \sin y, \sinh x \cos y, 1]$
 $\therefore \mathbf{F} = [\cosh x \sin y, \sinh x \cos y, 1]$

c) $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} [\cosh x \sin y] + \frac{\partial}{\partial y} [\sinh x \cos y] + \frac{\partial}{\partial z} [1]$
 $= \sinh x \sin y - \sinh x \sin y + 0$
 $= 0$
 $\therefore \text{SOLENOIDAL}$
 $\nabla \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cosh x \sin y & \sinh x \cos y & 1 \end{vmatrix}$
 $= [\cosh x \cos y - \cosh x \cos y, 0 - 0, \cosh x \sin y - \sinh x \cos y] = (0, 0, 0)$
 $\therefore \text{IRROTATIONAL}$

Question 7

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $f(r)$ is a twice differentiable function, show that

$$\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}, \quad r \neq 0.$$

proof

The handwritten proof shows the following steps:

- Define $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$.
- Express the Laplacian $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right)$.
- Expand the divergence: $\nabla^2 f = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right)$.
- Use the chain rule: $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{df}{dr} \cdot \frac{\partial r}{\partial x}$.
- Calculate $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, and $\frac{\partial r}{\partial z} = \frac{z}{r}$.
- Sum the terms: $\nabla^2 f = \frac{df}{dr} \left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right) + \frac{d^2 f}{dr^2} \left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right)$.
- Simplify using $x^2 + y^2 + z^2 = r^2$: $\nabla^2 f = \frac{df}{dr} \cdot r + \frac{d^2 f}{dr^2} \cdot r = r \frac{d^2 f}{dr^2} + \frac{df}{dr}$.
- Final result: $\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$.

Question 8

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla^2 \left[\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right) \right] = \frac{2}{|\mathbf{r}|^4}.$$

proof

$$\nabla^2 \left[\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right) \right] = \nabla^2 \left[\left(\frac{\partial}{\partial x} \frac{x}{x^2+y^2+z^2} + \frac{\partial}{\partial y} \frac{y}{x^2+y^2+z^2} + \frac{\partial}{\partial z} \frac{z}{x^2+y^2+z^2} \right) \right]$$

$$= \left(\frac{\partial}{\partial x} \frac{x}{x^2+y^2+z^2} \right) + \left(\frac{\partial}{\partial y} \frac{y}{x^2+y^2+z^2} \right) + \left(\frac{\partial}{\partial z} \frac{z}{x^2+y^2+z^2} \right)$$

$$= \frac{(x^2+y^2+z^2) - 2x^2}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2) - 2y^2}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2) - 2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2 - 2x^2 + x^2+y^2+z^2 - 2y^2 + x^2+y^2+z^2 - 2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2 - 2x^2 - 2y^2 - 2z^2 + x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{-x^2 - y^2 - z^2 + x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{0}{(x^2+y^2+z^2)^2} = 0$$

$$\nabla^2 \left[\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right) \right] = \frac{2}{|\mathbf{r}|^4}$$

Question 9

The scalar function φ is given below in terms of the non zero constants λ and μ .

$$\varphi(x, y, z) = e^x \sin(\lambda y + \lambda z) \cos(\mu y - \mu z).$$

Given that

$$\nabla \cdot (\nabla \varphi) = 0,$$

show, with a detailed method, that $\lambda^2 + \mu^2 = \frac{1}{2}$.

proof

$$\begin{aligned} \nabla^2 \phi &= 0 \quad \text{if} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \\ \nabla \cdot (\nabla \phi) &= 0 \\ \phi(x, y, z) &= \cos(x) \cos(y) \cos(z) \\ \nabla^2 \phi &= \frac{\partial^2}{\partial x^2} \cos(x) \cos(y) \cos(z) + \frac{\partial^2}{\partial y^2} \cos(x) \cos(y) \cos(z) + \frac{\partial^2}{\partial z^2} \cos(x) \cos(y) \cos(z) \\ &= -\cos(x) \cos(y) \cos(z) - \cos(x) \cos(y) \cos(z) - \cos(x) \cos(y) \cos(z) \\ &= -3 \cos(x) \cos(y) \cos(z) \neq 0 \end{aligned}$$

Question 10

It is given that

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

- a) Use the above identity to find a simplified expression for $\nabla \wedge (\nabla \wedge \mathbf{F})$, where \mathbf{F} is a smooth vector field.

