

Created by T. Madas

SERIES

EXAM QUESTIONS

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Question 1 (**)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3}$

b) $\sum_{r=1}^{\infty} \frac{1}{2r+2^r}$

, divergent, convergent

Question 2 (**)

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} k \left(\frac{1}{2} \right)^k$$

Show a full method, justifying every step in the workings.

convergent

Question 3 ()**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{r=3}^{\infty} \frac{\sqrt{r}}{r-2}$

b) $\sum_{k=1}^{\infty} \frac{1}{(k^4 + 2)\sqrt{k}}$

divergent , convergent

Question 4 ()**

Determine whether the following series converges or diverges.

$$\sum_{r=1}^{\infty} \frac{1}{r^2 + 4r}$$

Show a full method, justifying every step in the workings.

convergent

Question 5 ()**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{r=1}^{\infty} \frac{3r^2}{r^4 + 2}$

b) $\sum_{k=1}^{\infty} \left[\frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k} \right]$

convergent , convergent

(a) $\sum_{r=1}^{\infty} \frac{3r^2}{r^4+2} < \sum_{r=1}^{\infty} \frac{3r^2}{r^4} = \sum_{r=1}^{\infty} \frac{3}{r^2}$ which converges by $\sum \frac{1}{r^p}$ with $p=2 > 1$
 $\therefore \sum_{r=1}^{\infty} \frac{3r^2}{r^4+2}$ is convergent

(b) $\sum_{k=1}^{\infty} \frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k} < \sum_{k=1}^{\infty} \frac{1}{6^k} = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots = \frac{\frac{1}{6}}{1 - \frac{1}{6}} = \frac{1}{5}$
 $\therefore \sum_{k=1}^{\infty} \frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k}$ is convergent

Question 6 ()**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Show a full method, justifying every step in the workings.

convergent

$\sum_{n=1}^{\infty} \frac{2^n}{n!} = \dots$ BY THE RATIO TEST

$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left[\frac{2^{n+1}}{2^n} \times \frac{n!}{(n+1)!} \right]$
 $= \lim_{n \rightarrow \infty} \left[2 \times \frac{1}{n+1} \right] = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by the ratio test

Question 7 (**)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n.$$

Show a full method, justifying every step in the workings,

convergent

Handwritten solution for Question 7:

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n \text{ by the ratio test}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 \left(\frac{1}{2}\right)^{n+1}}{n^2 \left(\frac{1}{2}\right)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 \times \frac{1}{2}$$

$$= \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$ converges by the ratio test

Question 8 (**)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}.$$

Show a full method, justifying every step in the workings.

divergent

Handwritten solution for Question 8:

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} > \sum_{n=1}^{\infty} \frac{n^n}{n^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ which diverges}$$

$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ diverges by comparison

Question 9 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2}$

b) $\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)}$

c) $\sum_{r=1}^{\infty} \frac{1}{r} \left(\frac{2}{3}\right)^r$

divergent, convergent, convergent

Handwritten solutions for Question 9:

a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2} = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} = \dots$ Diverges (because $1+1+\dots$)

b) $\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)} < \sum_{k=1}^{\infty} \frac{|\sin k|}{k(k+1)} < \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} < \sum_{k=1}^{\infty} \frac{1}{k^2}$
 $\therefore \sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)}$ converges (changes to $\frac{1}{k^2}$)

c) $\sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r < \sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$
 $\therefore \sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r$ converges

Question 10 (**)

Investigate the convergence or divergence of each of the following series justifying every step in the workings.

a) $\sum_{k=1}^{\infty} \left[\frac{\sqrt{k}}{k^2 + 4k + 1} \right]$

b) $\sum_{n=1}^{\infty} \left[\frac{3^n + 2}{2^n + 3} \right]$

, convergent, divergent

a) ATTEMPTING TO SHOW CONVERGENCE BY COMPARISON AS THE ONE HAVE $\sum_{k=1}^{\infty} O\left(\frac{1}{k^2}\right)$

$$\sum_{k=1}^{\infty} \left[\frac{\sqrt{k}}{k^2 + 4k + 1} \right] < \sum_{k=1}^{\infty} \left[\frac{\sqrt{k}}{k^2} \right] = \sum_{k=1}^{\infty} \left[\frac{1}{k^{3/2}} \right]$$

WHICH CONVERGES BY THE "p-TEST"

$$\sum_{k=1}^{\infty} \left[\frac{1}{k^p} \right] \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \leq 1 \end{cases}$$

b) BY COMPARISON WE HAVE

$$\sum_{k=1}^{\infty} \left[\frac{3^k + 2}{2^k + 3} \right] < \sum_{k=1}^{\infty} \left[\frac{3^k + 2}{2^k} \right] = \sum_{k=1}^{\infty} \left[\frac{3^k}{2^k} + \frac{2}{2^k} \right]$$

$$= \sum_{k=1}^{\infty} \left[\left(\frac{3}{2}\right)^k \right] + 2 \sum_{k=1}^{\infty} \left[\frac{1}{2^k} \right]$$

↑ ↑
DIVERGENT G.P. CONVERGENT G.P.

$\therefore \sum_{k=1}^{\infty} \left[\frac{3^k + 2}{2^k + 3} \right]$ DIVERGES

ALTERNATIVE

$$\lim_{k \rightarrow \infty} \left[\frac{3^k + 2}{2^k + 3} \right] = \lim_{k \rightarrow \infty} \left[\frac{\left(\frac{3}{2}\right)^k + \frac{2}{2^k}}{1 + \frac{3}{2^k}} \right] \sim \left(\frac{3}{2}\right)^k \text{ as } k \rightarrow \infty$$

IF A SERIES IS TO DIVERGE THE NECESSARY (BUT NOT SUFFICIENT) CONDITION IS THAT THE TERM LIMIT IS NON-ZERO (NOT THE CASE HERE)

Question 11 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1}$$

divergent

METHOD A

$$u_n = \frac{(n+1)^2}{n^2+1} > \frac{(n+1)^2}{n^2+2n+1} = \left(\frac{n+1}{n+1}\right)^2 = 1$$

Thus $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1} > \sum_{n=1}^{\infty} 1$ which is UNBOUNDED AS the ARITHMETIC SERIES

$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1}$ DIVERGES BY THE COMPARISON TEST //

METHOD B

$$u_n = \frac{(n+1)^2}{n^2+1} = \frac{n^2+2n+1}{n^2+1} = \frac{n^2+1}{n^2+1} + \frac{2n}{n^2+1} = 1 + \frac{2n}{n^2+1}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1} = \sum_{n=1}^{\infty} \left(1 + \frac{2n}{n^2+1}\right) = \sum_{n=1}^{\infty} 1 + \sum_{n=1}^{\infty} \frac{2n}{n^2+1}$$

which DIVERGES AS $\sum_{n=1}^{\infty} 1$ IS NOT BOUNDED //

Question 12 (**+)

$$u_r = \ln\left(\frac{r}{r+1}\right), r \in \mathbb{N}.$$

Show clearly that $\sum_{r=1}^{\infty} u_r$ is divergent.

proof

$$\begin{aligned} \sum_{r=1}^{\infty} \ln\left(\frac{r}{r+1}\right) &= \lim_{k \rightarrow \infty} \left[\sum_{r=1}^k \ln\left(\frac{r}{r+1}\right) \right] = \lim_{k \rightarrow \infty} \left[\ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \dots + \ln \frac{k}{k+1} \right] \\ &= \lim_{k \rightarrow \infty} \left[\ln \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{k}{k+1} \right) \right] \\ &= \lim_{k \rightarrow \infty} \left[\ln \left(\frac{1}{k+1} \right) \right] = - \lim_{k \rightarrow \infty} (\ln(k+1)) \text{ which is UNBOUNDED} \\ \therefore \sum_{r=1}^{\infty} u_r \text{ IS DIVERGENT} // \end{aligned}$$

Question 13 (**+)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2-1}$$

divergent

Handwritten solution for Question 13:

$$\sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2-1} > \sum_{n=2}^{\infty} \frac{1}{n}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is a divergent series (harmonic series), by the comparison test, the original series $\sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2-1}$ is also divergent.

Question 14 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \left[\frac{\sin^2 n}{n(n+1)} \right]$

b) $\sum_{n=1}^{\infty} \left[\frac{2n}{3n^2-4} \right]$

convergent, divergent

Handwritten solution for Question 14:

(a) $U_n = \frac{\sin^2 n}{n(n+1)} < \frac{1}{n^2} < \frac{1}{n^2}$
 Now $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (it converges) (SANDWICH TEST)
 $\therefore \sum_{n=1}^{\infty} \frac{\sin^2 n}{n(n+1)}$ CONVERGES BY THE COMPARISON TEST

(b) $U_n = \frac{2n}{3n^2-4} > \frac{2}{3n} > \frac{2}{3} \left(\frac{1}{n} \right)$
 Now $\sum_{n=1}^{\infty} \frac{2}{3n} = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (HARMONIC SERIES)
 $\therefore \sum_{n=1}^{\infty} \left(\frac{2n}{3n^2-4} \right)$ DIVERGES BY THE COMPARISON TEST

Question 15 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

b) $\sum_{r=1}^{\infty} \frac{\cos r}{2(2^{r-1}+1)}$

c) $\sum_{k=1}^{\infty} \frac{k}{k+1}$

convergent , convergent , divergent

Handwritten solutions for the three series using the comparison test:

a) $\sum_{n=1}^{\infty} \frac{n}{n^3+1} < \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$
 $\therefore \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges

b) $\sum_{r=1}^{\infty} \frac{\cos r}{2(2^{r-1}+1)} < \sum_{r=1}^{\infty} \frac{|\cos r|}{2(2^{r-1}+1)} < \sum_{r=1}^{\infty} \frac{1}{2(2^{r-1}+1)} < \sum_{r=1}^{\infty} \frac{1}{2 \cdot 2^{r-1}} = \sum_{r=1}^{\infty} \frac{1}{2^r} = 1 < \infty$
 $\therefore \sum_{r=1}^{\infty} \frac{\cos r}{2(2^{r-1}+1)}$ converges

c) $\sum_{k=1}^{\infty} \frac{k}{k+1} = \sum_{k=1}^{\infty} \left(1 - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} 1 - \sum_{k=1}^{\infty} \frac{1}{k+1}$
 $\therefore \sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges

Question 16 (*)**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n (n+1)^2}$$

Show a full method, justifying every step in the workings.

convergent

Handwritten solution for Question 16:

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n (n+1)^2} < \sum_{n=1}^{\infty} \frac{4^{n+1}}{4^n (n+1)^2} = \sum_{n=1}^{\infty} \frac{4 \times 4^n}{4^n (n+1)^2} = \sum_{n=1}^{\infty} \frac{4}{(n+1)^2}$$

$$< 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = 4 \times \frac{\pi^2}{6} = \frac{2}{3}\pi^2$$

∴ IT CONVERGES BY COMPARISON

Question 17 (*)**

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{(k \sin k)^2}{2^k}$$

Show a full method, justifying every step in the workings.

convergent

Handwritten solution for Question 17:

$$\sum_{k=1}^{\infty} \frac{(k \sin k)^2}{2^k} = \sum_{k=1}^{\infty} \frac{k^2 \sin^2 k}{2^k} < \sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

↑ BY THE RATIO TEST

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 \sin^2(k+1)}{2^{k+1}} \cdot \frac{2^k}{k^2 \sin^2 k} \right|$$

$$= \lim_{k \rightarrow \infty} \left(\frac{(k+1)^2}{k^2} \cdot \frac{2^k}{2^{k+1}} \right) = \lim_{k \rightarrow \infty} \left(\frac{(k+1)^2}{k^2} \cdot \frac{1}{2} \right)$$

$$= \frac{1}{2} < 1$$

∴ SERIES CONVERGES BY COMPARISON TEST (WORKING OUT BY RATIO TEST)

Question 18 (*)**

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$$

$$\frac{5}{2}$$

Handwritten solution for Question 18:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1+2^n}{3^n} &= \sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{2^n}{3^n} \right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3} \right)^n + \left(\frac{2}{3} \right)^n \right] \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3} \right)^n \quad \dots \text{these are both G.P.s} \\ &= \left(\frac{1}{3} + \frac{1}{9} + \dots \right) + \left(\frac{2}{3} + \frac{2^2}{9} + \dots \right) \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{1}{2} + 2 = \frac{5}{2} \end{aligned}$$

Question 19 (*)**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!}$$

Show a full method, justifying every step in the workings.

divergent

Handwritten solution for Question 19:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!} &= \dots \text{By the Ratio Test} \\ \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{5^{n+1} ((n+1)!)^2}{(2n+2)!}}{\frac{5^n (n!)^2}{(2n)!}} \right| = \lim_{n \rightarrow \infty} \left[\frac{5}{5} \times \frac{(n+1)^2}{(n+1)(n+2)} \right] \\ &= \lim_{n \rightarrow \infty} \left[5 \times \frac{(n+1)}{(n+2)} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{5n^2 + 10n + 5}{n^2 + 2n + 2} \right] = \frac{5}{1} > 1 \end{aligned}$$

\therefore Series Diverges

Question 20 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a)
$$\sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)}.$$

b)
$$\sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}}.$$

convergent, divergent

Handwritten solution for Question 20:

a) $\sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)} < \sum_{n=1}^{\infty} \frac{n-2}{2n^3} = \sum_{n=1}^{\infty} \frac{n-2}{2n^3} = \sum_{n=1}^{\infty} \frac{1}{2n^2} - \frac{1}{n^3}$
 $= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$ AND BOTH SERIES CONVERGE BY THE P-TEST
 $\therefore \sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)}$ IS CONVERGENT BY COMPARISON

b) $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}} > \sum_{k=1}^{\infty} \frac{2}{\sqrt{4k^2+4k+4}} = \sum_{k=1}^{\infty} \frac{2}{\sqrt{4(k^2+k+1)}} = \sum_{k=1}^{\infty} \frac{2}{2\sqrt{k^2+k+1}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+k+1}}$
 $= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2}}$ WHICH DIVERGES BY THE P-TEST
 $\therefore \sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}}$ DIVERGES

P-TEST: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ CONVERGES IF $p > 1$, DIVERGES IF $p \leq 1$

Question 21 (***)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

b) $\sum_{k=1}^{\infty} \frac{k^4}{(k+1)^6}$

c) $\sum_{r=1}^{\infty} \frac{(r+1)(2r+1)(3r+1)}{r^4}$

☐ , ☐ convergent , ☐ convergent , ☐ divergent

Handwritten solution for Question 21:

a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$... BY THE RATIO TEST ...
 AS ALL THE TERMS ARE POSITIVE WE MAY IGNORE NEGATIVE SIGNS
 $\lim_{n \rightarrow \infty} \left[\frac{u_{n+1}}{u_n} \right] = \lim_{n \rightarrow \infty} \left[\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} \right] = \lim_{n \rightarrow \infty} \left[\frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n} \right]$
 $= \lim_{n \rightarrow \infty} \left[\frac{10}{n+1} \right] = 0 < 1$
 SERIES CONVERGES BY THE RATIO TEST

b) $\sum_{k=1}^{\infty} \frac{k^4}{(k+1)^6}$...
 $\lim_{k \rightarrow \infty} \frac{k^4}{(k+1)^6} = \lim_{k \rightarrow \infty} \frac{k^4}{k^6} = \lim_{k \rightarrow \infty} \frac{1}{k^2} = 0$
 SERIES CONVERGES BY COMPARISON

c) $\sum_{r=1}^{\infty} \frac{(r+1)(2r+1)(3r+1)}{r^4}$...
 $\lim_{r \rightarrow \infty} \frac{(r+1)(2r+1)(3r+1)}{r^4} = \lim_{r \rightarrow \infty} \frac{r \times 2r \times 3r}{r^4} = \lim_{r \rightarrow \infty} \frac{6r^3}{r^4} = \lim_{r \rightarrow \infty} \frac{6}{r} = 0$
 SERIES CONVERGES BY COMPARISON

Question 22 (***)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty} \frac{(t!)^3}{(3t)!}$$

Show a full method, justifying every step in the workings.

convergent

Handwritten solution for Question 22:

$$\sum_{t=1}^{\infty} \frac{(t!)^3}{(3t)!} \dots \text{BY THE RATIO TEST}$$

$$\lim_{t \rightarrow \infty} \left| \frac{u_{t+1}}{u_t} \right| = \lim_{t \rightarrow \infty} \left| \frac{\left(\frac{(t+1)!}{(3t+3)!} \right)^3}{\left(\frac{(t!)^3}{(3t)!} \right)} \right| = \lim_{t \rightarrow \infty} \left[\frac{(t+1)!^3}{(3t+3)!^3} \times \frac{(3t)!}{(t!)^3} \right]$$

$$= \lim_{t \rightarrow \infty} \left[\frac{1}{(2t+3)(2t+2)(2t+1)} \times (t+1)^3 \right]$$

$$= \lim_{t \rightarrow \infty} \left[\frac{(t+1)^3}{(2t+3)(2t+2)(2t+1)} \right] = \frac{1}{27} < 1$$

\therefore SERIES CONVERGES BY THE RATIO TEST

Question 23 (***)The sum of the first n terms of an arithmetic series with first term a and common difference d , is denoted by S_n .

Simplify fully

$$S_n - 2S_{n+1} + S_{n+2}$$

$$S_n - 2S_{n+1} + S_{n+2} = d$$

Handwritten solution for Question 23:

$$\begin{aligned} S_{n+2} - S_{n+1} &= u_{n+2} = a + (n+1)d = a + nd + d \\ S_{n+1} - S_n &= u_{n+1} = a + nd = a + nd \end{aligned}$$

SUBTRACT, SIDE BY SIDE

$$S_{n+2} - 2S_{n+1} + S_n = d$$

Question 24 (***)

It is given that

$$\frac{1}{n} \sum_{r=1}^n x_r = 2 \quad \text{and} \quad \sqrt{\frac{1}{n} \sum_{r=1}^n (x_r)^2 - \frac{1}{n^2} \left(\sum_{r=1}^n x_r \right)^2} = 3.$$

Determine, in terms of n , the value of

$$\sum_{r=1}^n (x_r + 1)^2.$$

$$\boxed{}, \quad \sum_{r=1}^n (x_r + 1)^2 = 18n$$

Handwritten solution for Question 24:

$$\begin{aligned} \frac{1}{n} \sum_{r=1}^n x_r &= 2 \\ \Rightarrow \sum_{r=1}^n x_r &= 2n \\ \sqrt{\frac{1}{n} \sum_{r=1}^n (x_r)^2 - \frac{1}{n^2} \left(\sum_{r=1}^n x_r \right)^2} &= 3 \\ \Rightarrow \frac{1}{n} \sum_{r=1}^n (x_r)^2 - \frac{(2n)^2}{n^2} &= 9 \\ \Rightarrow \frac{1}{n} \sum_{r=1}^n (x_r)^2 - 4 &= 9 \\ \Rightarrow \frac{1}{n} \sum_{r=1}^n (x_r)^2 &= 13n \\ \therefore \sum_{r=1}^n (x_r + 1)^2 &= \sum_{r=1}^n [(x_r)^2 + 2x_r + 1] \\ &= \frac{1}{n} \sum_{r=1}^n (x_r)^2 + 2 \sum_{r=1}^n x_r + \sum_{r=1}^n 1 \\ &= 13n + 2(2n) + n \\ &= 18n \end{aligned}$$

Question 25 (***)

It is given that

$$\sum_{r=1}^{20} [f(r) - 10] = 200 \quad \text{and} \quad \sum_{r=1}^{20} [f(r) - 10]^2 = 2800.$$

Find the value of

$$\sum_{r=1}^{20} [f(r)]^2.$$

$$\boxed{}, \quad \sum_{r=1}^{20} [f(r)]^2 = 8800$$

MANIPULATE THE SUMS AS FOLLOWS

$$\begin{aligned} \sum_{r=1}^{20} (f(r) - 10) &= \left[\sum_{r=1}^{20} f(r) \right] - \sum_{r=1}^{20} 10 \\ 200 &= \left[\sum_{r=1}^{20} f(r) \right] - 10 \times 20 \\ \sum_{r=1}^{20} f(r) &= 400 \end{aligned}$$

NEXT WE HAVE

$$\begin{aligned} \sum_{r=1}^{20} (f(r) - 10)^2 &= \sum_{r=1}^{20} [f(r)^2 - 20f(r) + 100] \\ 2800 &= \sum_{r=1}^{20} [f(r)^2] - 20 \sum_{r=1}^{20} f(r) + 100 \sum_{r=1}^{20} 1 \\ 2800 &= \sum_{r=1}^{20} [f(r)^2] - 20 \times 400 + 100 \times 20 \\ 2800 &= \sum_{r=1}^{20} [f(r)^2] - 6000 \\ \sum_{r=1}^{20} [f(r)^2] &= 8800 \end{aligned}$$

Question 26 (***)

It is given that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{4n^2}, \quad x \in \mathbb{R}, \quad x > 0.$$

Determine the range of possible values of x .

$$\boxed{}, \quad 0 < x < \frac{1}{5}$$

• THIS CAN BE DONE BY THE RATIO TEST
 THE n^{th} TERM OF THE SERIES IS GIVEN BY $u_n = \frac{(5x)^n}{4n^2}$
 • BY THE RATIO TEST (WHENEVER WE CAN AS THE TERMS ARE POSITIVE)

$$\frac{u_{n+1}}{u_n} = \frac{(5x)^{n+1}}{4(n+1)^2} \times \frac{4n^2}{(5x)^n} = \frac{5x n^2}{(n+1)^2}$$

$$= \frac{5x}{1 + \frac{2}{n} + \frac{1}{n^2}}$$
 • THIS WILL CONVERGE IF

$$\Rightarrow \frac{u_{n+1}}{u_n} \rightarrow L, \quad 0 \leq L < 1, \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow 5x \rightarrow L, \quad 0 \leq L < 1$$

(SINCE $1 + \frac{2}{n} + \frac{1}{n^2} \rightarrow 1$ as $n \rightarrow \infty$)

$$\Rightarrow 0 \leq 5x < 1$$

$$\underline{0 < x < \frac{1}{5}}$$

LEAVING THE EQUAL CASE

Question 27 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6}$$

divergent

Handwritten solution showing the divergence of the series using the comparison test:

$$\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6} > \sum_{n=3}^{\infty} \frac{1}{n}$$

The harmonic series $\sum_{n=3}^{\infty} \frac{1}{n}$ is known to diverge. Therefore, by the comparison test, the original series $\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6}$ also diverges.

Question 28 (***)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a)
$$\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!}$$

b)
$$\sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{\sqrt{t+1}}$$

convergent, convergent

Handwritten solution for Question 28:

(a) $\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!}$... By Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^{n+1} ((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{e^n (n!)^2} \right| = \lim_{n \rightarrow \infty} \left[\frac{e^{n+1} (2n)!}{e^n (2n+2)!} \cdot \frac{(n!)^2}{(n+1)!^2} \right]$$

$$= \lim_{n \rightarrow \infty} \left[e \times \frac{1}{(2n+2)(2n+1)} \times (n!)^2 \right] = \lim_{n \rightarrow \infty} \left[\frac{e (n!)^2}{2(2n+1)(2n+2)} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{e (n+1)}{2(2n+1)} \right] = \frac{e}{4} < 1$$

\therefore Series converges by the Ratio Test

(b) $\sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{\sqrt{t+1}}$... Terms alternate in sign

$$\lim_{t \rightarrow \infty} \left(\frac{1}{\sqrt{t+1}} \right) = 0$$

\therefore Series converges by Alternating Series Test

Question 29 (***)

By using an algebraic method, find the value of

$$99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2$$

5000

Method A

REGROUP THE TERMS

$$99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2$$

$$= (99^2 + 95^2 + \dots + 3^2) - (97^2 + 93^2 + \dots + 1^2)$$

$$= \sum_{k=1}^{50} (4k-1)^2 - \sum_{k=1}^{50} (4k-3)^2 \quad (\text{WRITE IN SUMMATION NOTATION})$$

$$= \sum_{k=1}^{50} [(4k-1)^2 - (4k-3)^2] \quad (\text{COMBINE SUMMATIONS})$$

$$= \sum_{k=1}^{50} (4k-1+4k-3)(4k-1-4k+3) \quad (\text{DIFFERENCE OF SQUARES})$$

$$= \sum_{k=1}^{50} (8k-4) \times 2$$

$$= \sum_{k=1}^{50} (16k-8)$$

$$= 16 \sum_{k=1}^{50} k - 8 \sum_{k=1}^{50} 1 \quad (\text{LINEARITY OF THE SUMMATION})$$

$$= 16 \times \frac{1}{2} \times 50 \times 51 - 8 \times 50$$

$$= 5000 - 200$$

$$= 5000$$

Method B

REGROUP THE TERMS AS PAIRS

$$= 99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2$$

$$= (99^2 - 97^2) + (95^2 - 93^2) + (91^2 - 89^2) + \dots + (3^2 - 1^2)$$

$$= (99-97)(99+97) + (95-93)(95+93) + (91-89)(91+89) + \dots + (3-1)(3+1)$$

$$= 2(99) + 2(98) + 2(96) + \dots + 2(4)$$

$$= 2[4 + 12 + 20 + \dots + (50 + 100 + 196)]$$

$$= 2 \times 4[1 + 3 + 5 + \dots + 49 + 47 + 69]$$

\rightarrow \therefore ARITHMETIC PROGRESSION WITH $a=1$
 $d=2$
 $n=50$

$u_n = a + (n-1)d$
 $49 = 1 + (n-1) \times 2$
 $48 = 1 + 2n - 2$
 $50 = 2n$
 $n = 25$

$$= 8 \times \sum_{k=1}^{25} (1+49)$$

$$= 8 \times \frac{25 \times 50}{2}$$

$$= 5000$$

AS REQUEST

Question 30 (***)

Evaluate, showing clearly your method

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}}$$

$$\boxed{}, \boxed{\frac{7}{12}}$$

SPLIT THE SUMMATION AS REQUEST

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\frac{3^n - 2}{4^{n+1}} \right] &= \sum_{n=1}^{\infty} \left[\frac{3^n}{4^{n+1}} - \frac{2}{4^{n+1}} \right] = \sum_{n=1}^{\infty} \left[\frac{3^n}{4^{n+1}} \right] - \sum_{n=1}^{\infty} \left[\frac{2}{4^{n+1}} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{3}{4 \times 4^n} \right] - \sum_{n=1}^{\infty} \left[\frac{2}{4 \times 4^n} \right] \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n - \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{4} \right)^n \end{aligned}$$

THIS IS A G.P. a = 3 r = 3/4 S _∞ = 3/1 - 3/4 = 3	THIS IS A G.P. a = 1/4 r = 1/4 S _∞ = 1/4 - 1/4 = 1/3
--	--

PUTTING ALL THE CHANCES TOGETHER

$$\sum_{n=1}^{\infty} \left[\frac{3^n - 2}{4^{n+1}} \right] = \frac{1}{4} \times 3 - \frac{1}{2} \times \frac{1}{3} = \frac{3}{4} - \frac{1}{6} = \frac{7}{12}$$

Question 31 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{k=1}^{\infty} \frac{k+2}{4k^2+5}$$

divergent

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k+2}{4k^2+5} &> \sum_{k=1}^{\infty} \frac{k+2}{4k^2+4k+16} = \sum_{k=1}^{\infty} \frac{k+2}{4(k^2+4k+16)} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{k+2}{k^2+4k+16} \\ &= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k+2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{WHICH DIVERGES} \\ \therefore \sum_{k=1}^{\infty} \frac{k+2}{4k^2+5} &\text{ DIVERGES} \end{aligned}$$

Question 32 (***)

Show clearly that

$$\sum_{r=1}^n \left[2r(2r^2 - 3r - 1) + n + 1 \right] = (n+1)^2 (n-1)^2.$$

proof

$$\begin{aligned} \sum_{r=0}^n \left[2r(2r^2 - 3r - 1) + (n+1) \right] &= \sum_{r=0}^n (4r^3 - 6r^2 - 2r + n + 1) \\ &= 4 \sum_{r=0}^n r^3 - 6 \sum_{r=0}^n r^2 - 2 \sum_{r=0}^n r + \sum_{r=0}^n (n+1) \\ &= 4 \sum_{r=1}^n r^3 - 6 \sum_{r=1}^n r^2 - 2 \sum_{r=1}^n r + (n+1) \sum_{r=0}^n 1 \\ &\text{NOW USE STANDARD RESULTS} \\ &= 4 \times \frac{1}{4} n^2 (n+1)^2 - 6 \times \frac{1}{6} n(n+1)(2n+1) - 2 \times \frac{1}{2} n(n+1) + (n+1)(n+1) \\ &= n^2 (n+1)^2 - n(n+1)(2n+1) - n(n+1) + (n+1)^2 \\ &= n(n+1) [n(n+1) - (2n+1) - 1] + (n+1)^2 \\ &= n(n+1) [n^2 + n - 2n - 2] + (n+1)^2 \\ &= n(n+1) (n^2 - n - 2) + (n+1)^2 \\ &= n(n+1)(n-2) + (n+1)^2 \\ &= (n+1)^2 [n(n-2) + 1] \\ &= (n+1)^2 (n^2 - 2n + 1) \\ &= (n+1)^2 (n-1)^2 \quad \text{✓ REQUIRED} \end{aligned}$$

Question 33 (***)

Consider the infinite series

$$1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \times 2^2} - \frac{x^6}{6^2 \times 4^2 \times 2^2} + \frac{x^8}{8^2 \times 6^2 \times 4^2 \times 2^2} - \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2} x^{2n}$$

$$\begin{aligned} & 1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \times 2^2} - \frac{x^6}{6^2 \times 4^2 \times 2^2} + \frac{x^8}{8^2 \times 6^2 \times 4^2 \times 2^2} - \dots \\ &= 1 - \frac{x^2}{2^2(1)^2} + \frac{x^4}{2^2(2!)^2} - \frac{x^6}{2^2(3!)^2} + \frac{x^8}{2^2(4!)^2} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} \end{aligned}$$

Question 34 (***)

Show clearly by an algebraic method that

$$40^2 - 39^2 + 38^2 - 37^2 + \dots + 2^2 - 1^2 = 820.$$

proof

$$\begin{aligned} & 40^2 - 39^2 + 38^2 - 37^2 + \dots + 2^2 - 1^2 \\ &= (40+39)(40-39) + (38+37)(38-37) + \dots + (2+1)(2-1) \\ &= 79 + 75 + 71 + \dots + 7 + 3 \\ &= 3 + 7 + \dots + 71 + 75 + 79 \quad \leftarrow \text{ARITHMETIC PROGRESSION} \\ &= \frac{20}{2} [2 \times 3 + 19 \times 4] \quad \leftarrow \begin{cases} a=3 \\ d=4 \\ n=20 \end{cases} \\ &= 10 \times (6 + 76) \\ &= 820 \end{aligned}$$

Question 35 (****)

By justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt[3]{n^2}}.$$

convergent

Handwritten solution for Question 35:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt[3]{n^2} [\sqrt{n+1} + \sqrt{n}]} = \sum_{n=1}^{\infty} \frac{(n+1) - n}{\sqrt[3]{n^2} [(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}]} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2} [(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}]} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2} (2\sqrt{n})} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{6}}}$$

WHICH CONVERGES SINCE

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ CONVERGES IF } p > 1$$

DIVERGES IF $p \leq 1$

Question 36 (****)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^{n+1}}.$$

Show a full method, justifying every step in the workings.

You may assume without proof the value of $\lim_{n \rightarrow \infty} \left[\left(\frac{n}{n+1} \right)^n \right]$.

convergent

Handwritten solution for Question 36:

$$\sum_{n=1}^{\infty} \frac{n!}{n^{n+1}} < \sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{Now by the Ratio Test...}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}} \right| = \lim_{n \rightarrow \infty} \left[\frac{(n+1)!}{n!} \times \frac{n^{n+1}}{(n+1)^{n+2}} \right]$$

$$= \lim_{n \rightarrow \infty} \left[(n+1) \times \frac{n^{n+1}}{(n+1)^{n+2}} \right] = \lim_{n \rightarrow \infty} \left[\frac{n^{n+1}}{(n+1)^{n+1}} \right] = \lim_{n \rightarrow \infty} \left[\left(\frac{n}{n+1} \right)^{n+1} \right]$$

$$= \frac{1}{e} < 1$$

\therefore Series converges.

Notes: $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e}$
 $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{n+1} = \frac{1}{e}$

Question 37 (****)

The sum of the first n terms of a series with general term u_n is given by the expression

$$S_n = n^2(n+1)(n+2).$$

a) Find the first term of the series.

b) Show clearly that ...

i. ... $u_n = n(n+1)(4n-1)$

ii. ... $\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2).$

$$u_1 = 6$$

(a) $S_1 = 1^2(1+1)(1+2)$
 $S_1 = u_1 = 1^2(1+1)(1+2) = 1 \times 2 \times 3 = 6$

(b) (i) $u_n = S_n - S_{n-1} = n^2(n+1)(n+2) - (n-1)^2n(n+1)$
 $= n(n+1)[n(n+2) - (n-1)^2]$
 $= n(n+1)(n^2+2n - n^2 + 2n - 1)$
 $= n(n+1)(4n-1)$ As required

(ii) $\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n = (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2)$
 $= 4n^2(2n+1)2(n+1) - n^2(n+1)(n+2)$
 $= 8n^2(n+1)(n+1) - n^2(n+1)(n+2)$
 $= n^2(n+1)[8(n+1) - (n+2)]$
 $= n^2(n+1)(7n+6)$
 $= 3n^2(n+1)(5n+2)$ As required

Question 38 (****)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty} \sqrt[4]{2^t + 5^t}.$$

Show a full method, justifying every step in the workings.

divergent

$\sum_{t=1}^{\infty} \sqrt[4]{2^t + 5^t}$
 Consider A term in series form
 $\sqrt[4]{2^t + 5^t} > \sqrt[4]{2^t} = \sqrt[4]{2} \times \sqrt[4]{2^{t-1}} = \sqrt[4]{2} \times \sqrt[4]{2^{t-1}} = 2 \sqrt[4]{2}$
 As $t \rightarrow \infty$ the general term does not tend to zero
 \therefore Series cannot converge

Question 39 (****)

$$\sum_{r=1}^n (ar^2 + br + c) \equiv n^3 + 5n^2 + 6n,$$

where a , b and c are integer constants.Determine the value of a , b and c .

$$a=3, b=7, c=2$$

$\sum_{r=1}^n ar^2 + br + c = a \sum_{r=1}^n r^2 + b \sum_{r=1}^n r + c \sum_{r=1}^n 1$
 $= a \frac{n(n+1)(2n+1)}{6} + b \frac{n(n+1)}{2} + cn$
 $= \frac{a}{6} n [2n^2 + 3n + 1] + \frac{b}{2} n [2n + 1] + \frac{c}{1} n$
 $= \frac{a}{6} n [2n^2 + 3n + 1] + \frac{b}{2} n [2n + 1] + \frac{c}{1} n$
 $= \frac{a}{6} n^3 + \frac{a}{2} n^2 + \frac{a}{6} n + \frac{b}{2} n^2 + \frac{b}{2} n + cn$
 $= \frac{a}{6} n^3 + \left(\frac{a}{2} + \frac{b}{2} \right) n^2 + \left(\frac{a}{6} + \frac{b}{2} + c \right) n$
 Now $\left(\frac{a}{6} n^3 + \left(\frac{a}{2} + \frac{b}{2} \right) n^2 + \left(\frac{a}{6} + \frac{b}{2} + c \right) n \right) \equiv n^3 + 5n^2 + 6n$
 $\frac{a}{6} = 1 \quad \frac{a}{2} + \frac{b}{2} = 5 \quad \frac{a}{6} + \frac{b}{2} + c = 6$
 $\therefore a=3 \quad b=7 \quad c=2$

Question 40 (****)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=3}^{\infty} \left[\frac{\sqrt{n}}{n-2} \right]$

b) $\sum_{n=1}^{\infty} \left[\frac{\sqrt{n}}{n+2} \right]$

divergent, divergent

(a) $\sum_{n=3}^{\infty} \frac{\sqrt{n}}{n-2} > \sum_{n=3}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ which diverges by the p-test

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4n+4} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{(\sqrt{n}+2)^2}$

$> \sum_{n=1}^{\infty} \frac{\sqrt{n}}{(\sqrt{n}+2)^2} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n}$

$= \sum_{n=1}^{\infty} \frac{1}{4\sqrt{n}}$ which diverges by the p-test

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p \geq 1 \\ \text{diverges if } p < 1 \end{cases}$

Question 41 (****)

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = -33200.$$

proof

$$\begin{aligned}
 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= \sum_{r=1}^{40} (2r-1)^3 - \sum_{r=1}^{40} (2r)^3 = \sum_{r=1}^{40} ((2r-1)^3 - (2r)^3) \\
 &= \sum_{r=1}^{40} (8r^3 - 12r^2 + 6r - 1) = 8 \sum_{r=1}^{40} r^3 - 12 \sum_{r=1}^{40} r^2 + 6 \sum_{r=1}^{40} r - \sum_{r=1}^{40} 1 \\
 &= 8 \times \frac{1}{4} \times 20 \times 21 \times 41 + 6 \times \frac{1}{2} \times 20 \times 21 - 20 \\
 &= -34400 + 1260 - 20 = -33200 \quad \text{As required}
 \end{aligned}$$

$$\begin{aligned}
 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= (1^3 + 2^3 + 3^3 + \dots + 40^3) - 2(2^3 + 4^3 + \dots + 40^3) \\
 &= \sum_{r=1}^{40} r^3 - 2 \times 2^2 \sum_{r=1}^{20} (1^3 + 2^3 + \dots + 20^3) \\
 &= \sum_{r=1}^{40} r^3 - 16 \sum_{r=1}^{20} r^3 \\
 &= \frac{1}{4} \times 20 \times 21^2 - 16 \times \frac{1}{4} \times 20 \times 21^2 \\
 &= 67200 - 70560 \\
 &= -33200
 \end{aligned}$$

Question 42 (****)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}.$$

divergent

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{1+\sqrt{n}} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{(1+\sqrt{n})(1-\sqrt{n})} \\
 &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{1-n} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{n-1} \\
 &> \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{n} = \frac{1}{2} + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) \\
 &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{n-1} - \sum_{n=2}^{\infty} \frac{1}{n} \\
 &\quad \text{Diverges}
 \end{aligned}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} \text{ DIVERGES}$$

Question 43 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}.$

b) $\sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4}}.$

c) $\sum_{k=1}^{\infty} \frac{k+10}{k^2+10}.$

d) $\sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{4m^2+1}$

divergent, divergent, divergent, convergent

Handwritten solutions for Question 43:

a) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} > 3 \sum_{n=1}^{\infty} \frac{1}{n+1} = 3 \sum_{n=2}^{\infty} \frac{1}{n}$
 $\therefore \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ DIVERGES BY COMPARISON WITH THE HARMONIC SERIES WHICH IS KNOWN TO DIVERGE

b) $\sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4}} > \sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4r+4}} = \sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{(r+2)^2}} = \sum_{r=1}^{\infty} \frac{1}{(r+2)^{2/3}}$
 $= \sum_{r=3}^{\infty} \frac{1}{r^{2/3}} > \sum_{r=3}^{\infty} \frac{1}{r} \quad \text{WHICH DIVERGES}$
 $\therefore \sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4}}$ DIVERGES BY COMPARISON

c) $\sum_{k=1}^{\infty} \frac{k+10}{k^2+10} > \sum_{k=1}^{\infty} \frac{k}{k^2+10} > \sum_{k=1}^{\infty} \frac{k}{k^2+8k+16} = \sum_{k=1}^{\infty} \frac{k}{(k+4)^2}$
 $= \sum_{k=1}^{\infty} \frac{k+4-4}{k^2+8k+16} = \sum_{k=1}^{\infty} \frac{k+4}{k^2+8k+16} - \sum_{k=1}^{\infty} \frac{4}{k^2+8k+16}$
 $= \sum_{k=1}^{\infty} \frac{k+4}{k^2+8k+16} - \sum_{k=1}^{\infty} \frac{1}{(k+4)^2}$
 $\therefore \sum_{k=1}^{\infty} \frac{k+10}{k^2+10}$ DIVERGES BY COMPARISON

d) $\sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{4m^2+1} < \sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{4m^2} < \sum_{m=1}^{\infty} \frac{\sqrt{4m+4}}{4m^2} < \sum_{m=1}^{\infty} \frac{\sqrt{4m+4}}{4m^2}$
 $= \sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{m^2} < \sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{m^2} = \sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{m^2} = \sum_{m=1}^{\infty} \frac{1}{m^{3/2}}$
 $< \sum_{m=1}^{\infty} \frac{1}{m^{3/2}}$ WHICH CONVERGES BY THE P-TEST
 $\therefore \sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{4m^2+1}$ CONVERGES BY COMPARISON

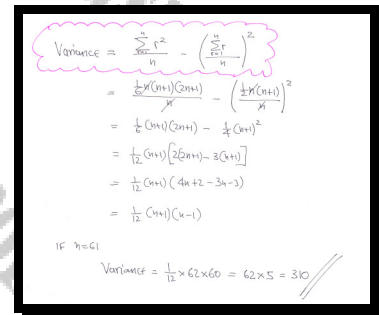
Question 44 (****)

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n} \sum_{r=1}^n r^2 - \left[\frac{1}{n} \sum_{r=1}^n r \right]^2.$$

Determine a simplified expression $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\boxed{\text{Var}(n) = \frac{1}{12}(n^2 - 1)}, \quad \boxed{\text{Var}(61) = 310}$$



Handwritten derivation of the variance formula for the first n natural numbers:

$$\begin{aligned} \text{Variance} &= \frac{\sum_{r=1}^n r^2}{n} - \left(\frac{\sum_{r=1}^n r}{n} \right)^2 \\ &= \frac{\frac{1}{6}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n} \right)^2 \\ &= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{4}(n+1)^2 \\ &= \frac{1}{12}n(n+1)[2(2n+1) - 3(n+1)] \\ &= \frac{1}{12}n(n+1)(4n+2-3n-3) \\ &= \frac{1}{12}n(n+1)(n-1) \end{aligned}$$

If $n=61$

$$\text{Variance} = \frac{1}{12} \times 61 \times 60 = 61 \times 5 = 310 //$$

Question 45 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$.

b) $\sum_{r=1}^{\infty} \frac{r!}{10^r}$.

c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

d) $\sum_{m=2}^{\infty} \frac{1}{m \ln m}$.

divergent, divergent, convergent, divergent

Handwritten solutions for Question 45:

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} > \sum_{n=2}^{\infty} \frac{1}{n}$ Initial Diverges
 $\therefore \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ is DIVERGENT BY COMPARISON

(b) $\sum_{r=1}^{\infty} \frac{r!}{10^r}$ BY THE RATIO TEST $\lim_{r \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| = \lim_{r \rightarrow \infty} \left| \frac{(r+1)!}{10^{r+1}} \cdot \frac{10^r}{r!} \right| = \lim_{r \rightarrow \infty} \left[\frac{(r+1)!}{r!} \cdot \frac{10^r}{10^{r+1}} \right]$
 $= \lim_{r \rightarrow \infty} \left[\frac{(r+1)!}{r!} \cdot \frac{1}{10} \right] = \lim_{r \rightarrow \infty} \left[\frac{(r+1) \cdot r!}{r!} \cdot \frac{1}{10} \right] = \lim_{r \rightarrow \infty} \left[\frac{(r+1)}{10} \right] = \infty > 1$
 $\therefore \sum_{r=1}^{\infty} \frac{r!}{10^r}$ DIVERGES BY THE RATIO TEST

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ • TERMS APPROACHES TO 0
 $\bullet \lim_{k \rightarrow \infty} \left(\frac{1}{\sqrt{k}} \right) = 0$
 \therefore SERIES CONVERGES BY THE ALTERNATING SERIES TEST

(d) $\sum_{m=2}^{\infty} \frac{1}{m \ln m}$ • $\lim_{m \rightarrow \infty} \left(\frac{1}{m \ln m} \right) = 0$
 $\bullet \int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{a \rightarrow \infty} \int_2^a \frac{1}{x \ln x} dx = \lim_{a \rightarrow \infty} \left[\ln |\ln x| \right]_2^a$
 $= \lim_{a \rightarrow \infty} \left[\ln |\ln a| - \ln |\ln 2| \right] = \infty$ WITHIN IS UNBOUNDED
 $\therefore \sum_{m=2}^{\infty} \frac{1}{m \ln m}$ DIVERGES BY THE INTEGRAL TEST

Question 46 (****)

Consider the infinite series

$$1 - \frac{x^2}{2} + \frac{x^4}{4 \times 2} - \frac{x^6}{6 \times 4 \times 2} + \frac{x^8}{8 \times 6 \times 4 \times 2} - \dots$$

- a) Write the above series in Sigma notation, in its simplest form.

Next consider another infinite series

$$x + \frac{x^3}{3} + \frac{x^5}{5 \times 3} + \frac{x^6}{7 \times 5 \times 3} + \frac{x^9}{9 \times 7 \times 5 \times 3} + \dots$$

- b) Also, write this series in Sigma notation, in its simplest form.

[You are not required to investigate the convergence or the sum of these series.]

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2^r r!} x^{2r}, \quad \sum_{r=0}^{\infty} \frac{2^r r!}{(2r+1)!} x^{2r}$$

Handwritten solution for Question 46:

a) $1 - \frac{x^2}{2} + \frac{x^4}{4 \times 2} - \frac{x^6}{6 \times 4 \times 2} + \frac{x^8}{8 \times 6 \times 4 \times 2} - \dots$
 $\dots = 1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} - \dots = \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r r!} x^{2r}$

b) $x + \frac{x^3}{3 \times 5 \times 7} + \frac{x^6}{6 \times 5 \times 4 \times 3 \times 2} + \dots$
 $= x + \frac{x^3}{3! \cdot 5 \cdot 7} + \frac{x^6}{6! \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \dots$
 $= x + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \dots = \sum_{r=0}^{\infty} \frac{2^r r!}{(2r+1)!} x^{2r}$

Question 47 (****)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n}$$

Show a full method, justifying every step in the workings.

divergent

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n} = \sum_{n=1}^{\infty} \frac{\frac{(2n)!}{n!(n)!}}{2^n} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n \cdot n!}$$

BY THE RATIO TEST

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{\binom{2(n+1)}{n+1}}{2^{n+1}}}{\frac{\binom{2n}{n}}{2^n}} \right| = \lim_{n \rightarrow \infty} \left[\frac{2^n \binom{2(n+1)}{n+1}}{2 \binom{2n}{n}} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \cdot \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} \right] = \lim_{n \rightarrow \infty} \left[\frac{(n+1)(2n+1)}{(n+1)^2} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{2n+1}{n+1} \right] = 2 > 1$$

\therefore SERIES DIVERGES

Question 48 (****)

$$\sum_{r=1}^n \left[\binom{n}{r} x^r (1+x+x^2)^{n-r} \right]$$

Simplify fully the above sum, into a summation free expression

$$(x+1)^{2n} - (x^2+x-1)^n$$

Let $S = \sum_{r=1}^n \binom{n}{r} x^r (1+x+x^2)^{n-r}$

Now add to both sides $\binom{n}{0} x^0 (1+x+x^2)^{n-0} = (1+x+x^2)^n$

Thus

$$S + (1+x+x^2)^n = \sum_{r=0}^n \binom{n}{r} x^r (1+x+x^2)^{n-r}$$

$$S + (1+x+x^2)^n = (1+x+x^2)^n \sum_{r=0}^n \binom{n}{r} x^r (1+x+x^2)^{-r}$$

$$S + (1+x+x^2)^n = (1+x+x^2)^n (1+x+x^2)^{-n} \sum_{r=0}^n \binom{n}{r} x^r$$

$$S + (1+x+x^2)^n = (1+x+x^2)^n (1+x+x^2)^{-n} (1+x)^n$$

$$S + (1+x+x^2)^n = (1+x)^n$$

$$S = (1+x)^n - (1+x+x^2)^n$$

i.e. $\sum_{r=1}^n \binom{n}{r} x^r (1+x+x^2)^{n-r} = (1+x)^n - (1+x+x^2)^n$

Question 49 (****+)

Consider the infinite series

$$x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{1 \times 3} + \frac{x^{\frac{5}{2}}}{(1 \times 2)(3 \times 5)} - \frac{x^{\frac{7}{2}}}{(1 \times 2 \times 3)(3 \times 5 \times 7)} + \frac{x^{\frac{9}{2}}}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9)} - \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=0}^{\infty} \left[\frac{(-2)^r}{(2r+1)!} x^{\frac{1}{2}+r} \right]$$

Handwritten solution for Question 49:

$$x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{1 \times 3} + \frac{x^{\frac{5}{2}}}{(1 \times 2)(3 \times 5)} - \frac{x^{\frac{7}{2}}}{(1 \times 2 \times 3)(3 \times 5 \times 7)} + \frac{x^{\frac{9}{2}}}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9)} - \dots$$

Looking at the fifth term (leaving 25)

$$\frac{1}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9)} = \frac{1}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17 \times 19 \times 21 \times 23 \times 25)}$$

$$= \frac{2^8 (1 \times 2 \times 3 \times 4)}{(3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17 \times 19 \times 21 \times 23 \times 25)} = \frac{2^8}{5!}$$

Thus

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} x^{\frac{1}{2}+r}}{(2r-1)!} = \sum_{r=1}^{\infty} \frac{(-2)^{r-1}}{(2r-1)!} x^{\frac{1}{2}+r-1}$$

Let

$$\sum_{r=0}^{\infty} \frac{(-2)^r}{(2r+1)!} x^{\frac{1}{2}+r}$$

Question 50 (****+)

Consider the infinite series

$$x - \frac{2}{3}(2x^2) + \frac{2 \times 2}{3 \times 5}(3x^3) - \frac{2 \times 2 \times 2}{3 \times 5 \times 7}(4x^4) + \frac{2 \times 2 \times 2 \times 2}{3 \times 5 \times 7 \times 9}(5x^5) - \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \left[\frac{2^{2n} \times (-1)^n \times (n+1)! \times x^{n+1}}{(2n+1)!} \right]$$

Handwritten solution for Question 50:

Looking at the first term (ignore the x)

$$\frac{2 \times 2 \times 2 \times 2}{3 \times 5 \times 7 \times 9} \times (5x^5) = \frac{2^4 \times 5!}{3 \times 5 \times 7 \times 9} = \frac{2^4 \times 4! \times 5}{(3 \times 5 \times 7 \times 9) \times 5} = 2^4 \times 5x^5$$

$$= \frac{2^4 \times 4!}{5!} \times 5x^5 = 2^4 \times (5x^5)$$

$$= \frac{2^4 \times 4!}{5!} \times 5x^5 \quad (n=5)$$

$\therefore \sum_{k=1}^{\infty} \frac{2^{2k} \times (k-1)! \times 2k \times (-1)^{k-1}}{(2k-1)!} = \sum_{k=1}^{\infty} \frac{2^{2k} \times 1! \times 2k \times (-1)^{k-1}}{(2k-1)!}$

$\therefore \sum_{n=0}^{\infty} \frac{2^{2n} \times n! \times (n+1) \times (-1)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2^{2(n+1)} \times 1! \times (n+1)}{(2n+1)!}$

Question 51 (****+)

Use a suitable method to sum the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)}$$

$\frac{1}{4}$

Handwritten solution for Question 51:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)} &= \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{1}{n(n+2)} = \dots \text{PARTIAL FRACTIONS OR INSPECTION} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{A}{n} - \frac{B}{n+2} \right] = \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} - \frac{(-1)^{n+1}}{n+2} \right] \\ &= \frac{1}{2} \left[1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{6} - \frac{1}{5} + \frac{1}{7} - \frac{1}{6} + \dots \right] \\ &= \frac{1}{2} \left[1 - \frac{1}{2} \right] = \frac{1}{4} \end{aligned}$$

Question 52 (****+)

Consider the infinite series

$$1 + \frac{2}{1 \times 1} + \frac{6}{(1 \times 2)(1 \times 3)} + \frac{10}{(1 \times 2 \times 3)(1 \times 3 \times 5)} + \frac{15}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=0}^{\infty} \frac{(r+1)(r+2)2^{r-1}}{(2r)!}$$

Handwritten solution for Question 52:

1 + $\frac{2}{1 \times 1} + \frac{6}{(1 \times 2)(1 \times 3)} + \frac{10}{(1 \times 2 \times 3)(1 \times 3 \times 5)} + \frac{15}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} + \dots$

NUMERATORS ARE TRIANGLE NUMBERS $1, 2, 3, 6, 10, 15, 21, 28, \dots$ i.e. $\frac{1}{2}r(r+1)$

Denominators are the fifth term

$$\frac{15}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} = \frac{15 \times \frac{1}{2} \times 4 \times 5}{(1 \times 2 \times 3 \times 4) \times (1 \times 3 \times 5 \times 7 \times 2)} = \frac{15 \times 2^2}{(1 \times 2 \times 3 \times 4) \times 6!} = \frac{15 \times 2^2}{6!}$$

$\therefore \sum_{r=0}^{\infty} \frac{\frac{1}{2}r(r+1) \times 2^{r-1}}{[2(r-1)]!} = \sum_{r=1}^{\infty} \frac{r(r+1) \times 2^{r-2}}{(2r-2)!}$

or $\sum_{r=0}^{\infty} \frac{\frac{1}{2}r(r+1) \times 2^r}{(2r)!} = \sum_{r=0}^{\infty} \frac{r(r+1) \times 2^{r-1}}{(2r)!}$

Question 53 (****+)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3+4}$$

convergent

Handwritten solution for Question 53 using the comparison test:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3+4} &< \sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3} = \sum_{n=1}^{\infty} \frac{(2n+3)^{\frac{1}{2}}}{n^2} \\ &< \sum_{n=1}^{\infty} \frac{(4n+4)^{\frac{1}{2}}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(n+1)^{\frac{1}{2}}}{n^2} \\ &< 2 \sum_{n=1}^{\infty} \frac{(n+1)^{\frac{1}{2}}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(2n)^{\frac{1}{2}}}{n^2} \\ &= 2\sqrt{2} \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} = 2\sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \end{aligned}$$

Withal concludes by the p-test

$$\sum \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Question 54 (****+)

A sequence is generated by the function

$$u_r(\theta) \equiv r \sin(\theta + r\pi), \quad r \in \mathbb{N}.$$

Find an expression or the value, whichever is appropriate, for each of the series

a) $\sum_{r=1}^{40} u_r(\theta).$

b) $\sum_{r=1}^{40} \left[u_r\left(\frac{\pi}{6}\right) \right]^2.$

$$20\sin\theta, \quad 5535$$

Handwritten solution for Question 54:

a) $\sum_{r=1}^{40} r \sin(\theta + r\pi) = \sin(\theta + \pi) + 2\sin(\theta + 2\pi) + 3\sin(\theta + 3\pi) + \dots$
 $= -\sin\theta + 2\sin\theta - 3\sin\theta + 4\sin\theta - \dots$
 $= (-\sin\theta + 2\sin\theta) + (-3\sin\theta + 4\sin\theta) + \dots$
 $= \sin\theta + \sin\theta + \dots + \sin\theta$
 $= 20 \text{ terms } (10 \text{ pairs})$
 $= 20\sin\theta$

b) $\sum_{r=1}^{40} r^2 \sin^2\left(\theta + r\pi\right) = \left[\sin^2(\theta + \pi) + 2^2 \sin^2(\theta + 2\pi) + 3^2 \sin^2(\theta + 3\pi) + \dots \right]$
 $= \sin^2\theta + 4\sin^2\theta + 9\sin^2\theta + \dots + 1600\sin^2\theta$
 $= \sin^2\theta (1 + 4 + 9 + \dots + 1600)$
 $= \sin^2\theta \times \frac{40 \times 41 \times 81}{6}$
 $= \frac{1}{4} \times \frac{40}{2} \times 41 \times 81$
 $= \frac{1}{2} \times 41 \times 81$
 $= 5 \times 41 \times 27$
 $= 5535$