The smooth vector fields \mathbf{E} and \mathbf{H} , satisfy the following relationships.

- $\nabla \cdot \mathbf{E} = 0$
- $\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$
- $\nabla \cdot \mathbf{H} = 0$
- $\nabla \wedge \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$

- b) Show that \mathbf{E} and \mathbf{H} , satisfy the wave equation.

$$\nabla^2 \mathbf{U} = \frac{\partial^2 \mathbf{U}}{\partial t^2}.$$

$$\nabla \wedge (\nabla \wedge \mathbf{F}) \equiv \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

Handwritten solution for part (a):

Let $\mathbf{A} = \nabla$, $\mathbf{B} = \nabla$, $\mathbf{C} = \mathbf{F}$

$\nabla \wedge (\nabla \wedge \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

For \mathbf{E} :

$\nabla \cdot \mathbf{E} = 0$ $\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$

$\nabla \wedge (\nabla \wedge \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$

$\nabla \wedge (-\frac{\partial \mathbf{H}}{\partial t}) = -\frac{\partial}{\partial t} (\nabla \wedge \mathbf{H})$

$-\frac{\partial}{\partial t} (\frac{\partial \mathbf{E}}{\partial t}) = -\frac{\partial^2 \mathbf{E}}{\partial t^2}$

$-\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla^2 \mathbf{E}$

$\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$

For \mathbf{H} :

$\nabla \cdot \mathbf{H} = 0$ $\nabla \wedge \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$

$\nabla \wedge (\nabla \wedge \mathbf{H}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}$

$\nabla \wedge (\frac{\partial \mathbf{E}}{\partial t}) = \frac{\partial}{\partial t} (\nabla \wedge \mathbf{E})$

$\frac{\partial}{\partial t} (-\frac{\partial \mathbf{H}}{\partial t}) = -\frac{\partial^2 \mathbf{H}}{\partial t^2}$

$-\frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla^2 \mathbf{H}$

$\nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$

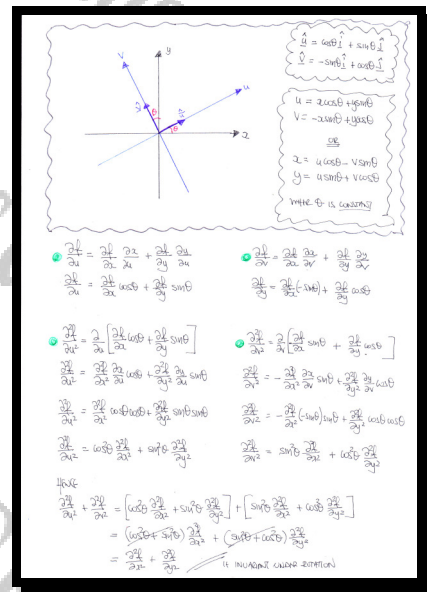
Question 11

The Laplacian operator ∇^2 in the standard two dimensional Cartesian system of coordinates is defined as

$$\nabla^2(\) \equiv \frac{\partial^2}{\partial x^2}(\) + \frac{\partial^2}{\partial y^2}(\).$$

Show that the two dimensional Laplacian operator is invariant under rotation.

proof



Question 12

It is given that if \mathbf{F} is a smooth vector field, then

$$\nabla \wedge (\nabla \wedge \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

- a) By using index summation notation, or otherwise, prove the validity of the above vector identity.

Maxwell's equations for the electric field \mathbf{E} and magnetic field \mathbf{B} , satisfy

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

where ρ is the electric charge density, \mathbf{J} is the current density, and μ_0 and ϵ_0 are positive constants.

- b) Show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$.