Question 55 (****+)

Consider the infinite series

$$1 + \frac{1}{1 \times 5} + \frac{1}{(1 \times 2)(5 \times 8)} + \frac{1}{(1 \times 2 \times 3)(5 \times 8 \times 11)} + \frac{1}{(1 \times 2 \times 3 \times 4)(5 \times 8 \times 11 \times 14)} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=1}^{\infty} \frac{\Gamma\left(\frac{5}{3}\right)}{3^{r-1} \times (r-1)! \times \Gamma\left(\frac{3r+2}{3}\right)} = \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{5}{3}\right)}{3^r \times r! \times \Gamma\left(\frac{3r+5}{3}\right)}$$

Handwritten solution for Question 55:

$$1 + \frac{1}{1 \times 5} + \frac{1}{(1 \times 2)(5 \times 8)} + \frac{1}{(1 \times 2 \times 3)(5 \times 8 \times 11)} + \dots$$

Look at the first term (r=1):

$$\frac{1}{(1 \times 2 \times 3)(5 \times 8 \times 11)} = \frac{1}{4! \times 3^2 \times \left(\frac{5}{3} + \frac{2}{3}\right)} = \frac{\Gamma\left(\frac{5}{3}\right)}{4! \times 3^2 \times \Gamma\left(\frac{5}{3}\right) \times \frac{5}{3} \times \frac{8}{3} \times \frac{11}{3}}$$

$$= \frac{\Gamma\left(\frac{5}{3}\right)}{4! \times 3^2 \times \Gamma\left(\frac{5}{3}\right)} \times \text{variable}$$

Thus:

$$\sum_{r=1}^{\infty} \frac{\Gamma\left(\frac{5}{3}\right)}{(r-1)! \times 3^{r-1} \times \Gamma\left(\frac{3r+2}{3}\right)} = \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{5}{3}\right)}{r! \times 3^r \times \Gamma\left(\frac{3r+5}{3}\right)}$$

Question 56 (****+)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}.$

b) $\sum_{k=1}^{\infty} k^2 e^{-k^2}.$

c) $\sum_{r=1}^{\infty} \frac{(-1)^r}{\ln(r+1)}.$

d) $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}.$

divergent, convergent, convergent, convergent

a) $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}} + \sum_{n=5}^{\infty} \frac{1}{2 + \sqrt{n}} = A + \sum_{n=5}^{\infty} \frac{1}{2 + \sqrt{n}}$
 $= A + \sum_{n=5}^{\infty} \frac{1}{2 + \sqrt{n}} > A + \sum_{n=5}^{\infty} \frac{1}{2 + \sqrt{n}}$
 $= A + \sum_{n=5}^{\infty} \left(\frac{1}{2 + \sqrt{n}} - \frac{1}{2 + \sqrt{n-1}} \right) = A + \sum_{n=5}^{\infty} \frac{1}{(2 + \sqrt{n})(2 + \sqrt{n-1})}$
 $\therefore \sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$ is divergent by the comparison test

b) $\sum_{k=1}^{\infty} k^2 e^{-k^2}$ by the ratio test $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 e^{-(k+1)^2}}{k^2 e^{-k^2}} \right|$
 $= \lim_{k \rightarrow \infty} \left[\left(\frac{k+1}{k} \right)^2 e^{-2k-1} \right]$
 $= \lim_{k \rightarrow \infty} \left[\left(1 + \frac{1}{k} \right)^2 e^{-2k-1} \right] = 1 \times 0 = 0 < 1$
 \therefore Series converges by the ratio test

c) $\sum_{r=1}^{\infty} \frac{(-1)^r}{\ln(r+1)}$
 \bullet Terms alternate in sign
 $\bullet \lim_{r \rightarrow \infty} \frac{1}{\ln(r+1)} = 0$
 \therefore Series converges by the alternating series test

d) $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}$
 $\bullet \lim_{m \rightarrow \infty} \left[\frac{1}{m(\ln m)^2} \right] = 0$
 $\bullet \int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{a \rightarrow \infty} \int_2^a \frac{1}{x(\ln x)^2} dx$
 $= \lim_{a \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_2^a = \lim_{a \rightarrow \infty} \left[-\frac{1}{\ln a} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$
 \therefore Series converges by the integral test

Question 57 (****+)

The following convergent series S is given below

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta \dots$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{\sin \theta}{10 + 6 \cos \theta}$$

proof

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta + \dots$$

$$C = \cos \theta - \frac{1}{3} \cos 2\theta + \frac{1}{9} \cos 3\theta - \frac{1}{27} \cos 4\theta$$

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta$$

$$C + iS = (\cos \theta + i \sin \theta) - \frac{1}{3} (\cos 2\theta + i \sin 2\theta) + \frac{1}{9} (\cos 3\theta + i \sin 3\theta) - \frac{1}{27} (\cos 4\theta + i \sin 4\theta) + \dots$$

$$C + iS = e^{i\theta} - \frac{1}{3} e^{i2\theta} + \frac{1}{9} e^{i3\theta} - \frac{1}{27} e^{i4\theta} + \dots$$

GEOMETRIC PROGRESSION WITH FIRST TERM $e^{i\theta}$, COMMON RATIO $(-\frac{1}{3} e^{i\theta})$

$$S_{\infty} \text{ formula: } \frac{a}{1-r} = \frac{e^{i\theta}}{1 - (-\frac{1}{3} e^{i\theta})} = \frac{e^{i\theta}}{1 + \frac{1}{3} e^{i\theta}} = \frac{3e^{i\theta}}{3 + e^{i\theta}} = \frac{3e^{i\theta}(1 - e^{i\theta})}{(3 + e^{i\theta})(1 - e^{i\theta})} = \frac{3e^{i\theta} - 3e^{i2\theta}}{9 - 3e^{i\theta} + e^{i\theta} - e^{i2\theta}} = \frac{3e^{i\theta} - 3e^{i2\theta}}{10 - 6\cos \theta}$$

$$= \frac{3(e^{i\theta} - e^{i2\theta})}{10 - 6\cos \theta} = \frac{3(e^{i\theta} - e^{i2\theta})}{10 - 6\cos \theta}$$

THE REQUIRED RESULT IS THE IMAGINARY PART OF THE EXPRESSION, I.E. $\sum_{n=1}^{\infty} (-\frac{1}{3})^{n-1} \sin n\theta = \frac{\sin \theta}{10 + 6\cos \theta}$

Question 58 (****+)

A sequence of positive integers is generated by

$$u_n = 3^n - 1, \quad n = 1, 2, 3, 4, \dots$$

- a) Write down the first seven terms of this sequence.
- b) Verify that

$$u_{n+1} = 3u_n + 2.$$

- c) Show clearly that ...

i. $\dots \frac{1}{u_{n+1}} < \frac{1}{3} \times \frac{1}{u_n}.$

ii. $\dots \frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \frac{1}{2186} + \dots < \frac{1}{8} \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots \right]$

- d) Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{u_n} < \frac{11}{16}.$$

2, 8, 26, 80, 242, 728, 2186, ...

(a) $u_n = 3^n - 1$ values: 2, 8, 26, 80, 242, 728, 2186, ...

(b) $u_{n+1} = 3(3^n - 1) + 2 = 3 \cdot 3^n - 3 + 2 = 3^{n+1} - 1$

(c) i) $\frac{1}{u_{n+1}} = \frac{1}{3u_n + 2} < \frac{1}{3u_n} = \frac{1}{3} \times \frac{1}{u_n}$

ii) $\frac{1}{u_2} < \frac{1}{3} \times \frac{1}{u_1} \Rightarrow \frac{1}{8} < \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
 $\frac{1}{u_3} < \frac{1}{3} \times \frac{1}{u_2} \Rightarrow \frac{1}{26} < \frac{1}{3} \times \frac{1}{8} = \frac{1}{24}$
 $\frac{1}{u_4} < \frac{1}{3} \times \frac{1}{u_3} \Rightarrow \frac{1}{80} < \frac{1}{3} \times \frac{1}{26} = \frac{1}{78}$
 $\frac{1}{u_5} < \frac{1}{3} \times \frac{1}{u_4} \Rightarrow \frac{1}{242} < \frac{1}{3} \times \frac{1}{80} = \frac{1}{240}$
 $\frac{1}{u_6} < \frac{1}{3} \times \frac{1}{u_5} \Rightarrow \frac{1}{728} < \frac{1}{3} \times \frac{1}{242} = \frac{1}{726}$

$\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \dots < \frac{1}{24} + \frac{1}{78} + \frac{1}{240} + \dots < \frac{1}{24} \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right]$

(d) $\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \dots < \frac{1}{24} \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right]$
 $\left(\frac{1}{2} + \frac{1}{6} \right) + \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right) < \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$
 $\sum_{n=1}^{\infty} \frac{1}{u_n} < \frac{11}{16}$

Question 59 (****+)

$$u_n = \frac{\sqrt{n} + 1}{\sqrt{n^3 - n}}, \quad n \in \mathbb{N}, \quad n \geq 5.$$

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of

$$\sum_{n=5}^{\infty} u_n.$$

convergent

Handwritten solution for Question 59:

$$u_n = \frac{\sqrt{n} + 1}{\sqrt{n^3 - n}} = \frac{(\sqrt{n} + 1)(\sqrt{n^3 - n})}{(\sqrt{n^3 - n})(\sqrt{n^3 - n})} = \frac{n^2 + n\sqrt{n^3 - n}}{n^3 - n} = \frac{n^2 + 3n\sqrt{n}}{n^3 - n^2}$$

$$> \frac{n^2 + 3n\sqrt{n}}{n^3} = \frac{1}{n} + \frac{3}{n^{5/2}} + \frac{1}{n^2}$$

$\therefore \sum_{n=5}^{\infty} u_n > \sum_{n=5}^{\infty} \frac{1}{n} + 3 \sum_{n=5}^{\infty} \frac{1}{n^{5/2}} + \sum_{n=5}^{\infty} \frac{1}{n^2}$

(DIVERGES) (CONVERGES) (CONVERGES)

By the p-test $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \leq 1 \end{cases}$

$\therefore \sum_{n=5}^{\infty} u_n$ is DIVERGES

Question 60 (****+)

Show clearly that

$$\sum_{n=1}^{\infty} \frac{4n-3}{n!} = e + 3.$$

proof

Handwritten solution for Question 60:

$$\sum_{n=1}^{\infty} \frac{4n-3}{n!} = \sum_{n=1}^{\infty} \left(\frac{4n}{n!} - \frac{3}{n!} \right) = 4 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - 3 \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$= 4 \left[1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right] - 3 \left[1 + \frac{1}{1!} + \frac{1}{2!} + \dots \right]$$

$$= 4e - 3(e-1) = e + 3$$

Question 61 (****+)

Consider the infinite series

$$x + \frac{x^3}{3^2} + \frac{x^5}{5^2 \times 3^2} + \frac{x^7}{7^2 \times 5^2 \times 3^2} + \frac{x^9}{9^2 \times 7^2 \times 5^2 \times 3^2} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \left[\frac{2^{2n} (n!)^2}{[(2n+1)!]^2} x^{2n+1} \right]$$

$$\begin{aligned} & x + \frac{x^3}{3^2} + \frac{x^5}{5^2 \times 3^2} + \frac{x^7}{7^2 \times 5^2 \times 3^2} + \dots \\ &= x \left(\frac{2^0}{(3 \times 2)^2} x^2 + \left(\frac{2^1 \times 1^2}{(5 \times 3 \times 2)^2} \right) x^4 + \left(\frac{2^2 \times 1^2 \times 2^2}{(7 \times 5 \times 3 \times 2)^2} \right) x^6 + \dots \right) \\ &= x \left(\frac{2^0 (1!)^2}{(3 \times 2)^2} x^2 + \left(\frac{2^1 (2 \times 1)^2}{(5 \times 3 \times 2)^2} \right) x^4 + \left(\frac{2^2 (3 \times 2 \times 1)^2}{(7 \times 5 \times 3 \times 2)^2} \right) x^6 + \dots \right) \\ &= \sum_{n=0}^{\infty} \frac{2^n (n!)^2}{(2n+1)!^2} x^{2n+1} \end{aligned}$$

Question 62 (****+)

Show clearly that

$$\sum_{r=0}^{\infty} \frac{r+4}{(r+2)!} = 3e - 5.$$

proof

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{r+4}{(r+2)!} &= \frac{4}{2!} + \frac{5}{3!} + \frac{6}{4!} + \frac{7}{5!} + \frac{8}{6!} + \dots \\ \text{HENCE WE MAY MANIPULATE IT AS FOLLOWS} \\ \sum_{r=0}^{\infty} \frac{(r+2)+2}{(r+2)!} &= \sum_{r=0}^{\infty} \frac{r+2}{(r+2)!} + \sum_{r=0}^{\infty} \frac{2}{(r+2)!} = \sum_{r=0}^{\infty} \frac{1}{(r+1)!} + 2 \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \\ \text{WRITE A FEW TERMS FROM EACH SUM} \\ &= \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \right] + 2 \left[\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \right] \\ &= \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \right] + 2 \left[\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \right] \\ &= [-1 + e] + 2[-1 + e] \\ &= 3e - 5 \end{aligned}$$

Question 63 (****+)

$$I_n = \int_0^{\ln 2} \tanh^n x \, dx, \quad n \in \mathbb{N}.$$

By considering a reduction formula for I_n , or otherwise, show clearly that

$$\sum_{r=1}^{\infty} \frac{1}{2r} \left(\frac{3}{5}\right)^{2r} = \ln\left(\frac{5}{4}\right).$$

proof

$$I_n = \int_0^{\ln 2} \tanh^n x \, dx = \int_0^{\ln 2} \tanh^{n-2} x \tanh^2 x \, dx = \int_0^{\ln 2} \tanh^{n-2} x (1 - \operatorname{sech}^2 x) \, dx$$

$$= \int_0^{\ln 2} \tanh^{n-2} x \, dx - \int_0^{\ln 2} \tanh^{n-2} x \operatorname{sech}^2 x \, dx = I_{n-2} - \int_0^{\ln 2} \tanh^{n-2} x \operatorname{sech}^2 x \, dx$$

$$= I_{n-2} - \frac{1}{n-1} \left[\tanh^{n-1} x \right]_0^{\ln 2} = I_{n-2} - \frac{1}{n-1} (\tanh(\ln 2))^{n-1}$$

Now $\tanh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} = \frac{2-1}{2+1} = \frac{1}{3}$

$$\therefore I_n = I_{n-2} - \frac{1}{n-1} \left(\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{n-1} \left(\frac{1}{3}\right)^{n-1} = I_{n-2} - I_n$$

$n=3: \frac{1}{2} \left(\frac{1}{3}\right)^2 = I_1 - I_3$
 $n=5: \frac{1}{4} \left(\frac{1}{3}\right)^4 = I_3 - I_5$
 $n=7: \frac{1}{6} \left(\frac{1}{3}\right)^6 = I_5 - I_7$
 $n=9: \frac{1}{8} \left(\frac{1}{3}\right)^8 = I_7 - I_9$
 \vdots

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{1}{3}\right)^{2n} = I_1 = \int_0^{\ln 2} \tanh x \, dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x} \, dx$$

$$= \left[\ln |\cosh x| \right]_0^{\ln 2} = \ln(\cosh(\ln 2)) - \ln(\cosh 0)$$

$$= \ln \left[\frac{1}{2} e^{\ln 2} + \frac{1}{2} e^{-\ln 2} \right] - \ln 1 = \ln \left[1 + \frac{1}{4} \right]$$

$$= \ln \frac{5}{4}$$

Question 64 (****+)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \left[\frac{n+2}{n^2+2} \right]$

b) $\sum_{n=1}^{\infty} \left[\frac{2^n + n^2}{3^n} \right]$

divergent, convergent

(a) $\sum_{n=1}^{\infty} \frac{n+2}{n^2+2} > \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$
 $\therefore \sum_{n=1}^{\infty} \frac{n+2}{n^2+2} > \sum_{n=1}^{\infty} \frac{1}{n}$ (DIVERGENT)
 $\Rightarrow \sum_{n=1}^{\infty} \frac{n+2}{n^2+2}$ IS DIVERGENT
 (b) $\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} + \sum_{n=1}^{\infty} \frac{n^2}{3^n} = A + \sum_{n=1}^{\infty} \frac{n^2}{3^n}$
 $< A + \sum_{n=2}^{\infty} \frac{2^n}{3^n} = A + 2 \sum_{n=2}^{\infty} \frac{2^{n-1}}{3^n} = A + 2 \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^{n-1}$
 $= A + 2 \times \text{CONVERGENT G.P.}$
 $\therefore \sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n}$ IS CONVERGENT BY COMPARISON
 ALTERNATIVE:
 $\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \frac{n^2}{3^n}$
 (CONVERGENT) BY RATIO TEST
 RATIO TEST: $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1} + (n+1)^2}{3^{n+1}}}{\frac{2^n + n^2}{3^n}} \right| = \lim_{n \rightarrow \infty} \left[\frac{2^{n+1} + (n+1)^2}{3^{n+1}} \times \frac{3^n}{2^n + n^2} \right] = \frac{1}{3} < 1$
 $\therefore \sum_{n=1}^{\infty} \frac{n^2}{3^n}$ CONVERGES BY THE RATIO TEST
 $\therefore \sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n}$ IS CONVERGENT

Question 65 (****+)

By considering the Mclaurin expansion of $\ln\left(\frac{1+x}{1-x}\right)$ find the value of

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r},$$

giving the final answer as the natural logarithm of an integer.

ln 3

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots$
 $= 2x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$
 $\therefore 2 \sum_{r=0}^{\infty} \frac{x^{2r+1}}{2r+1} = \ln\left(\frac{1+x}{1-x}\right)$
 $\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \sum_{r=0}^{\infty} \frac{(1/2)^{2r+1}}{2r+1} = \sum_{r=0}^{\infty} \frac{(1/2)^{2r+1}}{2r+1} = 2 \sum_{r=0}^{\infty} \frac{(1/2)^{2r+1}}{2r+1}$
 $= 2 \sum_{r=0}^{\infty} \frac{(1/2)^{2r+1}}{2r+1} = \ln\left(\frac{1+1/2}{1-1/2}\right) = \ln\left(\frac{3}{1}\right) = \ln 3$

Question 66 (****+)

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2}.$$

You may assume the series converges.

1

$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n^2 + 2n + 1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$
 Although we have repeated factors the partial fractions can easily be done by inspection
 $= \sum_{n=1}^{\infty} \left[\frac{1}{n^3} - \frac{1}{(n+1)^3} \right]$
 $= \left(\frac{1}{1^3} - \frac{1}{2^3} \right) + \left(\frac{1}{2^3} - \frac{1}{3^3} \right) + \left(\frac{1}{3^3} - \frac{1}{4^3} \right) + \left(\frac{1}{4^3} - \frac{1}{5^3} \right) + \dots$
 $= 1$

Question 67 (****+)

Sum each of the following double series.

a) $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{2^{m+n}} \right].$

b) $\sum_{m=0}^{\infty} \sum_{n=0}^m \left[\frac{1}{2^{m+n}} \right].$

,

4

,

8/3

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{2^{m+n}} \right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{2^m \cdot 2^n} \right] = \sum_{m=0}^{\infty} \left[\frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{2^n} \right]$$

$$= \sum_{m=0}^{\infty} \left[\frac{1}{2^m} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) \right]$$

$$= \sum_{m=0}^{\infty} \left[\frac{1}{2^m} \times \left(\frac{1}{1 - \frac{1}{2}} \right) \right] = \sum_{m=0}^{\infty} \left(\frac{1}{2^m} \times 2 \right)$$

$$= \sum_{m=0}^{\infty} \left(\frac{2}{2^m} \right) = (2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots)$$

$$= \frac{2}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$$

ALTERNATIVE

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{2^{m+n}} \right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{2^m \cdot 2^n} \right] = \left[\sum_{m=0}^{\infty} \frac{1}{2^m} \right] \left[\sum_{n=0}^{\infty} \frac{1}{2^n} \right]$$

$$= \left[\sum_{m=0}^{\infty} \frac{1}{2^m} \right]^2 = \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right]^2$$

$$= \left[\frac{1}{1 - \frac{1}{2}} \right]^2 = \left(\frac{1}{\frac{1}{2}} \right)^2 = 2^2 = 4$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^m \left[\frac{1}{2^{m+n}} \right] = \sum_{m=0}^{\infty} \sum_{n=0}^m \left[\frac{1}{2^m \cdot 2^n} \right] = \sum_{m=0}^{\infty} \left[\frac{1}{2^m} \sum_{n=0}^m \left(\frac{1}{2^n} \right) \right]$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^m} \left[1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^m} \right]$$

$$\text{HAI TRIANGLE}$$

$$\sum_{n=0}^m \left(\frac{1}{2^n} \right) = \frac{1 - \left(\frac{1}{2} \right)^{m+1}}{1 - \frac{1}{2}}$$

$$= \sum_{m=0}^{\infty} \left[\frac{1}{2^m} \times \frac{1 - \left(\frac{1}{2} \right)^{m+1}}{1 - \frac{1}{2}} \right]$$

$$= \sum_{m=0}^{\infty} \left[\frac{1}{2^m} \times 2 \times \left(1 - \left(\frac{1}{2} \right)^{m+1} \right) \right]$$

$$= \sum_{m=0}^{\infty} \left[\frac{2}{2^m} \left(1 - \frac{1}{2^{m+1}} \right) \right]$$

$$= \sum_{m=0}^{\infty} \left(\frac{2}{2^m} \right) - \sum_{m=0}^{\infty} \left(\frac{2}{2^{m+1}} \right)$$

$$= (2 + 1 + \frac{1}{2} + \dots) - (1 + \frac{1}{2} + \frac{1}{4} + \dots)$$

$$= \frac{2}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{2}}$$

$$= \frac{2}{\frac{1}{2}} - \frac{1}{\frac{1}{2}} = 4 - \frac{2}{1} = \frac{8}{3}$$

Consider the infinite series

Write the above series in Sigma notation, in its simplest form.

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} \Gamma\left(\frac{1}{3}\right)}{(r-1)! \times \Gamma\left(\frac{3r-2}{3}\right)} (x^2)^{r-1} = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{1}{3}\right)}{r! \times \Gamma\left(\frac{3r+1}{3}\right)} x^{2r}$$

$$1 - \frac{3a^2}{1+x} + \frac{9x^2}{(1+x)^3} - \frac{27x^4}{(1+x)^5} + \frac{81x^8}{(1+x)(1+x^2)(1+x^4)(1+x^8)} + \dots$$

Question 69 (****+)

$$\sum_{n=1}^{\infty} \left[\frac{n \pm 2}{n^2 \pm 2} \right].$$

Use a comparison test to show that all four series described by the above expression are divergent.

proof

Handwritten mathematical proof showing four cases (a, b, c, d) for the divergence of the series $\sum_{n=1}^{\infty} \left[\frac{n \pm 2}{n^2 \pm 2} \right]$. Each case uses the comparison test with a known divergent series like $\frac{1}{n}$ or $\frac{1}{n^2}$.

(a) $\sum_{n=1}^{\infty} \frac{n+2}{n^2+2}$ is compared to $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\frac{n+2}{n^2+2} > \frac{1}{n}$ for all n , and $\sum \frac{1}{n}$ diverges, $\sum \frac{n+2}{n^2+2}$ diverges.

(b) $\sum_{n=1}^{\infty} \frac{n-2}{n^2+2}$ is compared to $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\frac{n-2}{n^2+2} > \frac{1}{n}$ for all n , and $\sum \frac{1}{n}$ diverges, $\sum \frac{n-2}{n^2+2}$ diverges.

(c) $\sum_{n=1}^{\infty} \frac{n+2}{n^2-2}$ is compared to $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\frac{n+2}{n^2-2} > \frac{1}{n}$ for all n , and $\sum \frac{1}{n}$ diverges, $\sum \frac{n+2}{n^2-2}$ diverges.

(d) $\sum_{n=1}^{\infty} \frac{n-2}{n^2-2}$ is compared to $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\frac{n-2}{n^2-2} > \frac{1}{n}$ for all n , and $\sum \frac{1}{n}$ diverges, $\sum \frac{n-2}{n^2-2}$ diverges.

Question 70 (****+)

By showing a detailed method, sum the following series.

$$\frac{\pi^2}{2^2 2!} - \frac{\pi^4}{2^4 4!} + \frac{\pi^6}{2^6 6!} - \frac{\pi^8}{2^8 8!} + \dots + \frac{(-1)^{n+1} \pi^{2n}}{2^{2n} (2n)!} + \dots$$

1

$$S = \frac{\pi^2}{2! 2^2} - \frac{\pi^4}{4! 2^4} + \frac{\pi^6}{6! 2^6} - \frac{\pi^8}{8! 2^8} + \dots + \frac{\pi^{2n} (-1)^{n+1}}{(2n)! 2^{2n}} + \dots$$

• RECALL THE EXPANSION OF $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = 1 - \cos(x)$$

• LET $x = \frac{\pi}{2}$

$$\frac{(\frac{\pi}{2})^2}{2!} - \frac{(\frac{\pi}{2})^4}{4!} + \frac{(\frac{\pi}{2})^6}{6!} - \frac{(\frac{\pi}{2})^8}{8!} + \dots = 1 - \cos(\frac{\pi}{2})$$

$$\therefore \frac{\pi^2}{2^2 2!} - \frac{\pi^4}{2^4 4!} + \frac{\pi^6}{2^6 6!} - \frac{\pi^8}{2^8 8!} + \dots = 1$$

Question 71 (****+)

By showing a detailed method, sum the following series.

$$\frac{2}{1} + \frac{3}{2} + \frac{4}{4} + \frac{5}{8} + \frac{6}{16} + \frac{7}{32} \dots$$

6

• LET $S = \frac{2}{1} + \frac{3}{2} + \frac{4}{4} + \frac{5}{8} + \frac{6}{16} + \frac{7}{32} + \dots + \frac{n+1}{2^{n+1}} + \dots$
 (MULTIPLY THROUGH BY $\frac{1}{2}$)

$$\frac{1}{2}S = \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots$$

• 'LINE UP' THE TWO EXPRESSIONS AS FOLLOWS

$$S = 2 + \frac{3}{2} + \frac{4}{4} + \frac{5}{8} + \frac{6}{16} + \frac{7}{32} + \dots$$

$$-\frac{1}{2}S = -1 - \frac{3}{4} - \frac{4}{8} - \frac{5}{16} - \frac{6}{32} - \dots$$

$$\Rightarrow \frac{1}{2}S = 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

THIS IS A GEOMETRIC PROGRESSION WITH $a = \frac{1}{2}$
 $r = \frac{1}{2}$

$$\therefore S_n = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\Rightarrow \frac{1}{2}S = 2 + 1$$

$$\Rightarrow \frac{1}{2}S = 3$$

$$\Rightarrow S = 6$$

Question 72 (****+)

The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^n r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^n (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

, 5135

Handwritten solution for Question 72:

Define the individual components as simplified form

- $f(n) = \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$
- $g(n) = 1 + \sum_{r=1}^n (2r+1) = 1 + 2\sum_{r=1}^n r + \sum_{r=1}^n 1$

$$= 1 + 2 \times \frac{1}{2}n(n+1) + n$$

$$= 1 + n(n+1) + n = 1 + n^2 + n + n$$

$$= n^2 + 2n + 1 = (n+1)^2$$

Twice use above

$$\sum_{n=1}^{39} \frac{f(n)}{g(n)} = \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2$$

$$= \frac{1}{4} \times \frac{1}{6}n(n+1)(2n+1) \Big|_{n=1}^{39}$$

$$= \frac{1}{24} \times 39 \times 40 \times 41$$

$$= 5135$$

Question 73 (****+)Find the Maclaurin expansion of $\arctan x$, and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function f .

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

START WITH DIFFERENTIATION & INTEGRATION

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$\frac{d}{dx}(\arctan x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

INTEGRATING WITH RESPECT TO x , ASSUMING INTEGRATION CONSTANT

$$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$\arctan x = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{2n+1}}{2n+1} \right] + C$$

USE $x=0$ $0 = 0 + C$

$$\arctan x = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{2n+1}}{2n+1} \right]$$

FINALLY SUBSTITUTE $x=1$

$$\arctan 1 = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \right]$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \right]$$

$$\pi = \sum_{n=0}^{\infty} \left[\frac{4(-1)^n}{2n+1} \right] \quad \text{Hence } f(n) = \frac{4(-1)^n}{2n+1}$$

Question 74 (****)

It is given that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = k^n,$$

where n and k are positive integer constants.a) By considering the binomial expansion of $(1+x)^n$, determine the value of k .b) By considering the coefficient of x^n in

$$(1+x)^n (1+x)^n \equiv (1+x)^{2n},$$

simplify fully

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

$$\boxed{}, \boxed{k=2}$$

a) $(1+x)^n = \binom{n}{0} 1^n x^0 + \binom{n}{1} 1^{n-1} x^1 + \binom{n}{2} 1^{n-2} x^2 + \dots + \binom{n}{n} 1^n x^n$
 $\Rightarrow (1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n$
 LET $x=1$
 $\Rightarrow (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$
 $\Rightarrow \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

b) $(1+x)^n (1+x)^n \equiv (1+x)^{2n}$
 $\Rightarrow \left[\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n \right] \left[\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n \right]$
 $\equiv \binom{2n}{0} + \binom{2n}{1} x + \binom{2n}{2} x^2 + \dots + \binom{2n}{2n} x^{2n}$
 LOOKING AT THE COEFFICIENT OF x^n ON BOTH SIDES
 $\binom{n}{0} \binom{n}{n} + \binom{n}{1} \binom{n}{n-1} + \binom{n}{2} \binom{n}{n-2} + \dots + \binom{n}{n} \binom{n}{0} = \binom{2n}{n}$
 BUT FROM THE DEFINITION OF BINOMIAL COEFFICIENTS
 $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ e.g. $\binom{5}{3} = \frac{5!}{3!2!}$, $\binom{4}{2} = \frac{4!}{2!2!}$, $\binom{3}{2} = \frac{3!}{2!1!}$, $\binom{2}{2} = \frac{2!}{2!0!}$...
 THEREFORE
 $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$

Question 75 (*****)

By considering the binomial expansion of

$$\frac{1}{(1 - \cos \theta)^2},$$

sum each of the following series.

$$\bullet \sum_{r=1}^{\infty} \left[\frac{r}{2^{r-1}} \right] \cdot$$

$$\bullet \sum_{r=1}^{\infty} \left[\frac{r}{(-2)^{r-1}} \right] \cdot$$

$$\square, \quad \sum_{r=1}^{\infty} \left[\frac{r}{2^{r-1}} \right] = 4, \quad \sum_{r=1}^{\infty} \left[\frac{r}{(-2)^{r-1}} \right] = \frac{4}{9}$$

Start by a series substitution, $z = \cos \theta$

$$\frac{1}{(1 - \cos \theta)^2} = \frac{1}{(1 - z)^2} = (1 - z)^{-2}$$

$$= 1 + \binom{-2}{1}(-z) + \frac{\binom{-2}{2}(-z)^2}{1 \times 2 \times 1} + \dots + O(z^5)$$

$$= 1 + 2z + 3z^2 + 4z^3 + \dots \quad |z| < 1$$

$$= 1 + 2\cos \theta + 3\cos^2 \theta + 4\cos^3 \theta + \dots \quad |\cos \theta| < 1$$

$$= \sum_{r=1}^{\infty} r(\cos \theta)^{r-1}$$

Now $\sum_{r=1}^{\infty} \frac{r}{2^{r-1}} = \dots$ is the above expansion with $\cos \theta = \frac{1}{2}$
 $(z = \frac{1}{2})$

$$= \frac{1}{(1 - \frac{1}{2})^2} = \frac{1}{(\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4 //$$

And $\sum_{r=1}^{\infty} \frac{r}{(-2)^{r-1}} = \dots$ is the above expansion with $\cos \theta = -\frac{1}{2}$
 $(z = -\frac{1}{2})$

$$= \frac{1}{(1 - (-\frac{1}{2}))^2} = \frac{1}{(\frac{3}{2})^2} = \frac{1}{\frac{9}{4}} = \frac{4}{9} //$$

Question 76 (****)

$$f(x) \equiv \frac{1-x}{1+x+x^2+x^3}, \quad -1 < x < 1.$$

Show that $f(x)$ can be written in the form

$$f(x) = g(x) \sum_{r=0}^{\infty} (x^{4r}),$$

where $g(x)$ is a simplified function to be found.

$$\boxed{}, \quad \boxed{g(x) = (1-x)^2}$$

$$f(x) = \frac{1-x}{1+x+x^2+x^3} = \frac{1-x}{(1+x)(1+x^2)} = \frac{1-x}{(1+x)(1+x^2)}$$

$$= \frac{(1-x)(1-x)}{(1-x)(1+x)(1+x^2)} = \frac{(1-x)^2}{(1-x^2)(1+x^2)}$$

$$= \frac{(1-x)^2}{1-x^4}$$

NOW WE HAVE STANDARD FORM, OR THE SUM TO INFINITY OF A G.P.

$$f(x) = \frac{1-x}{1-x^4} = 1+x^4+x^8+\dots$$

$$\dots = (1-x^4)^{-1} = 1+x^4+x^8+\dots$$

$$\dots = (1-x^4)^{-1} \sum_{r=0}^{\infty} x^{4r}$$

LONGER ALTERNATIVE

$$f(x) = \frac{1-x}{(1+x)(1+x^2)} = \dots = \frac{1-x}{(1+x)(1+x^2)} \dots \text{NOW PARTIAL FRACTIONS}$$

$$\frac{1-x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$1-x \equiv A(1+x^2) + (1+x)(Bx+C)$$

if $x = -1 \Rightarrow 2 = 2A \Rightarrow A = 1$
 if $x = 0 \Rightarrow 1 = A + C \Rightarrow C = 0$
 if $x = 1 \Rightarrow 0 = 2A + 2B \Rightarrow B = -1$

HAVE WE HAVE

$$f(x) = \frac{1}{1+x} - \frac{x}{1+x^2}$$

$$f(x) = (1-x+x^2-x^3+\dots) - x(1-x^2+x^4-x^6+\dots)$$

$$f(x) = 1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+\dots$$

$$f(x) = (1-x+x^2) + (x^4-2x^5+x^6) + (x^8-2x^9+x^{10}) + \dots$$

$$f(x) = (1-x+x^2) + x^4(1-2x+x^2) + x^8(1-2x+x^2) + \dots$$

$$f(x) = (1-x+x^2) [1+x^4+x^8+\dots]$$

$$f(x) = (1-x)^2 \sum_{r=0}^{\infty} x^{4r}$$

As before

Question 77 (****)

The sum to infinity S of the convergent geometric series is given by

$$S = 1 + x + x^2 + x^3 + x^4 + \dots, \quad |x| < 1,$$

By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty} \left[\frac{1}{r \times 2^r} \right]$$

You may assume that integration and summation commute.

$$\boxed{}, \ln 2$$

• WRITE THE GEOMETRIC SERIES COMPACTLY

$$\Rightarrow \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots \quad |x| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1$$

• IN ORDER TO PRODUCE THE REQUIRED SERIES WE KNOW THE LHS IS

$$\Rightarrow \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

$$\Rightarrow \int_0^{\frac{1}{2}} \sum_{n=1}^{\infty} x^{n-1} dx = \int_0^{\frac{1}{2}} \frac{1}{1-x} dx$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{x^n}{n} \right]_0^{\frac{1}{2}} = \left[-\ln|1-x| \right]_0^{\frac{1}{2}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{\left(\frac{1}{2}\right)^n}{n} \right] = \left[-\ln\left(1-\frac{1}{2}\right) \right]_0^{\frac{1}{2}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{1}{n \times 2^n} \right] = -\ln \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{1}{n \times 2^n} \right] = \ln 2$$

• ALTERNATIVE SOLUTION USING STANDARD EXPANSIONS

(EXPANDING THE EXPRESSION OF $\ln(1-x)$)

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

Let $x = \frac{1}{2}$

$$\ln \frac{1}{2} = -\left[\frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right)^2 + \frac{1}{3} \left(\frac{1}{2}\right)^3 + \frac{1}{4} \left(\frac{1}{2}\right)^4 + \dots \right]$$

$$-\ln 2 = -\left[\frac{1}{2} + \frac{1}{2 \times 2^2} + \frac{1}{3 \times 2^3} + \frac{1}{4 \times 2^4} + \dots \right]$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{n \times 2^n} \right] = \ln 2$$

Question 78 (*****)

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{r!} = 2e.$$

proof

Handwritten proof showing the sum $\sum_{r=1}^{\infty} \frac{r^2}{r!} = 2e$. The proof uses the identity $\frac{r^2}{r!} = \frac{r}{(r-1)!} = \frac{(r-1)+1}{(r-1)!} = \frac{(r-1)}{(r-1)!} + \frac{1}{(r-1)!}$. This splits the sum into two parts: $\sum_{r=1}^{\infty} \frac{(r-1)}{(r-1)!} = \sum_{k=0}^{\infty} \frac{k}{k!} = e$ and $\sum_{r=1}^{\infty} \frac{1}{(r-1)!} = \sum_{k=0}^{\infty} \frac{1}{k!} = e$. The final result is $e + e = 2e$.

Question 79 (***)**

Investigate the convergence or divergence of each of the following two series using standard tests and justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \left[\frac{1}{n(n+3)} \right]$

b) $\sum_{n=4}^{\infty} \left[\frac{1}{n(n-3)} \right]$

You may not conclude simply by summing each the series.

, convergent , convergent

q) $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n^2+3n} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
WHICH CONVERGES BY COMPARISON

b) $\sum_{n=4}^{\infty} \frac{1}{n(n-3)} = \sum_{n=4}^{\infty} \frac{1}{n^2-3n}$
NOW USING THE DERIV
DERIVED EXPRESSION, NOT THE
 $\leq \sum_{n=4}^{\infty} \frac{1}{n^2} = \frac{1}{4} - \frac{1}{9}$
 $= 4 \left[\frac{1}{2^2} - \frac{1}{4} - \frac{1}{4} \right]$
 $= \frac{21}{3} - 1 - \frac{1}{4}$
 $= \frac{20}{3} - 1 - \frac{1}{4}$
i.e. CONVERGES BY COMPARISON

NOTES

- BOTH SERIES CAN BE INVESTIGATED VIA THE INTEGRAL TEST
- BOTH SERIES CAN BE SUMMED VERY EASILY VIA PARTIAL FRACTIONS
- ALSO $\sum_{n=4}^{\infty} \frac{1}{n(n-3)} = \sum_{n=1}^{\infty} \frac{1}{(n+3)^2}$ WHICH CONVERGES

CONSIDER THE GRAPH

$y = x^2 - 3x$
 $x^2 - 3x > \frac{1}{4}x^2$
 $4x^2 - 12x > x^2$
 $3x^2 - 12x > 0$
 $3x(x-4) > 0$
 $x < 0$ OR $x > 4$
Thus if $n > 4$
 $\frac{1}{n^2-3n} > \frac{1}{n^2}$

Question 80 (****)

The finite sum C is given below.

$$C = \sum_{r=0}^n \left[\binom{n}{r} (-1)^r \cos^n \theta \cos n\theta \right].$$

Given that $n \in \mathbb{N}$ determine the 4 possible expressions for C .

Give the answers in exact fully simplified form.

$$\boxed{\text{SF X}}, \quad \boxed{n = 4k, k \in \mathbb{N} : C = \cos n\theta \sin^n \theta}, \quad \boxed{n = 4k+1, k \in \mathbb{N} : C = \sin n\theta \sin^n \theta},$$

$$\boxed{n = 4k+2, k \in \mathbb{N} : C = -\cos n\theta \sin^n \theta}, \quad \boxed{n = 4k+3, k \in \mathbb{N} : C = -\sin n\theta \sin^n \theta}$$

Handwritten solution for Question 80:

$$C = \sum_{r=0}^n \binom{n}{r} (-1)^r \cos^n \theta \cos n\theta$$

$$C = \cos^n \theta \sum_{r=0}^n \binom{n}{r} (-1)^r \cos n\theta$$

$$C = \cos^n \theta \left[\binom{n}{0} (-1)^0 \cos n\theta + \binom{n}{1} (-1)^1 \cos n\theta + \dots + \binom{n}{n} (-1)^n \cos n\theta \right]$$

$$C = \cos^n \theta \left[\cos n\theta - n \cos^{n-1} \theta \sin \theta + \dots + (-1)^n \sin^n \theta \cos n\theta \right]$$

$$C = \cos^n \theta \left[\cos n\theta + i \sin n\theta \right] = \cos^n \theta \left[e^{in\theta} \right] = \cos^n \theta e^{in\theta}$$

$$C = \cos^n \theta \left[\cos n\theta + i \sin n\theta \right] = \cos^n \theta \left[\cos n\theta + i \sin n\theta \right]$$

$$C = \cos^n \theta \left[\cos n\theta + i \sin n\theta \right] = \cos^n \theta \left[\cos n\theta + i \sin n\theta \right]$$

Final results:

- If $n = 4k, k \in \mathbb{N} \Rightarrow (-1)^n = 1 \Rightarrow C = \cos n\theta \sin^n \theta$
- If $n = 4k+1, k \in \mathbb{N} \Rightarrow (-1)^n = -1 \Rightarrow C = \sin n\theta \sin^n \theta$
- If $n = 4k+2, k \in \mathbb{N} \Rightarrow (-1)^n = 1 \Rightarrow C = -\cos n\theta \sin^n \theta$
- If $n = 4k+3, k \in \mathbb{N} \Rightarrow (-1)^n = -1 \Rightarrow C = -\sin n\theta \sin^n \theta$

Question 81 (*****)

$$f(x) \equiv \frac{2-3x}{(1-x)(1-2x)}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

Show that $f(x)$ can be written in the form

$$f(x) = \sum_{r=0}^{\infty} [x^r g(r)],$$

where $g(r)$ is a simplified function to be found.

$$\boxed{}, \quad g(r) = 2^r + 1$$

• SIMPLY BY REWRITING & SPLITTING INTO PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow f(x) = \frac{2-3x}{(1-x)(1-2x)} = (2-3x) \times \frac{1}{(1-x)(1-2x)}$$

$$\Rightarrow f(x) = (2-3x) \times \left[\frac{A}{1-x} + \frac{B}{1-2x} \right]$$

$$\Rightarrow f(x) = (2-3x) \times \left[\frac{2}{1-2x} - \frac{1}{1-x} \right]$$

• NEXT WE USE STANDARD EXPANSIONS FOR THE SUM TO INFINITY OF A G.P.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

TO OBTAIN:

$$\Rightarrow f(x) = (2-3x) \left[2(1 + 2x + 4x^2 + 8x^3 + \dots) - (1 + x + x^2 + x^3 + \dots) \right]$$

$$\Rightarrow f(x) = (2-3x) \left[\frac{2}{1-x} - \frac{1}{1-x} \right]$$

$$\Rightarrow f(x) = (2-3x) \sum_{r=0}^{\infty} (2^r - 1)x^r$$

$$\Rightarrow f(x) = 2 \sum_{r=0}^{\infty} (2^r - 1)x^r - 3 \sum_{r=0}^{\infty} (2^r - 1)x^{r+1}$$

• ADJUST THE FIRST SUMMATION SO IT STARTS FROM r=0 AGAIN

$$\Rightarrow f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^r - 1)x^r - 3 \sum_{r=1}^{\infty} (2^{r-1} - 1)x^r$$

$$\Rightarrow f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^r - 1)x^r - 3 \sum_{r=1}^{\infty} (2^{r-1} - 1)x^r$$

• NEXT ADJUST THE FIRST SUMMATION SO IT STARTS FROM r=0 AGAIN

$$f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^r - 1)x^r - 3 \sum_{r=1}^{\infty} (2^{r-1} - 1)x^r$$

$$f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^r - 1)x^r - 3 \sum_{r=1}^{\infty} (2^{r-1} - 1)x^r$$

$$f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^r - 1)x^r - 3 \sum_{r=1}^{\infty} (2^{r-1} - 1)x^r$$

$$f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^r - 1)x^r - 3 \sum_{r=1}^{\infty} (2^{r-1} - 1)x^r$$

• ADJUST THE SUMMATION SO THAT IT STARTS FROM r=1

$$f(x) = 2 + \sum_{r=1}^{\infty} (2^{r+1} - 3 \cdot 2^{r-1})x^r$$

$$f(x) = (2^{r+1} - 3 \cdot 2^{r-1})x^r + \sum_{r=1}^{\infty} (2^{r+1} - 3 \cdot 2^{r-1})x^r$$

$$f(x) = \sum_{r=0}^{\infty} (2^{r+1} - 3 \cdot 2^{r-1})x^r$$

ANOTHER APPROACH: THIS IS NOT AS FORMAL AS THE OTHERS

$$f(x) = (2-3x) \left(\frac{2}{1-x} - \frac{1}{1-2x} \right)$$

$$f(x) = 2 + 6x + 14x^2 + 30x^3 + \dots$$

$$f(x) = 2 + 6x + 14x^2 + 30x^3 + \dots$$

WHICH ONE WOULD YOU PREFER IS

$$\sum_{r=0}^{\infty} (2^{r+1} - 3 \cdot 2^{r-1})x^r$$

Question 82 (*****)

Show, by considering standard series, that

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}.$$

You may assume without proof that $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} \right] = \frac{1}{6} \pi^2$

, proof

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}$$
 • USING THE EXPANSION OF $\ln(1+x)$

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} dx$$

$$= \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n-1}}{n} dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^1 x^{n-1} dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{x^n}{n} \right]_{x=0}^1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots$$
 • NOW $\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$$

$$\frac{1}{2^2} \zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \dots = \frac{\pi^2}{24}$$

$$\left\{ \begin{aligned} \zeta(2) &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6} \\ -2 \times \frac{1}{2^2} \zeta(2) &= -\frac{2}{2^2} - \frac{2}{4^2} - \frac{2}{6^2} - \dots = -\frac{2\pi^2}{24} \end{aligned} \right\}$$
 • THEREFORE $\frac{1}{2^2} \zeta(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots = \frac{\pi^2}{12}$

$$\therefore \int_0^1 \frac{\ln(1+x)}{x} dx = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots = \frac{\pi^2}{12}$$

Question 83 (*****)

Show, by a detailed method, that

$$\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} \dots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r},$$

where A and B are constants to be found.

$$\boxed{}, \quad A = \frac{51}{2}, \quad B = -1$$

$$\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} + \dots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r}$$

METHOD: USES IN SIMILAR INTERPRETATION

$$\sum_{k=1}^{50} \frac{49-k}{(k+1)(k+2)}$$

REWRITE IN PARTIAL FRACTIONS

$$\sum_{k=1}^{50} \frac{49-k}{(k+1)(k+2)} = \sum_{k=1}^{50} \left(\frac{50}{k+1} - \frac{51}{k+2} \right)$$

$$= \left(\frac{50}{2} - \frac{51}{3} \right) + \left(\frac{50}{3} - \frac{51}{4} \right) + \dots + \left(\frac{50}{48} - \frac{51}{49} \right) + \left(\frac{50}{49} - \frac{51}{50} \right)$$

$$= 25 - \left[\frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} \right] - \frac{51}{50}$$

$$= 25 - \left[\frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} \right] - 1 - \frac{1}{50}$$

$$= 25 - \left[1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= 25 + \frac{1}{2} - \left[1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= \frac{51}{2} - \sum_{r=1}^{50} \frac{1}{r}$$

Hence $A = \frac{51}{2}$
 $B = -1$

Question 84 (*****)

$$S = 1 + \frac{2}{4} + \frac{2 \cdot 3}{4 \cdot 8} + \frac{2 \cdot 3 \cdot 4}{4 \cdot 8 \cdot 12} + \frac{2 \cdot 3 \cdot 4 \cdot 5}{4 \cdot 8 \cdot 12 \cdot 16} + \dots$$

By considering a suitable binomial series, or other wise, find the sum to infinity of S .

$$\boxed{}, S_{\infty} = \frac{16}{9}$$

MINIMISE THE SERIES STEP BY STEP

$$\begin{aligned} \Rightarrow S &= 1 + \frac{2}{4} + \frac{2 \cdot 3}{4 \cdot 8} + \frac{2 \cdot 3 \cdot 4}{4 \cdot 8 \cdot 12} + \dots \\ \Rightarrow S &= 1 + \frac{2}{4(1)} + \frac{2 \cdot 3}{4^2(1 \cdot 2)} + \frac{2 \cdot 3 \cdot 4}{4^3(1 \cdot 2 \cdot 3)} + \dots \\ \Rightarrow S &= 1 + \frac{2}{4^1(1)} + \frac{2 \cdot 3}{4^2(1 \cdot 2)} + \frac{2 \cdot 3 \cdot 4}{4^3(1 \cdot 2 \cdot 3)} + \dots \end{aligned}$$

FINALLY WE NEED TO TAKE ONE OF THE SERIES, IN ORDER TO FORM A CONVERGENT BINOMIAL EXPANSION

$$\begin{aligned} \Rightarrow S &= 1 + \frac{2}{4^1(1)} + \frac{2 \cdot 3}{4^2(1 \cdot 2)} + \frac{2 \cdot 3 \cdot 4}{4^3(1 \cdot 2 \cdot 3)} + \dots \\ \Rightarrow S &= \left(1 - \frac{1}{4}\right)^{-2} \\ \Rightarrow S &= \left(\frac{3}{4}\right)^{-2} \\ \Rightarrow S &= \left(\frac{4}{3}\right)^2 \\ \Rightarrow S &= \frac{16}{9} \end{aligned}$$

Question 85 (*****)

$$3 + 33 + 333 + 3333 + 33333 + \dots$$

Express the sum of the first n terms of the above series in sigma notation.

You are not required to sum the series.

$$S_n = \sum_{r=1}^n \left[\frac{1}{3} (10^r - 1) \right]$$

$$\begin{aligned} S &= 3 + 33 + 333 + 3333 + 33333 + \dots \\ S &= \left(\frac{3}{10} \times 1\right) + \left(\frac{3}{10} \times 10\right) + \left(\frac{3}{10} \times 100\right) + \left(\frac{3}{10} \times 1000\right) + \dots \\ S &= \frac{3}{10} [1 + 10 + 100 + 1000 + \dots] \\ S &= \frac{3}{10} [(10^0 - 1) + (10^1 - 1) + (10^2 - 1) + (10^3 - 1) + \dots] \\ S &= \frac{3}{10} [(10^0 + 10^1 + 10^2 + 10^3 + \dots) - (-1 - 1 - 1 - 1 - \dots)] \\ S &= \frac{3}{10} \sum_{r=1}^n (10^r - 1) \end{aligned}$$

Question 86 (*****)

$$S_n = (2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

, **proof**

START BY WRITING THE SERIES IN SUMMATION NOTATION

$$(2 \times 1!) + (5 \times 2!) + (10 \times 3!) + \dots + (n^2 + 1)n! = \sum_{r=1}^n [(r^2 + 1)r!]$$

TRY SOME DIFFERENCES INVOLVING FACTORIALS, TRYING TO OBTAIN THE SUMMATION

$$[(r+1)!] - r! = (r+1)r! - r! = r \times r!$$

AS THIS DOES NOT PRODUCE A QUANTIFIABLE TERM IN r WE MAY TRY

$$\begin{aligned} (r+2)! - r! &= (r+2)(r+1)r! - r! \\ (r+2)! - r! &= (r^2 + 3r + 2)r! - r! \\ (r+2)! - r! &= (r^2 + 3r + 1)r! \\ (r+2)! - r! &= (r^2 + 1)r! + 2r \times r! \end{aligned}$$

\uparrow
YOU WANTED $r \times r! = (r+1)! - r!$

$$\begin{aligned} (r+2)! - r! &= (r^2 + 1)r! + 2[(r+1)! - r!] \\ (r+2)! - r! &= (r^2 + 1)r! + 2(r+1)! - 2r! \\ (r+2)! &= (r^2 + 1)r! + 2(r+1)! - 2r! \\ (r+2)! - 3(r+1)! + 2r! &= (r^2 + 1)r! \end{aligned}$$

HENCE WE HAVE

$$[(r^2 + 1)r!] \equiv (r+2)! - 3(r+1)! + 2r!$$

WRITING THE IDENTITY JUST OBTAINED

$$[(r^2 + 1)r!] \equiv (r+2)! - 3(r+1)! + 2r!$$

$r=1$	$2 \times 1! =$	$3! - 3 \times 2! + 2 \times 1!$
$r=2$	$5 \times 2! =$	$4! - 3 \times 3! + 2 \times 2!$
$r=3$	$10 \times 3! =$	$5! - 3 \times 4! + 2 \times 3!$
$r=4$	$17 \times 4! =$	$6! - 3 \times 5! + 2 \times 4!$
\vdots	\vdots	\vdots
$r=n$	$(n^2 + 1)n! =$	$(n+2)! - 3(n+1)! + 2n \times n!$
$r=n$	$(n^2 + 1)n! =$	$(n+2)! - 3(n+1)! + 2n \times n!$

$$\sum_{r=1}^n [(r^2 + 1)r!] = (n+2)! - 3(n+1)! + 2n! + 2 \times 1!$$

$$= (n+2)(n+1)! - 3(n+1)! + 2n! + 2$$

$$= (n+2-3)(n+1)! + 2n! + 2$$

$$= n(n+1)! + 2n! + 2$$

Question 87 (****)

Consider the infinite series

$$1 + \frac{-1}{2 \times 1} x^2 + \frac{-1 \times 1}{4 \times 3 \times 2 \times 1} x^4 + \frac{-1 \times 1 \times 3}{6 \times 5 \times 4 \times 3 \times 2 \times 1} x^6 + \frac{-1 \times 1 \times 3 \times 5}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} x^8 + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \left[\frac{2^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{-\sqrt{\pi} \times (2n)!} x^{2n} \right]$$

Handwritten solution for Question 87. It shows the series terms and then uses the Gamma function to express the general term. The final result is $\sum_{n=0}^{\infty} \left[\frac{2^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{-\sqrt{\pi} \times (2n)!} x^{2n} \right]$.

Question 88 (****)

$$\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \frac{9}{1^2 + 2^2 + 3^2 + 4^2} + \frac{11}{1^2 + 2^2 + 3^2 + 4^2 + 5^2} + \dots$$

Show, by a detailed method, that the sum of the first 40 terms of the series shown above is $\frac{240}{41}$.

 , proof

Handwritten solution for Question 88. It shows the series terms and then uses the formula for the sum of the first n terms of an arithmetic series to find the sum of the first 40 terms. The final result is $\frac{240}{41}$.