- c) Given further that $\rho = 0$ and $\mathbf{J} = \mathbf{0}$, show also that

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

proof

a) consider the i th component of $\nabla \cdot (\nabla \wedge \mathbf{E})$

$$\begin{aligned} & [\epsilon_{ijk} \nabla_j (\epsilon_{ikl} \nabla_l E_k)] \\ &= \epsilon_{ijk} \epsilon_{ikl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} E_k \\ &= -\epsilon_{ijk} \epsilon_{ikl} \frac{\partial^2}{\partial x_j \partial x_l} E_k \\ &= -\left[\delta_{ij} \delta_{kl} - \delta_{il} \delta_{kj} \right] \frac{\partial^2}{\partial x_j \partial x_l} E_k \\ &= -[\delta_{ij} \delta_{kl} - \delta_{il} \delta_{kj}] \frac{\partial^2}{\partial x_j \partial x_l} E_k \\ &= -\delta_{ij} \delta_{kl} \frac{\partial^2}{\partial x_j \partial x_l} E_k + \delta_{il} \delta_{kj} \frac{\partial^2}{\partial x_j \partial x_l} E_k \\ &= -\frac{\partial^2}{\partial x_j \partial x_l} E_k + \frac{\partial^2}{\partial x_l \partial x_j} E_k \\ &= -\nabla^2 E_i + \frac{\partial^2}{\partial x_i \partial x_i} E_i \\ &= -[\nabla \cdot (\nabla \wedge \mathbf{E}) - \nabla^2 E_i] \\ &\therefore \nabla \cdot (\nabla \wedge \mathbf{E}) = \nabla \cdot (\nabla \cdot \mathbf{E}) - \nabla^2 E_i \end{aligned}$$

b) $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ (i)
 $\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (ii)
 $\nabla \cdot \mathbf{B} = 0$ (iii)
 $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (iv)

• contract with (i)
 $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$
 • differentiate w.r.t
 $\Rightarrow \frac{\partial}{\partial t} [\nabla \cdot \mathbf{E}] = \frac{\partial}{\partial t} \left[\frac{\rho}{\epsilon_0} \right]$
 $\Rightarrow \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}$
 • by (ii)
 $\Rightarrow \nabla \cdot \left[\frac{1}{\mu_0} \nabla \wedge \mathbf{B} - \frac{1}{\epsilon_0} \frac{\partial \mathbf{E}}{\partial t} \right] = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}$
 $\Rightarrow \frac{1}{\mu_0} \nabla \cdot (\nabla \wedge \mathbf{B}) - \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}$
 $\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 0$ as required

c) $\nabla \cdot \mathbf{E} = 0$ (i)
 $\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (ii)
 $\nabla \cdot \mathbf{B} = 0$ (iii)
 $\nabla \wedge \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (iv)

• contract with (ii)
 $\nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
 • take the curl of the equation
 $\Rightarrow \nabla \cdot (\nabla \wedge \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$
 • by the identity of part (a)
 $\Rightarrow \nabla \cdot (\nabla \wedge \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} [\nabla \cdot \mathbf{B}]$
 $\Rightarrow -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} [0]$
 $\Rightarrow \nabla^2 \mathbf{E} = 0$

• similarly starting with (iv)
 $\nabla \cdot \mathbf{B} = 0$
 • take the curl of the equation
 $\Rightarrow \nabla \cdot (\nabla \wedge \mathbf{B}) = \nabla \cdot \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$
 • by the identity of part (a)
 $\Rightarrow \nabla \cdot (\nabla \wedge \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} [\nabla \cdot \mathbf{E}]$
 $\Rightarrow -\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} [0]$
 $\Rightarrow \nabla^2 \mathbf{B} = 0$ as required