Question 89 (*****)

By showing a detailed method, sum the following series.

$$\frac{1}{2^2 2!} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \dots + \frac{1}{2^{2r} (2r)!} + \dots$$

$$\frac{1}{2} \left[e^{\frac{1}{4}} - e^{-\frac{1}{4}} \right]^2 = 2 \sinh^2 \left(\frac{1}{4} \right)$$

Handwritten solution for Question 89:

Let $S = \frac{1}{2^2 2!} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \dots$

Consider the series expansion of $e^x = \cosh x + \sinh x$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$

$\cosh(x) = 1 + \frac{1}{2! 2^2} + \frac{1}{4! 2^4} + \frac{1}{6! 2^6} + \frac{1}{8! 2^8} + \dots$

$\sinh(x) = x + \frac{1}{3! 2^3} + \frac{1}{5! 2^5} + \frac{1}{7! 2^7} + \frac{1}{9! 2^9} + \dots$

$\cosh(x) = 1 + S$

$S = \cosh(x) - 1$

$S = \frac{1}{2} e^{\frac{1}{2}} + \frac{1}{2} e^{-\frac{1}{2}} - 1$

$S = \frac{1}{2} (e^{\frac{1}{2}} + e^{-\frac{1}{2}}) - 1$

$S = \frac{1}{2} (e^{\frac{1}{2}} - e^{-\frac{1}{2}})^2$

$S = \frac{1}{2} (e^{\frac{1}{4}} - e^{-\frac{1}{4}})^2$

$S = 2 \sinh^2 \left(\frac{1}{4} \right)$

Question 90 (****)

A function is defined as

$$\lceil x \rceil \equiv \{\text{the greatest integer less or equal to } x\}.$$

The function f is defined as

$$f(n) = n \left\lceil \frac{3}{5} + \frac{3n}{100} \right\rceil, \quad n \in \mathbb{N}.$$

Determine the value of

$$\sum_{n=1}^{82} f(n).$$

$$\boxed{}, \boxed{5877}$$

$\lceil x \rceil \equiv \{\text{GREATEST INTEGER LESS OR EQUAL TO } x\}$

• $f(n) = n \left\lceil \frac{3}{5} + \frac{3n}{100} \right\rceil, \quad n \in \mathbb{N}$

• WE USED TO INVESTIGATE THE SIZE OF THE FIRST 82 TERMS
WE USED TO WORK IN GROUPS

$\bullet \frac{3}{5} + \frac{3n}{100} < 1$ $\frac{3n}{100} < \frac{2}{5}$ $3n < 40$ $n < \frac{40}{3} = 13\frac{1}{3}$	$\bullet \frac{3}{5} + \frac{3n}{100} \leq 2$ $\frac{3n}{100} \leq \frac{7}{5}$ $3n \leq 140$ $n \leq \frac{140}{3} = 46\frac{2}{3}$	$\bullet \frac{3}{5} + \frac{3n}{100} \leq 3$ $\frac{3n}{100} \leq \frac{12}{5}$ $3n \leq 240$ $n \leq 80$
---	---	---

\therefore THE FIRST 13 TERMS OF $\lceil \dots \rceil$ ARE ALL 1

THE 'THIRDS' OF $\lceil \dots \rceil$ FROM 14th TO 46th ARE 2

THE TERMS OF $\lceil \dots \rceil$ FROM 47th TO 79th ARE 3

[EXACTLY THE 80th, 81st, 82nd TERMS OF $\lceil \dots \rceil$ ARE 3]

• SUMMING UP THE SERIES

$$\sum_{n=1}^{82} f(n) = \sum_{n=1}^{82} n \left\lceil \frac{3}{5} + \frac{3n}{100} \right\rceil$$

$$= \left(\sum_{n=1}^{13} n \times 1 \right) + \left(\sum_{n=14}^{46} n \times 2 \right) + \left(\sum_{n=47}^{79} n \times 3 \right) + \left(\sum_{n=80}^{82} n \times 3 \right)$$

$46 - 13 = 33$ $a = 14$ $d = 1$ $L = 46$ $n = 33$	$79 - 46 = 33$ $a = 47$ $d = 1$ $L = 79$ $n = 33$	$82 - 79 = 3$ $a = 80$ $d = 1$ $L = 82$ $n = 3$
---	---	---

NO USED FOR REMAINING FOR LAST 3 TERMS

• USING $S_n = \frac{n}{2}(a+L)$

• $\sum_{n=1}^{13} n = \frac{13}{2}(1+14) = \frac{13}{2} \times 15 = 13 \times 15 = 195$

• $\sum_{n=14}^{46} 2n = \frac{33}{2}(14+46) = \frac{33}{2} \times 60 = 33 \times 30 = 990$

• $\sum_{n=47}^{79} 3n = \frac{33}{2}(47+79) = \frac{33}{2} \times 126 = 33 \times 63 = 2079$

• $\sum_{n=80}^{82} 3n = \frac{3}{2}(80+82) \times 3 = 243 \times 3 = 729$

$\therefore \sum_{n=1}^{82} f(n) = 195 + 990 + 2079 + 729$

$= \frac{4158}{2} + 729$
 $= \frac{5877}{1}$

Question 91 (****)

Use partial fractions and a suitable Mclaurin expansion to sum the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+3)}$$

$$\boxed{\frac{2}{3} \ln 2 - \frac{5}{18}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+3)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+3)} = \dots$$
 PARTIAL FRACTIONS BY INSPECTION

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{A}{n} + \frac{B}{n+3} \right] = \frac{1}{3} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} - \frac{(-1)^{n+1}}{n+3} \right]$$

$$= \frac{1}{3} \left[\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} - \frac{1}{100} + \dots \right) - \left(\frac{1}{4} - \frac{1}{7} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} - \frac{1}{100} + \dots \right) \right]$$

$$= \frac{1}{3} \left[\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} - \frac{1}{100} + \dots \right) - 2 \left(-\frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} - \frac{1}{100} + \dots \right) \right]$$

$$= \frac{1}{3} \left[1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \frac{1}{81} - \frac{1}{100} + \dots \right]$$

 Now $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \dots$
 $\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$
 $\ln 2 = \left(1 - \frac{1}{2} + \frac{1}{3} \right) + \left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right)$
 $\ln 2 - \frac{5}{6} = -\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

$$= \frac{1}{3} + \frac{2}{3} \left(\ln 2 - \frac{5}{6} \right)$$

$$= \frac{2}{3} \ln 2 - \frac{5}{18}$$

Question 92 (****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2.$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

$$\boxed{}, \quad f(n) = \frac{1}{12} n(n+2)(n+1)^2$$

Handwritten solution for Question 92:

1. $1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2$

2. SIMPLY BY WRITING THE SERIES IN SIGMA NOTATION

$$\sum_{r=1}^n [r(n+1-r)^2] = \sum_{r=1}^n [r \{ (n+1)^2 - 2(n+1)r + r^2 \}]$$

$$= \sum_{r=1}^n [r(n+1)^2 - 2r(n+1)r + r^3]$$

$$= (n+1)^2 \sum_{r=1}^n r - 2(n+1) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^3$$

3. USING SIMPLIFIED SUMMATION FORMULAS WE HAVE

$$\sum_{r=1}^n [r(n+1-r)^2] = \underbrace{(n+1)^2 \sum_{r=1}^n r}_{\frac{1}{2}n(n+1)^2} - \underbrace{2(n+1) \sum_{r=1}^n r^2}_{4(n+1)(\frac{1}{6}n(n+1)(2n+1))} + \underbrace{\sum_{r=1}^n r^3}_{(\frac{1}{4}n(n+1))^2}$$

$$= \frac{1}{2}n(n+1)^2 [6(n+1) - 4(2n+1) + 3n]$$

$$= \frac{1}{12}n(n+1)^2(n+2)$$

Question 93 (****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \frac{1^3+2^3+3^3+4^3}{1+3+5+7} + \dots$$

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$$\frac{1}{1+1 \cdot 3} + \frac{1}{1+3 \cdot 5} + \frac{1}{1+5 \cdot 7} + \frac{1}{1+7 \cdot 9} + \dots$$

START BY IMITATING THE ABOVE EXPRESSION COMPACTLY

$$\sum_{N=1}^{16} \left[\frac{\frac{4}{25} t^3}{\frac{4}{25} t^3 - \frac{8}{25}} \right] = \sum_{N=1}^{16} \left[\frac{\frac{4}{25} t^3}{\frac{4}{25} t^3 - \frac{8}{25}} \right]$$

USING STANDARD SUMMATION FORMULAS

$$= \sum_{N=1}^{16} \left[\frac{\frac{1}{4} N^2 (N+1)^2}{2N^2 (N+1) - N} \right] = \sum_{N=1}^{16} \frac{\frac{1}{4} N^2 (N+1)^2}{N^2 + N - N}$$

$$= \sum_{N=1}^{16} \left[\frac{\frac{1}{4} N^2 (N+1)^2}{N^2} \right] = \frac{1}{4} \sum_{N=1}^{16} (N+1)^2$$

STILL GOOD AND USE FORMULAS AGAIN OR "TRANSLATE" BY 1 STEP

$$\dots = \frac{1}{4} \sum_{K=2}^{17} K^2 = \frac{1}{4} \times \frac{1}{6} \times (17)(17+1)(17+2) \Big|_{K=17} - \frac{1}{4} \times$$

$$= \frac{1}{4} \times \frac{1}{6} \times 17 \times 18 \times 19 - \frac{1}{4}$$

$$= \frac{1}{4} \left[\frac{1}{6} \times 17 \times 18 \times 19 - 1 \right]$$

$$= \underline{\underline{646}}$$

100% MARK

Question 94 (*****)

$$S_n = 1 \times 3 + 3 \times 3^2 + 5 \times 3^3 + 7 \times 3^4 + \dots + (2n-1) \times 3^n$$

Find a simplified expression for S_n , giving the answer in the form $A + f(n) \times 3^{n+1}$, where A is an integer and $f(n)$ a linear function of n .

[The standard techniques used for the summation of a geometric series are useful in this question]

$$S_n = 3 + (n-1) \times 3^{n+1}$$

Handwritten solution for Question 94 using the method of differences:

$$\begin{aligned} S_n &= 1 \times 3 + 3 \times 3^2 + 5 \times 3^3 + 7 \times 3^4 + \dots + (2n-1) \times 3^n \\ -3S_n &= \quad \quad -1 \times 3^2 - 3 \times 3^3 - 5 \times 3^4 - \dots - (2n-3) \times 3^n - (2n-1) \times 3^{n+1} \\ \hline -2S_n &= 3 + [2 \times 3^2 + 2 \times 3^3 + 2 \times 3^4 + \dots + 2 \times 3^n] - (2n-1) \times 3^{n+1} \\ -2S_n &= 3 + 2[3^2 + 3^3 + 3^4 + \dots + 3^n] - (2n-1) \times 3^{n+1} \\ &\quad \text{G.P. } \begin{cases} a=3 \\ r=3 \\ n=n \end{cases} \\ -2S_n &= 3 + 2 \left[\frac{9(3^n - 1)}{3-1} \right] - (2n-1) \times 3^{n+1} \\ -2S_n &= 3 + 9(3^n - 1) - (2n-1) \times 3^{n+1} \\ -2S_n &= 3 + 3^{n+1} - 9 - (2n-1) \times 3^{n+1} \\ 2S_n &= 6 + (2n-1) \times 3^{n+1} - 3^{n+1} \\ 2S_n &= 6 + (2n-2) \times 3^{n+1} \\ S_n &= 3 + (n-1) \times 3^{n+1} \end{aligned}$$

Question 95 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=1}^{\infty} \left[\frac{2^r}{(r+1)!} \right]$$

$$\frac{1}{2}(e^2 - 3)$$

Handwritten solution for Question 95 using the series expansion of e^x :

$$\begin{aligned} S &= \sum_{r=1}^{\infty} \frac{2^r}{(r+1)!} = \frac{2}{2!} + \frac{4}{3!} + \frac{8}{4!} + \frac{16}{5!} + \frac{32}{6!} + \dots \\ \Rightarrow S &= \frac{2^1}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \frac{2^4}{5!} + \frac{2^5}{6!} + \dots \\ \Rightarrow 2S &= \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \dots \\ \Rightarrow 2S + 1 + \frac{2^1}{1!} &= 1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \dots \\ \Rightarrow 2S + 3 &= e^2 \\ \Rightarrow S &= \frac{e^2 - 3}{2} \end{aligned}$$

Question 96 (****)

The r^{th} term of a progression is given by

$$u_r = ak^{r-1},$$

where a and k are constants with $k \neq \pm 1$.

Show clearly that

$$\sum_{r=1}^n (u_r \times u_{r+1}) = \frac{a^2 k (1 - k^{2n})}{1 - k^2}.$$

proof

Handwritten proof showing the derivation of the formula for the sum of products of terms in a geometric progression:

$$\begin{aligned}
 u_r &= ak^{r-1} \Rightarrow \{u_1, u_2, u_3, u_4, \dots, u_n\} \\
 &\quad \{a, ak, ak^2, ak^3, \dots, ak^{n-1}\} \\
 \text{Hence} \\
 \sum_{r=1}^n (u_r u_{r+1}) &= u_1 u_2 + u_2 u_3 + u_3 u_4 + \dots + u_n u_{n+1} \\
 &= a(ak) + ak(ak^2) + ak^2(ak^3) + \dots + ak^{n-1}(ak^n) \\
 &= a^2 k + a^2 k^3 + a^2 k^5 + \dots + a^2 k^{2n-1} \\
 &= a^2 k [1 + k^2 + k^4 + \dots + k^{2n-2}] \\
 &\quad \text{G.P. with } a=1, r=n, \text{ terms } n \text{ terms} \\
 &= a^2 k \times \frac{1 - (k^2)^n}{1 - k^2} \\
 &= \frac{a^2 k (1 - k^{2n})}{1 - k^2} \quad \text{As required}
 \end{aligned}$$

Question 97 (*****)

It is given that the following series converges to a limit L .

$$\sum_{r=1}^{\infty} \left[\frac{2x-1}{x+2} \right]^r$$

Determine with full justification the range of possible values of L .

$$\boxed{}, \boxed{L > -\frac{1}{2}}$$

$\sum_{r=1}^{\infty} \left(\frac{2x-1}{x+2} \right)^r = L$

• FIRSTLY THIS IS A GEOMETRIC PROGRESSION, WITH COMMON RATIO r

$a = r = \frac{2x-1}{x+2}$

$S_{\infty} = \frac{a}{1-r} = \frac{\frac{2x-1}{x+2}}{1 - \frac{2x-1}{x+2}} = \frac{2x-1}{(x+2) - (2x-1)} = \frac{2x-1}{3-x}$

• NEXT WE REQUIRE THE RANGE OF VALUES OF x , FOR WHICH THE SUM TO INFINITY EXISTS

$|r| < 1$

$-1 < \frac{2x-1}{x+2} < 1$

$\Rightarrow \frac{2x-1}{x+2} < 1$ $\Rightarrow \frac{2x-1}{x+2} - 1 < 0$ $\Rightarrow \frac{2x-1-x-2}{x+2} < 0$ $\Rightarrow \frac{x-3}{x+2} < 0$	$\Rightarrow \frac{2x-1}{x+2} > -1$ $\Rightarrow \frac{2x-1}{x+2} + 1 > 0$ $\Rightarrow \frac{2x-1+x+2}{x+2} > 0$ $\Rightarrow \frac{3x+1}{x+2} > 0$
---	---

$\frac{x-3}{x+2}$

$\frac{3x+1}{x+2}$

$\therefore -\frac{1}{3} < x < 3$

• THIS WE HAVE

$U(x) = \frac{2x-1}{3-x}, \quad -\frac{1}{3} < x < 3$

$U'(x) = \frac{(2-x) \times 2 - (2x-1) \times (-1)}{(3-x)^2} = \frac{2(3-x) + (2x-1)}{(3-x)^2}$

$U'(x) = \frac{5}{(3-x)^2} > 0$ FOR THE ABOVE DOMAIN

• AS $U(x)$ IS AN INCREASING FUNCTION, THE MINIMUM & MAXIMUM CAN BE EASILY FOUND

$U(-\frac{1}{3}) = \frac{2(-\frac{1}{3})-1}{3-(-\frac{1}{3})} = \frac{-\frac{2}{3}-1}{\frac{10}{3}} = -\frac{1}{2}$

$U(3) = +\infty$

$\therefore L > -\frac{1}{2}$

Question 98 (****)

By considering the trigonometric identity for $\tan(A - B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right).$$

You may assume the series converges.

$$\boxed{}, \quad \boxed{\frac{\pi}{4}}$$

● CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A - B)$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{\tan[\arctan(n+1)] - \tan[\arctan n]}{1 + \tan[\arctan(n+1)] \tan[\arctan n]}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{(n+1) - n}{1 + (n+1)n}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{1}{n^2 + n + 1}$$

$$\arctan\left(\frac{1}{n^2 + n + 1}\right) = \arctan[\arctan(n+1) - \arctan n]$$

● HENCE THE SUMMATION NOW BECOMES

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right) = \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan n]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n$$

● WHICH NOW GIVES IN A LIMITING SENSE

$$\lim_{k \rightarrow \infty} \left[\sum_{n=1}^k \arctan(n+1) - \sum_{n=1}^k \arctan n \right]$$

$$= \lim_{k \rightarrow \infty} \begin{bmatrix} \arctan(k+1) & - & \arctan k \\ + & & + \\ \arctan k & - & \arctan(k-1) \\ + & & + \\ \arctan(k-1) & - & \arctan(k-2) \\ + & & + \\ \vdots & & \vdots \\ \arctan 3 & - & \arctan 2 \\ + & & + \\ \arctan 2 & - & \arctan 1 \end{bmatrix}$$

$$= \lim_{k \rightarrow \infty} [\arctan(k+1) - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 99 (****)

It is given that

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{r^2}.$$

$$\boxed{\frac{\pi^2}{6}}$$

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

Let $X = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

$$\frac{1}{2}X = \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{18^2} + \dots$$

$$\frac{5}{4}X = \left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{5^2} + \frac{1}{7^2}\right) + \left(\frac{1}{9^2} + \frac{1}{11^2}\right) + \dots$$

$$\frac{5}{4}X = \sum_{r=1}^{\infty} \frac{1}{r^2} - \sum_{r=1}^{\infty} \frac{1}{(2r)^2}$$

$$\frac{5}{4}X = \sum_{r=1}^{\infty} \frac{1}{r^2} - \frac{1}{4} \sum_{r=1}^{\infty} \frac{1}{r^2}$$

$$\frac{4}{3} \sum_{r=1}^{\infty} \frac{1}{r^2} = \sum_{r=1}^{\infty} \frac{1}{r^2}$$

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{4}{3} \times \frac{\pi^2}{8}$$

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

Let $A = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

$$\frac{1}{4}A = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$$

$$\frac{3}{4}A = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots\right)$$

$$\frac{3}{4}A = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$$

$$\frac{3}{4} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{8}$$

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{4}{3} \times \frac{\pi^2}{8}$$

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

Question 100 (****)

Evaluate the following expression

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{1}{3^{m+n}} \right].$$

Detailed workings must be shown.

$$\boxed{}, \boxed{\frac{9}{4}}$$

WORK 45: Focus

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \left(\frac{1}{3^{m+n}} \right) \right] &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \left(\frac{1}{3^n} \times \frac{1}{3^m} \right) \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \sum_{m=0}^{\infty} \left(\frac{1}{3^m} \right) \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right) \right] \end{aligned}$$

This is a GEOMETRIC PROGRESSION WITH $a=1$, $r=\frac{1}{3}$ & $S_{\infty} = \frac{a}{1-r}$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \times \frac{1}{1-\frac{1}{3}} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \times \frac{3}{2} \right] \\ &= \frac{3}{2} \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \right] \\ &= \frac{3}{2} \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right) \end{aligned}$$

Since G.P. is finite with $S_{\infty} = \frac{1}{1-\frac{1}{3}}$

$$= \frac{3}{2} \times \frac{3}{2}$$

$$= \frac{9}{4}$$

Question 101 (****)

$$S = 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} - \dots$$

Find the sum to infinity of S , by considering the binomial series expansion of $(1+x)^n$ for suitable values of x and n .

$$\boxed{}, \quad S_{\infty} = \sqrt{\frac{2}{3}}$$

START BY CREATING FRACTIONS IN THE DENOMINATORS

$$\begin{aligned} \Rightarrow S &= 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} - \dots \\ \Rightarrow S &= 1 - \frac{1}{4 \times 1} + \frac{1 \cdot 3}{4^2 (1 \times 2)} - \frac{1 \cdot 3 \cdot 5}{4^3 (1 \times 2 \times 3)} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4^4 (1 \times 2 \times 3 \times 4)} - \dots \\ \Rightarrow S &= 1 - \frac{2^0 (1)}{4 \times 1} + \frac{2^1 (1)(1)}{4^2 (1 \times 2)} - \frac{2^2 (1)(1)(1)}{4^3 (1 \times 2 \times 3)} + \frac{2^3 (1)(1)(1)(1)}{4^4 (1 \times 2 \times 3 \times 4)} - \dots \end{aligned}$$

CREATE WHAT LOOKS LIKE A BINOMIAL EXPANSION

$$\Rightarrow S = 1 - \frac{1}{4} \left(\frac{1}{2} \right)^0 + \frac{1}{4^2} \left(\frac{1}{2} \right)^1 - \frac{1}{4^3} \left(\frac{1}{2} \right)^2 + \frac{1}{4^4} \left(\frac{1}{2} \right)^3 - \dots$$

FINALLY DEAL WITH THE MINUS SIGNS

$$\begin{aligned} \Rightarrow S &= 1 + \frac{1}{4} \left(\frac{1}{2} \right)^0 - \frac{1}{4^2} \left(\frac{1}{2} \right)^1 + \frac{1}{4^3} \left(\frac{1}{2} \right)^2 - \frac{1}{4^4} \left(\frac{1}{2} \right)^3 + \dots \\ \Rightarrow S &= \left(1 + \frac{1}{2} \right)^{-1} \quad \left(\text{BY COMPARING THE EXPANSION } (1+x)^{-1} \text{ WITH } x = \frac{1}{2} \right) \\ \Rightarrow S &= \left(\frac{3}{2} \right)^{-1} \\ \therefore 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} - \dots &= \sqrt{\frac{2}{3}} \end{aligned}$$

Question 102 (****)

Show clearly that

$$\ln \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] = \sum_{n=1}^{\infty} \frac{\sin^{2n-1} x}{2n-1}.$$

proof

$$\begin{aligned} \ln \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] &= \ln \left[\frac{\tan \frac{\pi}{4} + \tan \frac{x}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{x}{2}} \right] = \ln \left[\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right] = \ln \left[\frac{1 + \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}}{1 - \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}} \right] = \ln \left[\frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right] \\ &= \ln \left[\frac{(\cos \frac{x}{2} + \sin \frac{x}{2})(\cos \frac{x}{2} + \sin \frac{x}{2})}{(\cos \frac{x}{2} - \sin \frac{x}{2})(\cos \frac{x}{2} + \sin \frac{x}{2})} \right] = \ln \left[\frac{\cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \right] \\ &= \ln \left[\frac{1 + \sin x}{\cos x} \right] = \ln(1 + \sin x) - \ln(\cos x) = \ln(1 + \sin x) - \frac{1}{2} \ln \cos x \\ &= \ln(1 + \sin x) - \frac{1}{2} \ln(1 - \sin^2 x) \end{aligned}$$

Now

$$\begin{aligned} \ln(1+y) &= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots \\ \ln(1-y) &= -y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots \end{aligned}$$

where $y = \sin x$

$$\begin{aligned} &= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \dots \\ &\quad - \left[-y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \dots \right] \\ &= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \frac{1}{7}y^7 - \frac{1}{8}y^8 + \dots \\ &\quad + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{4}y^4 - \frac{1}{5}y^5 + \frac{1}{6}y^6 - \frac{1}{7}y^7 + \dots \\ &= y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \dots \\ &= \sin x + \frac{1}{3} \sin^3 x + \frac{1}{5} \sin^5 x + \frac{1}{7} \sin^7 x + \dots \end{aligned}$$

as required

Question 103 (****)

The function f is defined as

$$f(n) = \frac{e^{-\lambda} \lambda^n}{n!},$$

where $n = 0, 1, 2, 3, 4, \dots$ and λ is a positive constant.