Show that the Laplacian operator in the standard two dimensional Polar system of coordinates is given by

$$\nabla^2(\quad) \equiv \frac{\partial^2}{\partial r^2}(\quad) + \frac{1}{r} \frac{\partial}{\partial r}(\quad) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}(\quad).$$

proof

[illegible]

$$\begin{aligned}
 &= \cos^2 \frac{2\pi}{3t} - \cos^2 \sin^2 \left[\frac{1}{12} \frac{2\pi}{3t} + \frac{1}{6} \frac{2\pi}{3t} \right] - \frac{2\pi}{3t} \left[-\sin \frac{2\pi}{3t} + \cos \frac{2\pi}{3t} \right] \\
 &\quad + \frac{2\pi}{3t} \left[\sin \frac{2\pi}{3t} + \sin \frac{2\pi}{3t} \right] \\
 &= \cos^2 \frac{2\pi}{3t} + \frac{\cos^2 \sin^2 \frac{2\pi}{3t}}{12} - \frac{\cos^2 \sin^2 \frac{2\pi}{3t}}{6} + \frac{2\pi}{3t} - \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \sin \frac{2\pi}{3t} \\
 &\quad + \sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} + \frac{\sin^2 \frac{2\pi}{3t}}{6} \\
 &= \cos^2 \frac{2\pi}{3t} + \frac{\sin^2 \frac{2\pi}{3t}}{12} + \frac{2 \cos \frac{2\pi}{3t} \sin \frac{2\pi}{3t}}{6} - \frac{2 \cos \frac{2\pi}{3t} \sin \frac{2\pi}{3t}}{6} + \frac{\sin^2 \frac{2\pi}{3t}}{6} \\
 &= \cos^2 \frac{2\pi}{3t} + \frac{\sin^2 \frac{2\pi}{3t}}{12} + \frac{2 \sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t}}{6} - \frac{2 \sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t}}{6} + \frac{\sin^2 \frac{2\pi}{3t}}{6} \\
 &\quad \text{PROVE} \\
 &\frac{2\pi}{3t} \left(\frac{2\pi}{3t} \right) = \left(\sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \right) \left(\sin \frac{2\pi}{3t} + \cos \frac{2\pi}{3t} \right) \\
 &= \sin \frac{2\pi}{3t} \left(\cos^2 \frac{2\pi}{3t} \right) + \sin \frac{2\pi}{3t} \left(\sin^2 \frac{2\pi}{3t} \right) + \cos \frac{2\pi}{3t} \left(\sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \right) + \cos \frac{2\pi}{3t} \left(\cos^2 \frac{2\pi}{3t} \right) \\
 &= \sin^2 \frac{2\pi}{3t} + \sin^2 \frac{2\pi}{3t} + \frac{2\pi}{3t} \left(\sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \right) + \frac{2\pi}{3t} \left(\sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \right) \\
 &\quad \text{PROVE THAT} \\
 &= \sin^2 \frac{2\pi}{3t} + \sin^2 \frac{2\pi}{3t} + \frac{2\pi}{3t} \left(\sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \right) + \frac{2\pi}{3t} \left(\sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t} \right) \\
 &\quad + \frac{2\pi}{3t} \left[-\sin \frac{2\pi}{3t} + \cos \frac{2\pi}{3t} \right] \\
 &= \sin^2 \frac{2\pi}{3t} - \sin^2 \frac{2\pi}{3t} + \frac{2\pi}{3t} \cos \frac{2\pi}{3t} + \sin^2 \frac{2\pi}{3t} + \frac{2\pi}{3t} \cos \frac{2\pi}{3t} + \cos^2 \frac{2\pi}{3t} \\
 &\quad - \frac{\cos^2 \frac{2\pi}{3t}}{12} + \frac{2\pi}{3t} + \frac{2\pi}{3t} \\
 &= \sin^2 \frac{2\pi}{3t} + \frac{\cos^2 \frac{2\pi}{3t}}{12} + \frac{2 \sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t}}{6} - \frac{2 \sin \frac{2\pi}{3t} \cos \frac{2\pi}{3t}}{6} + \frac{\cos^2 \frac{2\pi}{3t}}{12}
 \end{aligned}$$

$$= \cancel{\sin \theta} \frac{\partial \phi}{\partial r} + \frac{\cos \theta}{r^2} \frac{\partial \phi}{\partial \theta} + \frac{\sin \theta}{r} \frac{\partial \phi}{\partial \phi} - \frac{\sin \theta}{r^2} \frac{\partial \phi}{\partial \theta} + \frac{\cos \theta}{r} \frac{\partial \phi}{\partial \theta}$$