By showing a detailed method, prove that ...

a) ... $\sum_{n=0}^{\infty} [n f(n)] = \lambda.$

b) ... $\sum_{n=0}^{\infty} [n^2 f(n)] = \lambda^2 + \lambda.$

proof

Handwritten proof for Question 103:

Given $f(n) = \frac{e^{-\lambda} \lambda^n}{n!}$

a) $\sum_{n=0}^{\infty} n f(n) = \sum_{n=0}^{\infty} n \frac{e^{-\lambda} \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n-1)!}$
 Note: The first term is zero, so we start the summation from $n=1$.
 $= e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} = e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda \lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$
 Re-index the summation from zero:
 $= \lambda e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$

b) $\sum_{n=0}^{\infty} n^2 f(n) = \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda} \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{n \lambda^n}{(n-1)!}$
 Note: The first two terms are zero, so we start the summation from $n=2$.
 $= e^{-\lambda} \sum_{n=2}^{\infty} \frac{n \lambda^n}{(n-1)!} = \lambda e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$
 Re-index the summation back from zero:
 $= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda \lambda^{k-1}}{(k-1)!} = \lambda^2 e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$
 Re-index the summation back to zero:
 $= \lambda^2 e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = \lambda^2 e^{-\lambda} \cdot e^{\lambda} = \lambda^2$
 Therefore, $\sum_{n=0}^{\infty} n^2 f(n) = \lambda^2 + \lambda$

Find in exact simplified form an exact expression for the sum of the first n terms of the following series

$$1 + 11 + 111 + 1111 + 11111 + \dots$$

$$\boxed{}, \quad S_n = \frac{1}{81} [10^{n+1} - 10 - 9n]$$

• Let $S_n = 1 + 11 + 111 + 1111 + \dots + \underbrace{111\dots111}_{n \text{ digits}}$

$$\Rightarrow S_n = \left(\frac{1}{3} \times 9\right) + \left(\frac{1}{3} \times 99\right) + \left(\frac{1}{3} \times 999\right) + \dots + \left(\frac{1}{3} \times \underbrace{999\dots999}_{n \text{ digits}}\right)$$

$$\Rightarrow S_n = \frac{1}{3} \left[9 + 99 + 999 + \dots + \underbrace{999\dots999}_{n \text{ digits}} \right]$$

$$\Rightarrow S_n = \frac{1}{3} \left[(10^1 - 1) + (10^2 - 1) + (10^3 - 1) + \dots + (10^n - 1) \right]$$

$$\Rightarrow S_n = \frac{1}{3} \left[(10^1 + 10^2 + 10^3 + \dots + 10^n) - (1 + 1 + \dots + 1) \right]$$

Geometric Progression

$n = 10$
 $r = 10$
 $a = \frac{a(r^n - 1)}{r - 1}$

$$\Rightarrow S_n = \frac{1}{3} \left[\frac{10(10^n - 1)}{10 - 1} - (1 \times n) \right]$$

$$\Rightarrow S_n = \frac{1}{3} \left[\frac{10}{9} (10^n - 1) - n \right]$$

$$\Rightarrow S_n = \frac{1}{3} \left[\frac{10}{9} (10^n - 1) - 9n \right]$$

$$\Rightarrow S_n = \frac{1}{3} \left[\frac{10^{n+1}}{9} - 10 - 9n \right]$$

$$\Rightarrow S_n = \frac{1}{81} \left[10^{n+1} - 9n - 10 \right]$$

• ALTERNATIVE BY LOOKING AT DIFFERENT PATTERNS
 • Let $S_n = 1 + 11 + 111 + 1111 + \dots + \underbrace{111\dots111}_{n \text{ digits}}$

$$\Rightarrow S_n = \begin{matrix} 10^0 & & & & \\ 10^1 + 10^0 & \leftarrow 1 \\ 10^2 + 10^1 + 10^0 & \leftarrow 11 \\ 10^3 + 10^2 + 10^1 + 10^0 & \leftarrow 111 \end{matrix}$$

$$\begin{matrix} 10^4 & 10^3 & 10^2 & 10^1 & 10^0 & \dots & 10^2 & 10^1 & 10^0 & & \dots & 10^2 & 10^1 & 10^0 & & \dots & 11\dots11 \\ \text{GP} & \text{GP} & \text{GP} & \text{GP} & & & \text{GP} & \text{GP} & & & & \text{GP} & \text{GP} & & & & \text{GP} \\ a=1 & a=1 & a=1 & a=1 & & & a=1 & a=1 & & & & a=1 & a=1 & & & & n \text{ digits} \\ r=10 & r=10 & r=10 & r=10 & & & r=10 & r=10 & & & & r=10 & r=10 & & & & \end{matrix}$$

 • HOW DOING ANY OF THESE G.P. VARIATIONS DOWN

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{1(10^n - 1)}{10 - 1} = \frac{1}{9}(10^n - 1)$$

$$\Rightarrow S_n = \sum \text{All These G.P.s}$$

$$\Rightarrow S_n = \frac{1}{9} [10^2 + 10^3 + 10^4 + \dots + \frac{1}{9} (10^{n+1} - 1) + \dots + \frac{1}{9} (10^2 + 10^1 + 10^0) + \frac{1}{9} (10^1 + 10^0) + \frac{1}{9} (10^0)]$$

$$\Rightarrow S_n = \frac{1}{9} \left[\frac{10(10^{n+1} + 10^2 + \dots + 10^0) - n \times 1}{10 - 1} \right]$$

$$\Rightarrow S_n = \frac{1}{9} \times \left[\frac{10(10^{n+1} - 1)}{10 - 1} - n \right] \quad (\text{Simplify the Sign G.P. as Given})$$

$$\Rightarrow S_n = \frac{1}{81} \left[\frac{10^{n+1}}{9} - 10 - n \right]$$

$$\Rightarrow S_n = \frac{1}{81} \left[10^{n+1} - 9n - 10 \right]$$

Question 105 (****)

The product operator \prod , is defined as

$$\prod_{i=1}^k [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \dots \times u_{k-1} \times u_k.$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right].$$

$$\frac{1}{8} \sqrt{\frac{5}{4}} - 1$$

● START BY WRITING A FEW TERMS EXPLICITLY / LOOK FOR A PATTERN

$$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] = \left[\prod_{r=1}^1 \left(\frac{8r-7}{40r} \right) \right] + \left[\prod_{r=1}^2 \left(\frac{8r-7}{40r} \right) \right] + \left[\prod_{r=1}^3 \left(\frac{8r-7}{40r} \right) \right] + \dots$$

$$= \frac{1}{40} + \frac{1}{40} \times \frac{3}{80} + \frac{1}{40} \times \frac{3}{80} \times \frac{5}{120} + \frac{1}{40} \times \frac{3}{80} \times \frac{5}{120} \times \frac{7}{160} + \dots$$

$$= \frac{1}{40} + \frac{1 \times 3}{40 \times 80} + \frac{1 \times 3 \times 5}{40 \times 80 \times 120} + \frac{1 \times 3 \times 5 \times 7}{40 \times 80 \times 120 \times 160} + \dots$$

$$= \frac{1}{40 \times 1} + \frac{1 \times 3}{40^2 \times (1 \times 2)} + \frac{1 \times 3 \times 5}{40^3 \times (1 \times 2 \times 3)} + \frac{1 \times 3 \times 5 \times 7}{40^4 \times (1 \times 2 \times 3 \times 4)} + \dots$$

● THIS RESEMBLES A BINOMIAL EXPANSION DUE TO THE FRACTIONALS AT THE DENOMINATOR
THE NEXT STEP IS TO "CREATE" NUMERATORS OF THE FORM $n(n-1)(n-2)(n-3) \dots$

● BY INSPECTION THIS WILL GIVE AS $-\frac{1}{8}, -\frac{3}{8}, -\frac{5}{8}, -\frac{7}{8}$

● THIS TRY AND ADJUST THE 36045

$$= \frac{1}{(40 \times 1)!} + \frac{1 \times 3}{(40 \times 2)!} + \frac{1 \times 3 \times 5}{(40 \times 3)!} + \frac{1 \times 3 \times 5 \times 7}{(40 \times 4)!} + \dots$$

● $\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] = \frac{1}{1!} \left(-\frac{1}{40} \right) + \frac{1}{2!} \left(-\frac{3}{40} \right) + \frac{1}{3!} \left(-\frac{5}{40} \right) + \frac{1}{4!} \left(-\frac{7}{40} \right) + \dots$

THIS IS A BINOMIAL EXPANSION WITH THE 1^{st} MISSING AT THE FIRST

$$= \left(1 - \frac{1}{40} \right)^{-1} - 1$$

$$= \left(\frac{39}{40} \right)^{-1} - 1$$

$$= \frac{40}{39} - 1$$

Question 106 (****)

Find the value of

$$\sum_{r=0}^{\infty} \left[\frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right].$$

Hint: Express $\sin^4 \theta$ in terms of $\sin^2 \theta$ and $\sin^2 2\theta$ only.

$$\boxed{}, \boxed{\frac{1}{2}}$$

• STARTING BY MANIPULATING THE SIN TO THE POWER 4

$$\sin^4 \theta = (\sin^2 \theta)^2 = \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta$$

$$= \frac{1}{4} - \frac{1}{2} (1 - 2\sin^2 \theta) + \frac{1}{4} (1 - \sin^2 2\theta)$$

$$= \frac{1}{4} - \frac{1}{2} + \sin^2 \theta + \frac{1}{4} - \frac{1}{4} \sin^2 2\theta$$

$$= \sin^2 \theta - \frac{1}{4} \sin^2 2\theta$$

• NOW WE HAVE BY OBSERVING THE SUM OF THE FIRST N TERMS

$$\sum_{r=0}^n \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sum_{r=0}^n \left[\frac{1}{4^r} \left(\sin^2(\pi \times 2^{r-2}) - \frac{1}{4} \sin^2(\pi \times 2^{r-1}) \right) \right]$$

$$= \sum_{r=0}^n \left[\frac{1}{4^r} \sin^2(\pi \times 2^{r-2}) - \frac{1}{4^{r+1}} \sin^2(\pi \times 2^{r-1}) \right]$$

$$= \frac{\sin^2 \pi}{4^0} - \frac{1}{4^1} \sin^2 \frac{\pi}{2}$$

$$= \frac{1}{4^1} \sin^2 \pi - \frac{1}{4^2} \sin^2 2\pi$$

$$= \frac{1}{4^2} \sin^2 2\pi - \frac{1}{4^3} \sin^2 4\pi$$

$$= \frac{1}{4^n} \sin^2(\pi \times 2^{n-2}) - \frac{1}{4^{n+1}} \sin^2(\pi \times 2^{n-1})$$

$$= \sin^2 \frac{\pi}{4} - \frac{1}{4^{n+1}} \sin^2(\pi \times 2^{n-1})$$

• HENCE WE HAVE

$$\sum_{r=0}^{\infty} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{4} = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

Question 107 (****)

Find the sum to infinity of the following series.

$$\frac{1}{1} - \frac{1}{1+4} + \frac{1}{1+4+9} - \frac{1}{1+4+9+16} + \frac{1}{1+4+9+16+25} + \dots$$

You may find the series expansion of $\arctan x$ useful in this question.

$$\boxed{}, \boxed{6(\pi-3)}$$

WRITE THE SERIES IN 'COMPACT' NOTATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1^2+2^2+\dots+n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\frac{n(n+1)(2n+1)}{6}}$$

REWRITE THE $\frac{1}{n(n+1)(2n+1)}$ TERM & SPLIT THE DIFF INTO SPECIAL FRACTIONS BY INSPECTING

$$\frac{1}{n(n+1)(2n+1)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{2n+1} = \frac{1}{n} + \frac{1}{n+1} - \frac{2}{2n+1}$$

HENCE USE THIS

$$\dots = \sum_{n=1}^{\infty} \left[6(-1)^{n+1} \left[\frac{1}{n} + \frac{1}{n+1} - \frac{2}{2n+1} \right] \right]$$

$$= 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} - 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

NOTE: CONSIDER EACH TERM OF THE SUMMATION SEPARATELY

- $6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 6 \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right] = 6 \ln 2$ (from Q10)
- $6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = 6 \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots \right]$
 $= -6 \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots \right]$
 $= -6 \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right]$
 $= -6 \ln 2$

NOTE: CONSIDER THE SERIES EXPANSION OF $\arctan x$

$$\Rightarrow \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$\Rightarrow \int \frac{1}{1+x^2} dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots + C$$

$$\Rightarrow \arctan x = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

Let $x=0 \Rightarrow C=0$

$$\Rightarrow \arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{2n-1} x^{2n-1} \right)$$

$$\Rightarrow \arctan 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$\Rightarrow \pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right) = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \pi = 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \pi = 24 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right]$$

$$\Rightarrow \pi = 24 + 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \pi - 24$$

FINALLY COLLECTING ALL THE RESULTS

$$\frac{1}{1} - \frac{1}{1+4} + \frac{1}{1+4+9} - \frac{1}{1+4+9+16} + \frac{1}{1+4+9+16+25} - \dots$$

$$= 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} - 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$= 6 \ln 2 + (-6 \ln 2) + (\pi - 24)$$

$$= \pi - 18 = 6(\pi - 3)$$

Question 108 (****)

$$f(x) \equiv \frac{1-7x}{(1+x)(1-3x)}, \quad -\frac{1}{3} < x < \frac{1}{3}.$$

Show that $f(x)$ can be written in the form

$$f(x) = 1 - \sum_{r=1}^{\infty} [x^r g(r)],$$

where $g(r)$ is a simplified function to be found.

$$\boxed{}, \quad \boxed{g(r) = 3^r + 2 \times (-1)^{r+1}}$$

$f(x) = \frac{1-7x}{(1+x)(1-3x)} \quad |x| < \frac{1}{3}$

• BY PARTIAL FRACTIONS OR DIRECT EXPANSIONS, GIVING AN INFINITE GEOMETRIC EXPANSION, OR THE SUM TO INFINITY OF A GEOMETRIC SERIES.

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{(1-3x)} = 1 + 3x + 3^2x^2 + 3^3x^3 + \dots$$

• THIS FOR ABOVE

$$\Rightarrow f(x) = (1-7x)(1+x)^{-1}(1-3x)^{-1}$$

$$\Rightarrow f(x) = (1-7x)(1+x+3x^2+3^2x^3+\dots)(1+3x+3^2x^2+3^3x^3+\dots)$$

$$\Rightarrow f(x) = (1-7x) \begin{bmatrix} 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6 + \dots \\ -x - 3x^2 - 9x^3 - 27x^4 - 81x^5 - 243x^6 - \dots \\ x^2 + 3x^3 + 9x^4 + 27x^5 + 81x^6 + \dots \\ -3x^3 - 3x^4 - 9x^5 - 27x^6 - \dots \\ x^4 + 3x^5 + 9x^6 + \dots \\ -x^5 - 3x^6 - \dots \end{bmatrix}$$

$$\Rightarrow f(x) = (1-7x)(1 + 2x + 7x^2 + 20x^3 + 41x^4 + 82x^5 + \dots)$$

$$\Rightarrow f(x) = \begin{bmatrix} 1 + 2x + 7x^2 + 20x^3 + 41x^4 + 82x^5 + \dots \\ -7x - 14x^2 - 49x^3 - 147x^4 - 427x^5 - \dots \end{bmatrix}$$

$$1 - 5x - 7x^2 - 29x^3 - 79x^4 - 245x^5 - \dots$$

• BY INSPECTION

↑ N/A ↑ 3^2 ↑ $3^2 \cdot 2$ ↑ 3^3 ↑ $3^3 \cdot 2$ ↑ 3^4

• IF/WHEN WE MAY WRITE

$$f(x) = 1 - \sum_{r=1}^{\infty} [3^r + 2(-1)^{r+1}]x^r$$

$$f(x) = 1 - \sum_{r=1}^{\infty} (3^r + 2(-1)^{r+1})x^r$$

Question 109 (****)

It is given that

$$\zeta(2) = \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$\boxed{\frac{\pi^2}{8}}$$

Handwritten solution for Question 109:

$\zeta(2) = \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$

$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

$\frac{1}{2} \zeta(2) = \frac{1}{2^2} + \frac{1}{2^2 \cdot 2} + \frac{1}{2^2 \cdot 3} + \frac{1}{2^2 \cdot 4} + \frac{1}{2^2 \cdot 5} + \dots$

$\frac{1}{4} \zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$

SUBTRACTING

$\zeta(2) - \frac{1}{4} \zeta(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

$\frac{3}{4} \zeta(2) = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$

$\frac{3}{4} \times \frac{\pi^2}{6} = \frac{1}{2} \times \frac{\pi^2}{(2r-1)^2}$

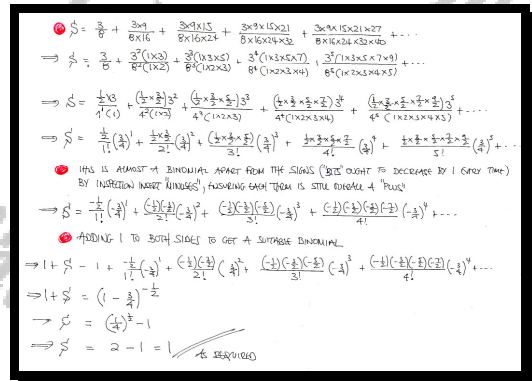
$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$

Question 110 (****)

$$S = \frac{3}{8} + \frac{3 \times 9}{8 \times 16} + \frac{3 \times 9 \times 15}{8 \times 16 \times 24} + \frac{3 \times 9 \times 15 \times 21}{8 \times 16 \times 24 \times 32} + \frac{3 \times 9 \times 15 \times 21 \times 27}{8 \times 16 \times 24 \times 32 \times 40} \dots$$

By considering a suitable binomial expansion, show that $S = 1$.

 , proof



Handwritten solution for Question 110:

$$S = \frac{3}{8} + \frac{3 \times 9}{8 \times 16} + \frac{3 \times 9 \times 15}{8 \times 16 \times 24} + \frac{3 \times 9 \times 15 \times 21}{8 \times 16 \times 24 \times 32} + \frac{3 \times 9 \times 15 \times 21 \times 27}{8 \times 16 \times 24 \times 32 \times 40} + \dots$$

$$\Rightarrow S = \frac{3}{8} + \frac{3^2(1 \times 3)}{8^2(1 \times 2)} + \frac{3^3(1 \times 3 \times 5)}{8^3(1 \times 2 \times 3)} + \frac{3^4(1 \times 3 \times 5 \times 7)}{8^4(1 \times 2 \times 3 \times 4)} + \dots$$

$$\Rightarrow S = \frac{3}{8} + \frac{(3 \times \frac{3}{2})^2}{4^2(1 \times 2)} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2})^2}{4^3(1 \times 2 \times 3)} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2})^2}{4^4(1 \times 2 \times 3 \times 4)} + \dots$$

$$\Rightarrow S = \frac{3}{8} + \frac{3 \times \frac{3}{2}}{2!} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2})}{3!} + \frac{3 \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2}}{4!} + \dots$$

THIS IS ALMOST A BINOMIAL EXPANSION FROM THE SIGNS (BUT) OUGHT TO DECREASE BY 1 (EVEN TIME)
BY INCREASING INCREASING, FUSING EACH TERM IS STILL REMAIN A PLUS!

$$\Rightarrow S = \frac{3}{8} + \frac{(3 \times \frac{3}{2})}{2!} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2})}{3!} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2})}{4!} + \dots$$

ADJUSTING 1 TO BOTH SIDES TO GET A SUITABLE BINOMIAL

$$\Rightarrow 1 + S - 1 = \frac{3}{8} + \frac{(3 \times \frac{3}{2})}{2!} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2})}{3!} + \frac{(3 \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2})}{4!} + \dots$$

$$\Rightarrow 1 + S = (1 + \frac{3}{8})^{\frac{1}{2}}$$


$$\Rightarrow S = (\frac{11}{8})^{\frac{1}{2}} - 1$$

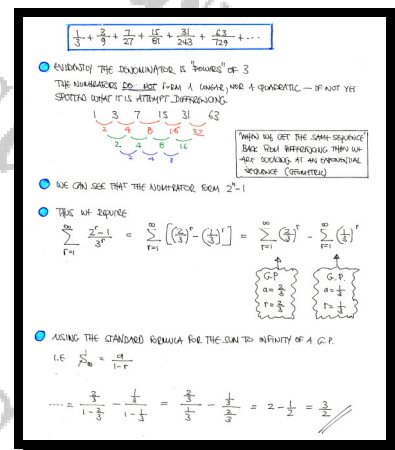
$$\Rightarrow S = 2 - 1 = 1$$

Question 111 (****)

Sum the following series of infinite terms.

$$\frac{1}{3} + \frac{3}{9} + \frac{7}{27} + \frac{15}{81} + \frac{31}{243} + \frac{63}{729} + \dots$$

 , $\frac{3}{2}$



Handwritten solution for Question 111:

$$\frac{1}{3} + \frac{3}{9} + \frac{7}{27} + \frac{15}{81} + \frac{31}{243} + \frac{63}{729} + \dots$$

ESTIMATE THE DENOMINATOR IS "POWERS" OF 3
THE NUMERATOR ARE NOT FROM A LINEAR AND QUADRATIC - IF NOT YET
SPOTTING WHAT IT IS ATTEMPT DIFFERENCES

1 3 7 15 31 63
2 4 8 16 32
2 4 8 16 32
2 4 8 16 32
2 4 8 16 32

WE CAN SEE THAT THE NUMERATOR FROM 2^{n-1}

THE WE REQUIRE

$$\sum_{r=1}^{\infty} \frac{2^{r-1}}{3^r} = \sum_{r=1}^{\infty} \left(\frac{2}{3} \right)^r = \sum_{r=1}^{\infty} \left(\frac{2}{3} \right)^r = \sum_{r=1}^{\infty} \left(\frac{2}{3} \right)^r$$

USING THE STANDARD FORMULA FOR THE SUM TO INFINITY OF A G.P.
I.E. $S_{\infty} = \frac{a}{1-r}$

$$\dots = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = \frac{2}{3} \times \frac{3}{1} = 2 - 1 = \frac{3}{2}$$

Question 112 (****)

Sum the following series of infinite terms.

$$\frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots$$

 ,

$\frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots$
 THE NUMERATOR IS THE FIBONACCI SEQUENCE, THE DENOMINATOR IS A G.P. WITH COMMON RATIO 2, OR $\frac{1}{2}$ FOR THE WHOLE FRACTIONAL TERM.
 THUS LET THE REQUIRED SUM BE S
 $\bullet S = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots$
 $\bullet \frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \dots$
 $\bullet \frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \dots$
 Adding: $\frac{1}{2}S = \frac{1}{2}$
 $S = 1$

Question 113 (****)

By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{2n^2}\right) \right].$$

$$\boxed{}, \quad \frac{\pi}{4}$$

Handwritten solution for Question 113:

Given: $\arctan(2n+1) - \arctan(2n-1) = \psi$

• TAKE TANGENTS ON BOTH SIDES

$$\tan[\arctan(2n+1) - \arctan(2n-1)] = \tan \psi$$

$$\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} = \tan \psi$$

$$\tan \psi = \frac{2}{1 + 4n^2 - 1} = \frac{1}{2n^2}$$

$$\psi = \arctan\left(\frac{1}{2n^2}\right)$$

• IF WE: $\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1)$

$n=1: \arctan\left(\frac{1}{2}\right) = \arctan 3 - \arctan 1$
 $n=2: \arctan\left(\frac{1}{8}\right) = \arctan 5 - \arctan 3$
 $n=3: \arctan\left(\frac{1}{18}\right) = \arctan 7 - \arctan 5$
 \vdots
 $n=k: \arctan\left(\frac{1}{2k^2}\right) = \arctan(2k+1) - \arctan(2k-1)$

• SUMMATION:

$$\sum_{n=1}^k \arctan\left(\frac{1}{2n^2}\right) = \arctan(2k+1) - \arctan 1$$

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \lim_{k \rightarrow \infty} [\arctan(2k+1) - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 114 (****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right].$$

$$\boxed{}, \quad \boxed{\frac{4199}{20}}$$

• SPILT MANIPULATING BY DIVISION EXERCISE BY PARTIAL FRACTIONS

$$\frac{n^3 - n^2 + 1}{n^2 - n} = \frac{n(n^2 - n) + 1}{n^2 - n} = n + \frac{1}{n^2 - n} = n + \frac{1}{n(n-1)}$$

$$= n + \frac{-1}{n-1} + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}$$

• THIS ONE OF THE

$$\frac{n^3 - n^2 + 1}{n^2 - n} \equiv n + \frac{1}{n-1} - \frac{1}{n}$$

IF $n=2$ $\frac{2^3 - 2^2 + 1}{2^2 - 2} = 2 + \frac{1}{2-1} - \frac{1}{2}$

IF $n=3$ $\frac{3^3 - 3^2 + 1}{3^2 - 3} = 3 + \frac{1}{3-1} - \frac{1}{3}$

IF $n=4$ $\frac{4^3 - 4^2 + 1}{4^2 - 4} = 4 + \frac{1}{4-1} - \frac{1}{4}$

IF $n=5$ $\frac{5^3 - 5^2 + 1}{5^2 - 5} = 5 + \frac{1}{5-1} - \frac{1}{5}$

\vdots

IF $n=20$ $\frac{20^3 - 20^2 + 1}{20^2 - 20} = 20 + \frac{1}{20-1} - \frac{1}{20}$

• ADDING

$$\sum_{n=2}^{20} \left[n + \frac{1}{n-1} - \frac{1}{n} \right] = \left[\sum_{n=2}^{20} n \right] + 1 - \frac{1}{20}$$

$$= \frac{19(21)}{2} + 1 - \frac{1}{20}$$

$$= 200 - \frac{1}{20}$$

$$= \frac{4199}{20}$$

$19 \times 21 = 399 + 19 = 419$

$200 - \frac{1}{20} = \frac{4199}{20}$

Question 115 (****)

$$\sum_{r=1}^{\infty} \left[\frac{1}{r^2} \right] = L.$$

It is given that the above infinite series converges to a limit L .

Find, in terms of L where appropriate, the limit of each of the following infinite series.

a) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$

b) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

c) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots$

d) $\frac{1}{1^2} + \frac{1}{2^2} - \frac{8}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{8}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{8}{9^2} + \dots$

$$\boxed{}, \boxed{\frac{1}{4}L}, \boxed{\frac{3}{4}L}, \boxed{\frac{1}{2}L}, \boxed{0}$$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots = L$

a) MULTIPLY THE GIVEN SERIES BY $\frac{1}{4} = \frac{1}{2^2}$
 $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \dots = \frac{1}{4}L$

b) SUBTRACT THE ANSWER FROM PART (a) FROM THE SERIES GIVEN
 $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots = L$
 $-\frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \dots = -\frac{1}{4}L$
 $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{3}{4}L$

c) SUBTRACT (a) FROM (b) OR SUBTRACT $2 \times$ (a) FROM THE SERIES GIVEN
 $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots = L$
 $-\frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \dots = -\frac{1}{2}L$
 $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots = \frac{1}{2}L$

d) FIRSTLY MULTIPLY THE SERIES GIVEN BY $\frac{1}{9} = \frac{1}{3^2}$
 $\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \frac{1}{15^2} + \dots = \frac{1}{9}L$

THEN BY SUBTRACTING 9 TIMES THE ABOVE SERIES FROM THE SERIES GIVEN WE OBTAIN
 $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \dots = L$
 $-\frac{1}{3^2} - \frac{1}{6^2} - \frac{1}{9^2} - \frac{1}{12^2} - \frac{1}{15^2} - \dots = -\frac{1}{9}L$
 $1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots = \frac{8}{9}L$

Question 116 (****)

Consider the infinite series

$$\frac{2}{2^2} \left(\frac{1}{2}\right) x^2 + \frac{3}{2^2 \times 4^2} \left(\frac{1}{2} + \frac{1}{4}\right) x^4 + \frac{4}{2^2 \times 4^2 \times 6^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right) x^6 + \frac{5}{2^2 \times 4^2 \times 6^2 \times 8^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right) x^8 + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=1}^{\infty} \left[\frac{n+1}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \sum_{m=1}^n \frac{1}{2m} \right] \quad \text{or} \quad \sum_{n=1}^{\infty} \sum_{m=1}^n \left[\frac{n+1}{2m(n!)^2} \left(\frac{x}{2}\right)^{2n} \right]$$

Handwritten solution for Question 116:

$$\begin{aligned} & \frac{2}{2^2} \left(\frac{1}{2}\right) x^2 + \frac{3}{2^2 \times 4^2} \left(\frac{1}{2} + \frac{1}{4}\right) x^4 + \frac{4}{2^2 \times 4^2 \times 6^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right) x^6 + \dots \\ &= \sum_{n=1}^{\infty} \frac{n+1}{2^n (n!)^2} \sum_{m=1}^n \frac{1}{2m} \left(\frac{x}{2}\right)^{2n} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{n+1}{2^n (n!)^2} \left(\frac{x}{2}\right)^{2n} \end{aligned}$$

Question 117 (****)

Determine the sum to infinity of the following series

$$\frac{10}{1!} + \frac{7}{2!} + \frac{4}{3!} + \frac{1}{4!} - \frac{2}{5!} - \frac{5}{6!} + \dots$$

10e - 13

Handwritten solution for Question 117:

$$\begin{aligned} & \frac{10}{1!} + \frac{7}{2!} + \frac{4}{3!} + \frac{1}{4!} - \frac{2}{5!} - \frac{5}{6!} + \dots = \sum_{n=1}^{\infty} \frac{(3-2n)}{n!} \\ &= \sum_{n=1}^{\infty} \frac{3}{n!} - 2 \sum_{n=1}^{\infty} \frac{n}{n!} = 3 \sum_{n=1}^{\infty} \frac{1}{n!} - 2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \\ &= 3 \sum_{k=1}^{\infty} \frac{1}{k!} - 2 \sum_{k=0}^{\infty} \frac{1}{k!} \\ & \quad \text{But } \sum_{k=0}^{\infty} \frac{1}{k!} = e \\ &= 3 \left[-1 + \sum_{k=1}^{\infty} \frac{1}{k!} \right] - 2e = 3 \left[-1 + \sum_{k=0}^{\infty} \frac{1}{k!} \right] - 2e \\ &= 3[-1 + e] - 2e = 3e - 3 - 2e = 10e - 13 \end{aligned}$$

Question 118 (****)

Consider the binomial infinite series expansion

$$(1+ax)^n,$$

where $a \in \mathbb{R}$, $n \in \mathbb{Q}$, $n \notin \mathbb{N}$.Show that the series converges if $|ax| < 1$.

proof

$(1+ax)^n \quad n \in \mathbb{Q} \quad n \notin \mathbb{N}$
 $(1+ax)^n = 1 + \frac{n}{1}(ax) + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \dots$
 Consider term $u_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} (ax)^r$
 For convergence by d'Alembert's ratio test
 $\lim_{r \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| \rightarrow L < 1$ for convergence
 $\therefore \lim_{r \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| = \lim_{r \rightarrow \infty} \left| \frac{n(n-1)(n-2)\dots(n-r+1)}{(r+1)!} (ax)^{r+1} \cdot \frac{r!}{n(n-1)(n-2)\dots(n-r+1) (ax)^r} \right|$
 $= \lim_{r \rightarrow \infty} \left| \frac{n-r}{r+1} (ax) \right| = \left| \frac{n-r}{r+1} \right| |ax|$
 $\lim_{r \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| = \lim_{r \rightarrow \infty} \left| \frac{n-r}{r+1} \right| |ax| = |ax|$
 \therefore For convergence $|ax| < 1$

Question 119 (*****)

The n^{th} term of a series is given recursively by

$$u_{n+1} = \frac{n}{2n+1} u_n, \quad n \in \mathbb{N}, \quad u_1 = 2.$$

- a) Show, by direct manipulation, that

$$u_n = \frac{2^n \times [(n-1)!]^2}{(2n-1)!}.$$

[you may not use proof by induction in this part]

- b) Determine whether $\sum_n u_n$ converges or diverges.

converges

a) $u_{n+1} = \frac{n}{2n+1} u_n, \quad u_1 = 2$

$$u_{n+1} = \frac{n}{2n+1} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-3} \times \dots \times \frac{2}{3} \times u_1$$

$$u_{n+1} = \frac{n!}{(2n+1)(2n-1)(2n-3)\dots 3 \times 1} \times 2$$

$$u_{n+1} = \frac{n! \times 2^{n+1}}{(2n+1)!}$$

$$u_n = \frac{n! \times 2^n}{(2n-1)!}$$

b) BY THE RATIO TEST, DIRECTLY FROM THE RECURSIVE RELATION

$$\frac{u_{n+1}}{u_n} = \frac{n}{2n+1}$$

As $n \rightarrow \infty$, $\frac{u_{n+1}}{u_n} \rightarrow \frac{1}{2} < 1$

\therefore SERIES CONVERGES

Question 120 (****)

Determine, in terms of k and n , a simplified expression

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right].$$

$$\boxed{}, \quad \frac{1}{n} \left(\frac{1}{k} \right)^n - \frac{1}{k}$$

• SPLIT BY PARTIAL FRACTIONS

$$\frac{r(r-1)-1}{r(r-1)} = \frac{1}{r} + \frac{8}{r-1}$$

$$\frac{r(r-1)-1}{r(r-1)} = \frac{A(r-1) + B}{r(r-1)}$$

$$\begin{aligned} \frac{1}{r} &\Rightarrow -1 = -A \Rightarrow A = 1 \\ \frac{8}{r-1} &\Rightarrow -1 = B \Rightarrow B = -1 \end{aligned}$$

• WRITE THE NEW FRACTION

$$\left(\frac{1}{r} \right)^n \frac{r(r-1)-1}{r(r-1)} = \left(\frac{1}{r} \right)^n + - \left(\frac{1}{r-1} \right)^n$$

• $r=2$ $\left(\frac{1}{2} \right)^n \frac{2(2-1)-1}{2(2-1)} = \left(\frac{1}{2} \right)^n + - \left(\frac{1}{2-1} \right)^n$

• $r=3$ $\left(\frac{1}{3} \right)^n \frac{3(3-1)-1}{3(3-1)} = \left(\frac{1}{3} \right)^n + - \left(\frac{1}{3-1} \right)^n$

• $r=4$ $\left(\frac{1}{4} \right)^n \frac{4(4-1)-1}{4(4-1)} = \left(\frac{1}{4} \right)^n + - \left(\frac{1}{4-1} \right)^n$

• $r=5$ $\left(\frac{1}{5} \right)^n \frac{5(5-1)-1}{5(5-1)} = \left(\frac{1}{5} \right)^n + - \left(\frac{1}{5-1} \right)^n$

• \dots

• $r=n$ $\left(\frac{1}{n} \right)^n \frac{n(n-1)-1}{n(n-1)} = \left(\frac{1}{n} \right)^n + - \left(\frac{1}{n-1} \right)^n$

• ADDING

$$\sum_{r=2}^n \left(\frac{1}{r} \right)^n \frac{r(r-1)-1}{r(r-1)} = \left(\frac{1}{2} \right)^n + - \frac{1}{1}$$

Question 121 (****)

Use an appropriate method to sum the following series

$$\sum_{r=1}^{\infty} \frac{r \times 2^r}{(r+2)!}$$

You may assume the series converges.

 , 1

STARTING FROM THE DEFINITION OF THE EXPONENTIAL SERIES

$$e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ where } x \neq 0$$

DIVIDE ABOVE BY x^2

$$\Rightarrow \frac{e^x}{x^2} = \sum_{n=0}^{\infty} \frac{x^{n-2}}{n!}$$

$$\Rightarrow \frac{e^x}{x^2} = \frac{1}{0!} + \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \dots$$

$$\Rightarrow \frac{e^x}{x^2} = \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} \right) + \sum_{r=1}^{\infty} \frac{x^r}{(r+2)!}$$

NEXT WE DIFFERENTIATE THE ABOVE EQUATION W.R.T x

$$\Rightarrow \frac{x^2 e^x - 2x e^x}{x^4} = -\frac{2}{x^3} - \frac{1}{x^2} + \sum_{r=1}^{\infty} \left[\frac{r x^{r-1}}{(r+2)!} \right]$$

$$\Rightarrow \frac{e^x (x-2)}{x^4} = -\frac{2}{x^3} - \frac{1}{x^2} + \sum_{r=1}^{\infty} \left[\frac{r x^{r-1}}{(r+2)!} \right]$$

LET $x=2$

$$\Rightarrow 0 = -\frac{1}{4} - \frac{1}{4} + \sum_{r=1}^{\infty} \left[\frac{r \times 2^{r-1}}{(r+2)!} \right]$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r \times 2^{r-1}}{(r+2)!} \right] = \frac{1}{2} \times 2$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r \times 2^r}{(r+2)!} \right] = 1$$

Question 122 (****)

The n^{th} term of a series is given recursively by

$$u_n = \frac{2n}{2n+1}u_{n-1}, \quad n \in \mathbb{N}, \quad u_0 = 1.$$

- a)** Show, by direct manipulation, that

$$u_n = \frac{4^n \times (n!)^2}{(2n+1)!}.$$

[you may not use proof by induction in this part]

- b)** Determine whether $\sum_{n=1}^{\infty} u_n$ converges or diverges.

diverges

q) $U_n = \frac{2n}{2n+1} U_{n-1} + U_n = 1$

$\rightarrow U_n = \frac{2n}{2n+1} + \frac{2n-2}{2n+1} U_{n-1}$

$\rightarrow U_n = \frac{2n}{2n+1} + \frac{2n-2}{2n+1} \cdot \frac{2n-4}{2n-3} U_{n-2}$

$\rightarrow U_n = \frac{2n}{(2n+1)(2n-1)} \times \frac{(2n-2)(2n-4)}{(2n-3)(2n-5)} \times \dots \times \frac{6 \times 4 \times 2}{3 \times 1} U_1$

$\rightarrow U_n = \frac{2^n n! \cdot (n-2)! \cdot (n-4)! \cdot \dots \times 6 \times 4 \times 2}{(2n+1)(2n-1)(2n-3) \dots (3 \times 1)} \times \frac{2 \times 1}{2} = 1$

$\rightarrow U_1 = \frac{2^1 n! \cdot (n-2)! \cdot (n-4)! \cdot \dots \times 6 \times 4 \times 2}{(2n+1)(2n-1)(2n-3) \dots (3 \times 1)} \times \frac{2 \times 1}{2} = 1$

$\rightarrow U_1 = \frac{2^1 n! \cdot \frac{n!}{(2n+1)!} \times \frac{n!}{(2n-1)!} \times \dots \times \frac{n!}{3!} \times \frac{n!}{1!}}{(2n+1)!}$

$\rightarrow U_1 = \frac{2^{n+1} \times n! \times n! \times (n+1)!}{(2n+1)!} = \frac{2^n \times n! \times n! \times 2(n+1)}{(2n+1)!}$

$\rightarrow U_1 = \frac{2^n \times n! \times n!}{(2n+1)!}$

$\Rightarrow U_1 = \frac{4^n \times (n!)^2}{(2n+1)!}$

As $n \rightarrow \infty$

b) $\bullet U_n = \frac{2n}{2n+1} U_{n-1}$

$U_n = \frac{2n}{2n+1} \rightarrow 1$ As $n \rightarrow \infty$

\therefore RATIO TEST FAILS

(GIVE TRY ALTERNATE TEST PLEASE)

\bullet BY BINOMIAL TEST

$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{n!}{n!} \left(\frac{n! \cdot (-1)^n}{n!} \right) \right]$

$= \lim_{n \rightarrow \infty} \left[n \left(\frac{2n+1}{2n} - 1 \right) \right]$

$= \lim_{n \rightarrow \infty} \left[n \left(\frac{2n+1}{2n} - 1 \right) \right]$

$= \lim_{n \rightarrow \infty} \left[\frac{2n+1}{2n} - 1 \right] = \frac{1}{2} < 1$

\therefore SERIES DIVERGES

Question 123 (****)

The following convergent series S is given below

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos \theta} \sin(\sin \theta).$$

, proof

Define series C & S' , based on complex numbers

$$C = \frac{\cos \theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \dots$$

$$S' = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

Combine to form a complex exponential series

$$C + iS' = \frac{1}{1!}(\cos \theta + i \sin \theta) - \frac{1}{2!}(\cos 2\theta + i \sin 2\theta) + \frac{1}{3!}(\cos 3\theta + i \sin 3\theta) - \dots$$

$$C + iS' = \frac{1}{1!}e^{i\theta} - \frac{1}{2!}e^{i2\theta} + \frac{1}{3!}e^{i3\theta} - \frac{1}{4!}e^{i4\theta} + \dots$$

Now consider some simple standard expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

$$z = \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} = 1 - e^{-x}$$

Hence we now have

$$C + iS' = (e^{i\theta}) - \frac{(e^{i\theta})^2}{2!} + \frac{(e^{i\theta})^3}{3!} - \frac{(e^{i\theta})^4}{4!} + \dots$$

$$C + iS' = 1 - e^{-e^{i\theta}}$$

$$C + iS' = 1 - e^{-(\cos \theta + i \sin \theta)}$$

$$C + iS' = 1 - e^{-\cos \theta} \times e^{-i \sin \theta}$$

$$C + iS' = 1 - e^{-\cos \theta} [\cos(\sin \theta) - i \sin(\sin \theta)]$$

$C + iS' = [1 - e^{-\cos \theta} \cos(\sin \theta)] + i[e^{-\cos \theta} \sin(\sin \theta)]$

Selecting imaginary part we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos^n(\theta)}{n!} = e^{-\cos \theta} \sin(\sin \theta)$$

Question 124 (****)

$$g(x) \equiv \sum_{r=0}^{\infty} f(x, r) = \frac{1-x}{\sqrt{1-x^2} \sqrt[3]{1-x^3}}, \quad -1 < x < 1.$$

Given that the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$, determine in exact simplified form a simplified expression of $f(x, r)$.

$$\boxed{}, \quad f(x, r) = \frac{(-x)^r}{r!}$$

The handwritten solution is divided into two columns. The left column shows the initial steps: identifying the first term of the series as $\frac{1}{5}x^5$, then using the binomial expansion for $(1-x)^{-1/2}$ and the geometric series for $(1-x^3)^{-1/3}$ to find the first few terms of $g(x)$. The right column shows the final step: comparing the coefficient of x^5 in the expansion of $g(x)$ with the coefficient of x^5 in the expansion of $\sum_{r=0}^{\infty} \frac{(-x)^r}{r!}$ to determine that $f(x, r) = \frac{(-x)^r}{r!}$.

Left Column:

- Identify the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$.
- Write $g(x) = \sum_{r=0}^{\infty} f(x, r) = \frac{1-x}{\sqrt{1-x^2} \sqrt[3]{1-x^3}}$.
- Expand $(1-x)^{-1/2}$ using the binomial theorem: $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{512}x^4 + \dots$
- Expand $(1-x^3)^{-1/3}$ using the binomial theorem: $(1-x^3)^{-1/3} = 1 + \frac{1}{3}x^3 + \frac{2}{9}x^6 + \dots$
- Multiply the two expansions to find the first few terms of $g(x)$: $g(x) = (1-x) \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{512}x^4 + \dots \right) \left(1 + \frac{1}{3}x^3 + \frac{2}{9}x^6 + \dots \right)$
- Collect like terms to find the first few terms of $g(x)$: $g(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{24}x^5 + \dots$

Right Column:

- Write $g(x) = \sum_{r=0}^{\infty} f(x, r) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!}$.
- Compare the coefficient of x^5 in the expansion of $g(x)$ with the coefficient of x^5 in the expansion of $\sum_{r=0}^{\infty} \frac{(-x)^r}{r!}$.
- From the expansion of $g(x)$, the coefficient of x^5 is $-\frac{1}{24}$.
- From the expansion of $\sum_{r=0}^{\infty} \frac{(-x)^r}{r!}$, the coefficient of x^5 is $\frac{(-1)^5}{5!} = -\frac{1}{120}$.
- Set the coefficients equal: $-\frac{1}{24} = -\frac{1}{120}$.
- Solve for $f(x, r)$: $f(x, r) = \frac{(-x)^r}{r!}$.

Question 125 (****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right]$$

$$\frac{1}{3}$$

● START BY PARTIAL FRACTIONS (BY ADDITION)

$$\frac{4r-1}{r(r-1)} = \frac{1}{r-1} + \frac{3}{r}$$

● SINCE WE KNOW THAT

$$\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r = \frac{1}{r-1} \left(-\frac{1}{3} \right)^r + \frac{3}{r} \left(-\frac{1}{3} \right)^r$$

• r=2: $\frac{7}{2 \cdot 1} \left(-\frac{1}{3} \right)^2 = \frac{1}{1} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2 = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$

• r=3: $\frac{11}{3 \cdot 2} \left(-\frac{1}{3} \right)^3 = \frac{1}{2} \left(-\frac{1}{3} \right)^3 + \frac{3}{3} \left(-\frac{1}{3} \right)^3 = -\frac{1}{54} - \frac{1}{27} = -\frac{1}{18}$

• r=4: $\frac{15}{4 \cdot 3} \left(-\frac{1}{3} \right)^4 = \frac{1}{3} \left(-\frac{1}{3} \right)^4 + \frac{3}{4} \left(-\frac{1}{3} \right)^4 = \frac{1}{81} + \frac{1}{36} = \frac{5}{108}$

• r=5: $\frac{19}{5 \cdot 4} \left(-\frac{1}{3} \right)^5 = \frac{1}{4} \left(-\frac{1}{3} \right)^5 + \frac{3}{5} \left(-\frac{1}{3} \right)^5 = -\frac{1}{324} - \frac{1}{270} = -\frac{1}{540}$

• ...

• r=n: $\frac{4n-1}{n(n-1)} \left(-\frac{1}{3} \right)^n = \frac{1}{n-1} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \left(-\frac{1}{3} \right)^n$

● FINALLY

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n-1} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \left(-\frac{1}{3} \right)^n \right]$$

$$\Rightarrow \sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \frac{1}{3}$$

Question 126 (*****)

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6.$$

proof

$$\begin{aligned} S_1 &= \sum_{r=1}^{\infty} \frac{1}{2^r} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots \\ 2S_1 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots \\ S_1 &= \frac{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots}{2} \quad \text{SUBTRACT} \\ S_1 &= 1 + \frac{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots}{2} \\ S_1 &= 1 + \frac{S_1}{2} \quad \leftarrow \text{STANDARD GP} \\ S_1 &= 2 + 2 \cdot \frac{S_1}{2} \\ S_1 &= 2 + 2T \quad \dots \\ T &= \sum_{r=1}^{\infty} \frac{r}{2^r} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots \\ 2T &= 1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots \\ T &= \frac{1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots}{2} \quad \text{SUBTRACT} \\ T &= 1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \\ T &= 1 + (GP \text{ where sum to infinity is } 1) \\ T &= 2 \\ \therefore S_1 &= 2 + 2T \\ S_1 &= 2 + 2 \times 2 \\ S_1 &= 6 \end{aligned}$$

ALTERNATIVE USING STANDARD RESULT FROM STATISTICS

Let X be a GEOMETRIC DISTRIBUTION with probability p , $E(X) = \frac{1}{p}$
 $\text{Var}(X) = \frac{1-p}{p^2}$

NOW SET THE PROBABILITY OF A SUCCESS TO BE $\frac{1}{2}$ (e.g. Tossing A Fair coin)
 AND X IS THE NUMBER OF FLIPS UNTIL YOU GET HEAD

X : 1 2 3 4 5 ...
 $P(X=x)$: $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{32}$...

THIS $E(X) = \left(\frac{1^2}{2}\right) + \left(2^2 \times \frac{1}{4}\right) + \left(3^2 \times \frac{1}{8}\right) + \left(4^2 \times \frac{1}{16}\right) + \left(5^2 \times \frac{1}{32}\right) + \dots$
 $= \sum_{r=1}^{\infty} \frac{r^2}{2^r}$

BUT $\text{Var}(X) = E(X) - (E(X))^2$
 $\frac{1 - \frac{1}{2}}{\frac{1}{4}} = \frac{E(X)}{\frac{1}{2}} - \left(\frac{1}{2}\right)^2$
 $2 = \frac{E(X)}{\frac{1}{2}} - 4$
 $\frac{E(X)}{\frac{1}{2}} = 6$

Question 127 (*****)

Show clearly that

$$1 + \frac{1}{24} + \frac{1.4}{24 \cdot 48} + \frac{1.4 \cdot 7}{24 \cdot 48 \cdot 72} + \frac{1.4 \cdot 7 \cdot 10}{24 \cdot 48 \cdot 72 \cdot 96} - \dots = \frac{2}{\sqrt[3]{7}}.$$

proof

$$\begin{aligned} S &= 1 + \frac{1}{24} + \frac{1 \cdot 4}{24 \cdot 48} + \frac{1 \cdot 4 \cdot 7}{24 \cdot 48 \cdot 72} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{24 \cdot 48 \cdot 72 \cdot 96} + \dots \\ S &= \left(1 + \frac{1}{24}\right) + \frac{1 \cdot 4}{24 \cdot 48} + \frac{1 \cdot 4 \cdot 7}{24 \cdot 48 \cdot 72} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{24 \cdot 48 \cdot 72 \cdot 96} + \dots \\ S &= 1 + \frac{3 \cdot \left(\frac{1}{2}\right)}{24 \cdot (1 \cdot 2)} + \frac{3 \cdot \left(\frac{1}{2}\right) \cdot 3}{24 \cdot (1 \cdot 2 \cdot 3)} + \frac{3 \cdot \left(\frac{1}{2}\right) \cdot 3 \cdot 4}{24 \cdot (1 \cdot 2 \cdot 3 \cdot 4)} + \dots \end{aligned}$$

NOW FOR BINOMIAL EXPANSION SERIES THE POWER MUST BE DECREASING
 i.e. $n(n-1)(n-2) \dots$ ETC
 THAT MEANS THE SERIES

$$S = 1 + \frac{C(3)}{1} \left(\frac{1}{2}\right) + \frac{C(3)(2)}{1 \cdot 2} \left(\frac{1}{2}\right)^2 + \frac{C(3)(2)(1)}{1 \cdot 2 \cdot 3} \left(\frac{1}{2}\right)^3 + \dots$$

THIS IS THE BINOMIAL SERIES EXPANSION OF
 $(1 - \frac{1}{2})^{-3}$ WITH $x = \frac{1}{2}$ (NOTE THIS SERIES CONVERGES FOR $-0.5 < x < 0.5$)

$$S = \left(1 - \frac{1}{2}\right)^{-3} = \left(\frac{1}{2}\right)^{-3} = \left(\frac{1}{2}\right)^{-3} = \frac{2}{\sqrt[3]{7}}$$

Question 128 (****)

$$S_n = \sum_{r=1}^n (r^2 \times 2^r)$$

Use the standard techniques for the summation of a geometric series, to show that

$$S_n = (n^2 - 2n + 3) \times 2^{n+1} - 6.$$

[You may not use proof by induction in this question.]

proof

Handwritten solution for the summation of a geometric series using the standard technique of multiplying by the common ratio and subtracting.

$$\begin{aligned} \sum_{r=1}^n r^2 2^r &= S_n = 1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + 5 \times 2^5 + \dots + n^2 \times 2^n \\ -2S_n &= -1 \times 2^1 - 2 \times 2^2 - 3 \times 2^3 - 4 \times 2^4 - \dots - (n-1) \times 2^n - n^2 \times 2^{n+1} \quad \text{+BD} \\ \hline -S_n &= -1 \times 2^1 + 3 \times 2^2 + 5 \times 2^3 + 7 \times 2^4 + \dots + (2n-1) \times 2^n - n^2 \times 2^{n+1} \\ \text{Hence } S_n &= \frac{1 \times 2^1 - 3 \times 2^2 + 5 \times 2^3 - 7 \times 2^4 + \dots - (2n-1) \times 2^n + n^2 \times 2^{n+1}}{-1} \\ \text{Addition} \quad S_n &= 1 \times 2 + 2 \times 2^2 + 2 \times 2^3 + 2 \times 2^4 + \dots + 2 \times 2^n + (-1)^{n-1} \times 2^n + n^2 \times 2^{n+1} \\ S_n &= 2 + 2 \left[2^2 + 2^3 + 2^4 + \dots + 2^n \right] + (n^2 - 2n + 1) \times 2^{n+1} \\ &\quad \text{GP with } a=4, r=2, n=n-1 \\ S_n &= 2 + 2 \left(\frac{4(2^{n-1}-1)}{2-1} \right) + (n^2 - 2n + 1) \times 2^{n+1} \\ S_n &= 2 + 2(2^n - 1) + (n^2 - 2n + 1) \times 2^{n+1} \\ S_n &= 2 + 2 \times 2^n - 2 + (n^2 - 2n + 1) \times 2^{n+1} \\ S_n &= (n^2 - 2n + 3) \times 2^{n+1} - 6 \end{aligned}$$

Question 129 (****)

By showing a detailed method, sum the following series.

$$\sum_{r=0}^9 \left[(r+1) \times 11^r \times 10^{9-r} \right].$$

You may leave the answer in index form.

$$\boxed{}, \sum_{r=0}^9 \left[(r+1) \times 11^r \times 10^{9-r} \right] = 10^{11}$$

$\sum_{r=0}^9 \left[(r+1) \times 11^r \times 10^{9-r} \right] = ?$
 • WRITE A FEW TERMS OUT AND LOOK FOR A PATTERN
 $\Rightarrow S = 1(11^0 \times 10^9) + 2(11^1 \times 10^8) + 3(11^2 \times 10^7) + \dots + 9(11^8 \times 10^1) + 10(11^9 \times 10^0)$
 $\Rightarrow S = 1(11^0 \times 10^9) + 2(11^1 \times 10^8) + 3(11^2 \times 10^7) + \dots + 9(11^8 \times 10^1) + 10(11^9 \times 10^0)$
 $\Rightarrow S = 10^9 \left[1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + \dots + 9\left(\frac{11}{10}\right)^8 + 10\left(\frac{11}{10}\right)^9 \right]$
 • THE SERIES ABOVE IS A GEOMETRIC SERIES - MULTIPLY BY $-\frac{11}{10}$
 $\Rightarrow -\frac{11}{10}S = 10^9 \left[-\left(\frac{11}{10}\right) - 2\left(\frac{11}{10}\right)^2 - \dots - 9\left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10} \right]$
 • ADD THE LAST TWO LINES ABOVE
 $\Rightarrow \left(1 - \frac{11}{10}\right)S = 10^9 \left[1 + \left(\frac{11}{10}\right) + \left(\frac{11}{10}\right)^2 + \dots + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10} \right]$
 THIS IS A G.P.
 $\frac{a=1}{r=\frac{11}{10}} \quad S_n = \frac{a(r^n - 1)}{r - 1}$
 $\Rightarrow -\frac{1}{10}S = 10^9 \times \frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1} + 10^9 \left(-10 \times \left(\frac{11}{10}\right)^{10}\right)$
 $\Rightarrow -\frac{1}{10}S = 10^9 \times \frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1} - 10^9 \times \left(\frac{11}{10}\right)^{10}$
 $\Rightarrow -\frac{1}{10}S = 10^9 \times \left(\frac{11}{10}\right)^{10} - 10^9 - 10^9 \times \left(\frac{11}{10}\right)^{10}$
 $\Rightarrow S = 10^{11}$

Question 130 (****)

Use the ratio test to show that the following series converges

$$\sum_{n=1}^{\infty} \left[\frac{5^n + 1}{n^n + 8} \right]$$

You may assume without proof that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-n} = \frac{1}{e}$.

 , proof

As all the terms are positive we may ignore modulus in the ratio test

$$\begin{aligned} \frac{U_{n+1}}{U_n} &= \frac{\frac{5^{n+1} + 1}{(n+1)^{n+1} + 8}}{\frac{5^n + 1}{n^n + 8}} = \frac{5^{n+1} + 1}{(n+1)^{n+1} + 8} \times \frac{n^n + 8}{5^n + 1} \\ &< \frac{5^{n+1} + 5}{(n+1)^{n+1} + 8} \times \frac{n^n + 8}{5^n + 1} \\ &= \frac{5(5^n + 1)}{(n+1)^{n+1} + 8} \times \frac{n^n + 8}{5^n + 1} \\ &= \frac{5(n^n + 8)}{(n+1)^{n+1} + 8} \\ &< \frac{5 \times n^n}{(n+1)^{n+1} + 8} \\ &< \frac{A \times n^n}{(n+1)^{n+1}} \quad (\text{for sufficiently large } n) \\ &= A \frac{n^n}{(n+1)^{n+1}} < \frac{1}{n+1} \\ &= \frac{A}{n+1} \times \left(\frac{n+1}{n} \right)^{-n} = \frac{A}{n+1} \times \left(1 + \frac{1}{n} \right)^{-n} \\ &= \frac{A}{n+1} \times e^{-1} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

$\therefore \sum_{n=1}^{\infty} \left[\frac{5^n + 1}{n^n + 8} \right]$ converges

Question 131 (****)

$$f(x) = \frac{1}{\sqrt{1-x}}, \quad -1 < x < 1.$$

a) By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x) = \sum_{r=0}^{\infty} \binom{2r}{r} \left(\frac{1}{4}x\right)^r.$$

b) Find a similar expression for $\frac{1}{\sqrt{16-x^2}}$ and show further that

$$\frac{x}{(16-x^2)^{\frac{3}{2}}} = \sum_{r=1}^{\infty} \binom{2r}{r} \left(\frac{1}{16}x\right) \left(\frac{1}{8}x\right)^{2r-1}.$$

c) Determine the exact value of

$$\sum_{r=1}^{\infty} \binom{2r}{r} \left(\frac{5}{32}\right)^r \left(\frac{4}{25}\right)^r.$$

$$\square, \frac{25}{108}$$

a) $(1-x)^{-\frac{1}{2}} = 1 + \frac{\frac{1}{2}(-x)}{1!} + \frac{\frac{1}{2}(\frac{3}{2})(-x)^2}{2!} + \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})(-x)^3}{3!} + \dots + \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})\dots(\frac{n-1}{2})(-x)^n}{n!} + \dots$
 REWRITE THIS COMBINATORICALLY → PROBABLY IT IS EASIER TO LEARN THIS! AT THE FRONT OF THE SLIDES!
 $= 1 + \sum_{k=1}^{\infty} \left(\frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})\dots(\frac{k-1}{2})}{k!} (-x)^k \right) = 1 + \sum_{k=1}^{\infty} \left(\frac{(2k)!}{2^{2k} k!} \frac{1}{(2k-1)!!} (-x)^k \right)$
 $= 1 + \sum_{k=1}^{\infty} \left(\frac{(1 \cdot 3 \cdot 5 \dots (2k-1))}{k!} (-x)^k \right) = 1 + \sum_{k=1}^{\infty} \left(\frac{(2k)!}{2^k k!} \frac{1}{(2k-1)!!} (-x)^k \right)$
 $= 1 + \sum_{k=1}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} (-x)^k \right) = 1 + \sum_{k=1}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} (-x)^k \right)$
 $= 1 + \sum_{k=1}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} (-x)^k \right) = 1 + \sum_{k=1}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} (-x)^k \right)$
 $= 1 + \sum_{k=1}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} (-x)^k \right) = 1 + \sum_{k=1}^{\infty} \left(\frac{(2k-1)!!}{2^k k!} (-x)^k \right)$

$$\begin{aligned} \text{Differentiate both sides} \quad & \frac{d}{dx} [(1-x^2)^{-\frac{1}{2}}] = \frac{d}{dx} \left[\frac{1}{\sqrt{1-x^2}} \cdot \left(\frac{1}{1-x^2} \right)^{-\frac{1}{2}} \right] \\ \Rightarrow & -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \cdot (-2x) = \frac{1}{\sqrt{1-x^2}} \cdot x + \left(\frac{1}{1-x^2} \right)^{-\frac{1}{2}} \cdot \frac{1}{2} \\ \Rightarrow & \frac{x}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \\ \Rightarrow & \frac{x}{\sqrt{1-x^2}} \left(\frac{1}{\sqrt{1-x^2}} \right) = \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \\ \Rightarrow & \frac{x}{\sqrt{1-x^2}} \left(\frac{1}{\sqrt{1-x^2}} \right)^{-1} = \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \\ & \boxed{\text{Ans: } \frac{1}{2\sqrt{1-x^2}}} \\ & \boxed{2 = \frac{1}{2}} \\ & \Rightarrow \sqrt{1-x^2} = \frac{1}{2} \\ & = \sqrt{1 - \left(\frac{1}{2} \right)^2} = \frac{\sqrt{3}}{2} \\ & = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2\sqrt{1-x^2}}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2\sqrt{1-x^2}}} = \frac{\sqrt{3}}{10} \end{aligned}$$

Question 132 (****)

Determine, in terms of n , a simplified expression

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right].$$

$$\boxed{\frac{5}{24}}, \quad \sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{6} - \frac{n+5}{(n+5)!}, \quad \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$$

• SIMPLY WITH SPECIAL FRACTIONS — NOTE THAT THE NUMERATOR IS A QUADRATIC IN r , SO USE PARTIAL FRACTIONS. REMARK

$$1.E \quad \frac{r^2 + 9r + 19}{(r+5)!} = \frac{A}{(r+5)} + \frac{B}{(r+5)!}$$

$$\Rightarrow r^2 + 9r + 19 = A(r+5) + B$$

$$\Rightarrow r^2 + 9r + 19 = Ar + 5A + B$$

$$\therefore B = 14 \quad A = -1$$

• HENCE BY THE METHOD OF DIFFERENCES

$$\frac{r^2 + 9r + 19}{(r+5)!} = \frac{1}{(r+5)!} - \frac{1}{(r+4)!}$$

$r=1$	$\frac{1+9+19}{6!} = \frac{1}{6!} - \frac{1}{5!}$
$r=2$	$\frac{4+18+19}{7!} = \frac{1}{7!} - \frac{1}{6!}$
$r=3$	$\frac{9+27+19}{8!} = \frac{1}{8!} - \frac{1}{7!}$
$r=4$	$\frac{16+36+19}{9!} = \frac{1}{9!} - \frac{1}{8!}$
\vdots	\vdots
$r=n-1$	$\frac{(n-1)^2 + 9(n-1) + 19}{(n+4)!} = \frac{1}{(n+4)!} - \frac{1}{(n+3)!}$
$r=n$	$\frac{n^2 + 9n + 19}{(n+5)!} = \frac{1}{(n+5)!} - \frac{1}{(n+4)!}$

• ADDING

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{4!} + \frac{1}{5!} - \left[\frac{1}{(n+4)!} + \frac{1}{(n+5)!} \right]$$

$$= \frac{5}{3!} + \frac{1}{5!} - \left[\frac{1}{(n+4)!} + \frac{1}{(n+5)!} \right]$$

• NOW PROCEED AS BEFORE

$$\therefore \sum_{r=1}^{\infty} \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{5}{3!} - \frac{1}{5!} = \frac{1}{6} - \frac{1}{120} = \frac{19}{120}$$

• BOTH ARE THE SAME

$$\rightarrow \sum_{r=1}^{\infty} \left[\frac{(r-3)^2 + 9(r-3) + 19}{(r+5)!} \right] = \frac{5}{5!} - \frac{1}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 - 6r + 1 + 9r - 27 + 19}{(r+5)!} \right] = \frac{5}{5!} - \frac{1}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 3r - 16}{(r+5)!} \right] = \frac{5}{5!} - \frac{1}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{1+7+11}{5!} + \frac{6}{5!} - \frac{1}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{24}{5!} - \frac{1}{(n+5)!}$$

$$\rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{1}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$$

Question 133 (**)**

The n^{th} term of a series is given recursively by

$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \quad n \in \mathbb{N}, \quad n \geq 1,$$

where a is a positive constant.

Given further that $A_0 = 1$, show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$

, proof

[illegible]

Question 134 (****)

By considering the series expansions of $\ln(1-x^2)$ and $\ln\left(\frac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty} \left[\left(\frac{1}{2r} + \frac{1}{2r+1} \right) \left(\frac{1}{4} \right)^r \right].$$

$$\boxed{}, \boxed{-1 + \frac{1}{2} \ln 12}$$

• STARTING WITH THE FOLLOWING SERIES EXPANSIONS

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad |x| < 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad |x| < 1$$

• NOW USING THE SUGGESTED SERIES

$$\ln(1-x^2) = -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 - \dots$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots - \left[-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \dots \right]$$

$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots$$

• LOOKING AT THE REQUIRED SERIES

$$\left(\frac{1}{2} + \frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{16} + \left(\frac{1}{6} + \frac{1}{7}\right)\frac{1}{64} + \left(\frac{1}{8} + \frac{1}{9}\right)\frac{1}{256} + \dots$$

$$= \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{16} + \frac{1}{6} \times \frac{1}{64} + \frac{1}{8} \times \frac{1}{256} + \dots$$

$$= \frac{1}{3} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{16} + \frac{1}{7} \times \frac{1}{64} + \frac{1}{9} \times \frac{1}{256} + \dots$$

• NOW MANIPULATE THE SUGGESTED EXPANSIONS AS FOLLOWS

$$-\frac{1}{2}\ln(1-x^2) = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \dots$$

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) = 1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 + \frac{1}{9}x^8 + \dots$$

• ADDING THESE RESULTS WE OBTAIN

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln(1-x^2) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right)x^2 + \left(\frac{1}{4} + \frac{1}{5}\right)x^4 + \left(\frac{1}{6} + \frac{1}{7}\right)x^6 + \dots$$

• THIS WE WANT

$$\sum_{r=1}^{\infty} \left(\frac{1}{2r} + \frac{1}{2r+1} \right) x^{2r} = \frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln(1-x^2) - 1, \quad |x| < 1$$

• LET $x = \frac{1}{2}$

$$\sum_{r=1}^{\infty} \left(\frac{1}{2r} + \frac{1}{2r+1} \right) \left(\frac{1}{4}\right)^r = \frac{1}{2 \times \frac{1}{2}} \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) - \frac{1}{2}\ln\left(1-\left(\frac{1}{2}\right)^2\right) - 1$$

$$= \frac{1}{1} [\ln 3 - \ln \frac{3}{2}] - 1$$

$$= \frac{1}{1} [\ln 3 + \ln \frac{2}{3}] - 1$$

$$= \frac{1}{1} \ln 2 - 1$$

Question 135 (****)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty} \left[\binom{2r}{r} \frac{2}{2r+1} \left(\frac{x}{2} \right)^{2r+1} \right]$$

□, proof

STARTING FROM THE BINOMIAL EXPANSION OF $(1-x^2)^{-\frac{1}{2}}$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}(x^2) + \frac{\frac{1}{2}(\frac{3}{2})}{2!}x^4 + \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})}{3!}x^6 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3 \times 5}{2! \times 8}x^4 + \frac{5 \times 3 \times 7}{3! \times 16}x^6 + O(x^8)$$

INTEGRATING TERM BY TERM

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1 \times 2}{2! \times 2}x^2 + \frac{1 \times 2 \times 3}{2! \times 8}x^4 + \frac{1 \times 2 \times 3 \times 5}{3! \times 16}x^6 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{2! \times 2}x^2 + \frac{4!}{2! \times 8}x^4 + \frac{6!}{3! \times 16}x^6 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{2! \times 2}x^2 + \frac{4!}{2! \times 8}x^4 + \frac{6!}{3! \times 16}x^6 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{2! \times 2}x^2 + \frac{4!}{2! \times 8}x^4 + \frac{6!}{3! \times 16}x^6 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{r=0}^{\infty} \left[\frac{(2r)!}{(r!)^2} \left(\frac{x^2}{2} \right)^r \right]$$

INTEGRATING BOTH SIDES, WITHIN THE RANGE OF CONVERGENCE

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \sum_{r=0}^{\infty} \left[\frac{(2r)!}{(r!)^2} \frac{x^{2r}}{2^r} \right] dx$$

$$\arcsin x = \sum_{r=0}^{\infty} \left[\frac{(2r)!}{(r!)^2} \frac{x^{2r+1}}{2^{r+1}} \right] + C$$

$$\arcsin 0 = \sum_{r=0}^{\infty} \left[\frac{(2r)!}{(r!)^2} \frac{0^{2r+1}}{2^{r+1}} \right] + C$$

$$\arcsin 0 = \sum_{r=0}^{\infty} \left[\frac{(2r)!}{(r!)^2} \frac{0^{2r+1}}{2^{r+1}} \right] + C$$

As $\arcsin 0 = 0$

Question 136 (****)

The product operator \prod , is defined as

$$\prod_{i=1}^k [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \dots \times u_{k-1} \times u_k.$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right].$$

$$\boxed{\frac{8\sqrt{5}}{4} - 1}$$

● START BY WRITING A FEW TERMS EXPLICITLY (LOOK FOR A PATTERN)

$$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] = \left[\prod_{r=1}^1 \left(\frac{8r-7}{40r} \right) \right] + \left[\prod_{r=1}^2 \left(\frac{8r-7}{40r} \right) \right] + \left[\prod_{r=1}^3 \left(\frac{8r-7}{40r} \right) \right] + \dots$$

$$= \frac{1}{40} + \frac{1}{40} \times \frac{3}{20} + \frac{1}{40} \times \frac{3}{20} \times \frac{11}{60} + \frac{1}{40} \times \frac{3}{20} \times \frac{11}{60} \times \frac{19}{120} + \dots$$

$$= \frac{1}{40} + \frac{1 \times 3}{40 \times 20} + \frac{1 \times 3 \times 11}{40 \times 20 \times 60} + \frac{1 \times 3 \times 11 \times 19}{40 \times 20 \times 60 \times 120} + \dots$$

$$= \frac{1}{40 \times 1} + \frac{1 \times 3}{40^2 \times (2)} + \frac{1 \times 3 \times 11}{40^3 \times (2 \times 3)} + \frac{1 \times 3 \times 11 \times 19}{40^4 \times (2 \times 3 \times 4)} + \dots$$

● THIS RESEMBLES A BINOMIAL EXPANSION DUE TO THE FRACTIONS AT THE DENOMINATOR
THE NEXT THING TO EXAMINE IS TO EXAMINE THE NUMERATOR OF THE FORM $1 \times (n-1) \times (n-2) \times (n-3) \times \dots$

● BY INSPECTION THIS WILL COVER AS $-\frac{1}{8}, -\frac{3}{8}, -\frac{11}{8}, -\frac{19}{8}$

● THIS TRY AND ADJUST THE SIGNATURE

$$= \frac{1}{(40)(1)!} + \frac{1 \times 3}{(40)^2 (2)!} + \frac{1 \times 3 \times 11}{(40)^3 (3)!} + \frac{1 \times 3 \times 11 \times 19}{(40)^4 (4)!} + \dots$$

● $\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] = \frac{-\frac{1}{8}}{1!} + \frac{(\frac{1}{40})(-\frac{3}{8})}{2!} + \frac{(\frac{1}{40})(\frac{3}{40})(-\frac{11}{8})}{3!} + \frac{(\frac{1}{40})(\frac{3}{40})(\frac{11}{40})(-\frac{19}{8})}{4!} + \dots$

THIS IS A BINOMIAL EXPANSION WITH THE 1^{st} MISSING AT THE FRONT

$$= \left(1 - \frac{1}{40}\right)^{-\frac{1}{8}} - 1$$

$$= \left(\frac{39}{40}\right)^{-\frac{1}{8}} - 1$$

$$= \frac{8\sqrt{5}}{4} - 1$$

Question 137 (****)

A sequence $u_1, u_2, u_3, u_5, u_6, \dots$ is generated by the recurrence relation

$$n^2 u_{n+1} = (n+1)u_n, \quad n = 1, 2, 3, 4, \dots$$

It is further given that

$$\sum_{n=1}^{\infty} u_n = 1.$$

Find in exact form the value of u_1 .

$$\boxed{u_1 = \frac{1}{2e}}$$

START BY RE-WRITING THE RECURSION & EXPANDING A FEW TERMS

$$\Rightarrow n^2 u_{n+1} = (n+1)u_n$$

$$\Rightarrow u_{n+1} = \frac{(n+1)}{n^2} u_n$$

- $u_1 = a$
- $u_2 = \frac{2}{1^2} a$
- $u_3 = \frac{3}{2^2} \times \frac{2}{1^2} a$
- $u_4 = \frac{4}{3^2} \times \frac{3}{2^2} \times \frac{2}{1^2} a$
- $u_5 = \frac{5}{4^2} \times \frac{4}{3^2} \times \frac{3}{2^2} \times \frac{2}{1^2} a$
- \vdots
- \vdots
- \vdots

ADD UP THESE TERMS (BOUNDED) UP TO ∞

$$\Rightarrow \sum_{n=1}^{\infty} u_n = a \left[1 + \frac{2}{1^2} + \frac{3 \times 2}{2^2} + \frac{4 \times 3 \times 2}{3^2 \times 2^2} + \frac{5 \times 4 \times 3 \times 2}{4^2 \times 3^2 \times 2^2} + \dots \right]$$

$$\Rightarrow 1 = a \left[1 + \frac{2!}{1!^2} + \frac{3!}{2!^2} + \frac{4!}{3!^2} + \frac{5!}{4!^2} + \dots \right]$$

$$\Rightarrow 1 = a \left[\frac{1!}{0!1!} + \frac{2!}{1!1!} + \frac{3!}{2!1!} + \frac{4!}{3!1!} + \frac{5!}{4!1!} + \dots \right]$$

$$\Rightarrow 1 = a \left[\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!} + \dots \right]$$

$$\Rightarrow 1 = a \sum_{n=0}^{\infty} \left(\frac{n!}{n!} \right)$$

MANIPULATE EVEN IN ORDER THE SIGMA NOTATION

$$\Rightarrow 1 = a \sum_{n=0}^{\infty} \left(\frac{n}{n!} + \frac{1}{n!} \right)$$

FOR THE FIRST SUMMATION, AS IT SEEMS A BIT HARD AGAIN

$$\Rightarrow 1 = a \left[\sum_{n=0}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} \right]$$

$$\Rightarrow 1 = a \left[\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} \right]$$

$$\Rightarrow 1 = 2a \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\Rightarrow 1 = 2a e$$

$$\Rightarrow a = \frac{1}{2e}$$

$\therefore u_1 = \frac{1}{2e}$

Question 138 (****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \dots$$

$$\boxed{\frac{1}{6}}$$

DEFINING THE SERIES IN SIGMA NOTATION

$$S_{\infty} = \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+1)!}$$

ATTEMPT SUMMATION BY THE METHOD OF DIFFERENCES

• TRY

$$\frac{1}{(r+3)(r+1)!} = \frac{A}{(r+3)!} + \frac{B}{(r+1)!}$$

$$1 = A + B(r+3)(r+2)$$

NO A & B CAN SATISFY THE ABOVE

• TRY NEXT

$$\frac{1}{(r+3)(r+1)!} = \frac{A}{(r+3)!} + \frac{B}{(r+2)!}$$

$$\Rightarrow \frac{1}{(r+3)(r+1)!} = \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow \frac{r+2}{(r+3)(r+2)(r+1)!} = \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow \frac{r+2}{(r+3)!} = \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow r+2 = A + B(r+3) + Br$$

$$\therefore B=1 \quad A=-1$$

HENCE WE NOW HAVE A SPLITTABLE IDENTITY

$$\frac{1}{(r+3)(r+1)!} = \frac{1}{(r+2)!} - \frac{1}{(r+3)!}$$

• $r=1$: $\frac{1}{4 \times 2!} = \frac{1}{3!} - \frac{1}{4!}$

• $r=2$: $\frac{1}{5 \times 3!} = \frac{1}{4!} - \frac{1}{5!}$

• $r=3$: $\frac{1}{6 \times 4!} = \frac{1}{5!} - \frac{1}{6!}$

• $r=4$: $\frac{1}{7 \times 5!} = \frac{1}{6!} - \frac{1}{7!}$

• \vdots

• $r=N$: $\frac{1}{(N+3)(N+1)!} = \frac{1}{(N+2)!} - \frac{1}{(N+3)!}$

$$\Rightarrow \sum_{r=1}^N \frac{1}{(r+3)(r+1)!} = \frac{1}{3!} - \frac{1}{(N+3)!}$$

$$\Rightarrow \lim_{N \rightarrow \infty} \left[\sum_{r=1}^N \frac{1}{(r+3)(r+1)!} \right] = \lim_{N \rightarrow \infty} \left[\frac{1}{3!} - \frac{1}{(N+3)!} \right]$$

$$\Rightarrow \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+1)!} = \frac{1}{3!} - \frac{1}{\infty}$$

Question 139 (****)

- a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \, dx.$$

- b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part (a).

[you may assume that integration and summation commute]

$$\boxed{}, \boxed{\frac{1}{6}}$$

a) SIMPLY BY INTEGRATION BY PARTS

Let $u = \arctan x$ and $dv = x^3$. Then $du = \frac{1}{1+x^2} dx$ and $v = \frac{x^4}{4}$.

$$\int_0^1 x^3 \arctan x \, dx = \left[\frac{x^4}{4} \arctan x \right]_0^1 - \int_0^1 \frac{x^4}{4} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{1}{4} \arctan 1 - \frac{1}{4} \int_0^1 \frac{x^4}{1+x^2} dx$$

$$= \frac{\pi}{16} - \frac{1}{4} \int_0^1 \frac{x^2(x^2+1) - x^2}{1+x^2} dx$$

$$= \frac{\pi}{16} - \frac{1}{4} \int_0^1 (x^2 - \frac{x^2}{1+x^2}) dx$$

$$= \frac{\pi}{16} - \frac{1}{4} \left[\frac{x^3}{3} - \arctan x \right]_0^1$$

$$= \frac{\pi}{16} - \frac{1}{4} \left(\frac{1}{3} - \frac{\pi}{4} \right)$$

$$= \frac{\pi}{16} - \frac{1}{12} + \frac{\pi}{16} = \frac{\pi}{8} - \frac{1}{12}$$

b) FIND THE EXPANSION OF ARCTAN

$\frac{d}{dx} (\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots$

INTEGRATE BOTH SIDES

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots + C$

$\arctan 0 = 0 \implies C = 0$

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

THIS IS YOUR ANSWER

$\int_0^1 x^3 \arctan x \, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+4} dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+5)} \left[\frac{x^{2n+5}}{2n+5} \right]_0^1$

NEED TO SUM THIS SERIES BY PARTIAL FRACTIONS

$\frac{1}{(2n+1)(2n+5)} = \frac{1}{4} \left(\frac{1}{2n+1} - \frac{1}{2n+5} \right)$ (BY INSPECTION)

THIS IS YOUR ANSWER

$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{2n+1} - \frac{1}{2n+5} \right)$

INTEGRATE THE PARTIAL

$n=0: \frac{1}{4} \left(\frac{1}{1} - \frac{1}{5} \right)$

$n=1: \frac{1}{4} \left(\frac{1}{3} - \frac{1}{7} \right)$

$n=2: \frac{1}{4} \left(\frac{1}{5} - \frac{1}{9} \right)$

$n=3: \frac{1}{4} \left(\frac{1}{7} - \frac{1}{11} \right)$

\vdots

$n=4: \frac{1}{4} \left(\frac{1}{9} - \frac{1}{13} \right)$

\vdots

$n=5: \frac{1}{4} \left(\frac{1}{11} - \frac{1}{15} \right)$

FINALLY WE HAVE THE RESULT

$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4} \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{2n+1} - \frac{1}{2n+5} \right)$

$= \frac{1}{4} \lim_{N \rightarrow \infty} \left[1 - \frac{1}{3} - \frac{1}{2N+3} + \frac{1}{2N+5} \right]$

$= \frac{1}{4} \times \left(1 - \frac{1}{3} \right)$

$= \frac{1}{4} \times \frac{2}{3}$

$= \frac{1}{6}$

Question 140 (*****)

Find the sum to infinity of the following series.

$$1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \frac{1}{9 \times 4^4} + \dots$$

$$\boxed{}, \boxed{\ln 3}$$

METHOD A - USING SERIES EXPANSIONS

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$

SUBTRACTING THE EXPANSIONS GIVES

$$\ln(1+x) - \ln(1-x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + O(x^7)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + O(x^9) \right]$$

$$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = 2 \sum_{k=0}^{\infty} \left(\frac{2^{2k+1}}{(2k+1)4^{k+1}} \right)$$

NOW WITHIN THE RANGE OF CONVERGENCE, LET $x = \frac{1}{2}$

$$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = 2 \sum_{k=0}^{\infty} \left(\frac{2^{2k+1}}{(2k+1)4^{k+1}} \right)$$

$$\ln\left(\frac{3}{1}\right) = 2 \sum_{k=0}^{\infty} \left(\frac{2^{2k+1}}{(2k+1)2^{2k+2}} \right)$$

$$\ln 3 = 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)2^k}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^k} = \ln 3$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)4^k} = \ln 3$$

$$\therefore 1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \frac{1}{9 \times 4^4} + \dots = \ln 3$$

METHOD B - ALTERNATING TECHNIQUES

CONSIDER

$$\int_0^{\frac{1}{2}} x^{-2k} dx = \left[\frac{1}{-2k+1} x^{-2k+1} \right]_0^{\frac{1}{2}} = \frac{1}{-2k+1} \left[\left(\frac{1}{2}\right)^{-2k+1} - 0 \right] = \frac{1}{(2k-1)2^{2k-1}}$$

$$= \frac{1}{(2k-1)2^{2k-1}} = \frac{1}{(2k-1)4^{k-1}}$$

NOW CONSIDER THE INFINITE SUM GIVEN

$$1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k+1)4^k}$$

$$= 2 \times \frac{1}{2} \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)4^k} \right] = 2 \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)4^k} \right] = 2 \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)4^k} \right]$$

INTEGRABLE SUMMATION & INTEGRATION

$$\dots = 2 \int_0^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} x^{2k} \right] dx = 2 \int_0^{\frac{1}{2}} [1 + x^2 + x^4 + x^6 + \dots] dx$$

$$= 2 \int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx = \int_0^{\frac{1}{2}} \frac{2}{(1-x)(1+x)} dx$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1-x} + \frac{1}{1+x} dx = \left[-\ln|1-x| + \ln|1+x| \right]_0^{\frac{1}{2}}$$

$$= \left(\ln \frac{3}{2} - \ln \frac{1}{2} \right) - \left(\ln 1 - \ln 1 \right) = \ln \frac{3}{2} - \ln \frac{1}{2} = \ln 3$$

Question 141 (****)

By showing a detailed method, sum the following series.

$$\sum_{n=0}^{\infty} \left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} \right].$$

$$\boxed{\frac{3}{2}}$$

START BY TRIGONOMETRIC IDENTITIES

$$\sum_{n=0}^{\infty} \frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} = \sum_{n=0}^{\infty} \frac{\frac{1}{2} + \frac{1}{2}\cos\left(\frac{1}{3}n\pi\right)}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + \cos\left(\frac{1}{3}n\pi\right)}{2^n}$$

SPLIT INTO A GEOMETRIC PROGRESSION AND ANOTHER SERIES

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2^n} \right) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{1}{3}n\pi\right)}{2^n}$$

$$= \frac{1}{2} \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right] + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{1}{3}n\pi\right)}{2^n}$$

$\sum_{n=0}^{\infty} r^n = \frac{a}{1-r}$ \downarrow USING COMPLEX NUMBERS

$$= \left[\frac{1}{2} \times \frac{1}{1-\frac{1}{2}} \right] + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\operatorname{Re}\left[e^{\frac{1}{3}n\pi i} \right]}{2^n}$$

$$= \left[\frac{1}{2} \times \frac{1}{\frac{1}{2}} \right] + \frac{1}{2} \operatorname{Re} \left[\sum_{n=0}^{\infty} \left(\frac{e^{\frac{1}{3}\pi i}}{2} \right)^n \right]$$

$$= 1 + \frac{1}{2} \operatorname{Re} \left[\sum_{n=0}^{\infty} \left(\frac{e^{\frac{1}{3}\pi i}}{2} \right)^n \right]$$

NOTE THAT THE SERIES CONVERGES SINCE $\left| \frac{e^{\frac{1}{3}\pi i}}{2} \right| = \frac{1}{2} < 1$

$$= 1 + \frac{1}{2} \operatorname{Re} \left[1 + \frac{1}{2} e^{\frac{1}{3}\pi i} + \frac{1}{4} e^{\frac{2}{3}\pi i} + \frac{1}{8} e^{\pi i} + \frac{1}{16} e^{\frac{4}{3}\pi i} + \dots \right]$$

APPLY TAKING THE SUM TO INFINITY OF A G.P.

$$= 1 + \frac{1}{2} \operatorname{Re} \left[\frac{1}{1 - \frac{e^{\frac{1}{3}\pi i}}{2}} \right]$$

$$= 1 + \operatorname{Re} \left[\frac{1}{2 - e^{\frac{1}{3}\pi i}} \right]$$

MANIPULATE THE EXPRESSION TO EXTRACT THE REAL PART

$$= 1 + \operatorname{Re} \left[\frac{2 - e^{\frac{1}{3}\pi i}}{(2 - e^{\frac{1}{3}\pi i})(2 - e^{\frac{2}{3}\pi i})} \right]$$

$$= 1 + \operatorname{Re} \left[\frac{2 - e^{\frac{1}{3}\pi i}}{4 - 2e^{\frac{1}{3}\pi i} - 2e^{\frac{2}{3}\pi i} + 1} \right]$$

$$= 1 + \operatorname{Re} \left[\frac{2 - e^{\frac{1}{3}\pi i}}{5 - 4\left(\frac{1}{2}e^{\frac{1}{3}\pi i} + \frac{1}{2}e^{\frac{2}{3}\pi i}\right)} \right]$$

$$= 1 + \operatorname{Re} \left[\frac{2 - (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})}{5 - 4\cosh\left(\frac{1}{3}\pi i\right)} \right]$$

$$= 1 + \operatorname{Re} \left[\frac{2 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}{5 - 4\cosh\frac{\pi}{3}} \right] \quad \cosh(\pi i) = \cos \pi$$

$$= 1 + \operatorname{Re} \left[\frac{\frac{3}{2} + i\frac{\sqrt{3}}{2}}{5} \right]$$

$$= 1 + \frac{\frac{3}{2}}{5}$$

$$= 1 + \frac{3}{10}$$

$$\therefore \sum_{n=0}^{\infty} \left(\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} \right) = \frac{3}{2}$$

Question 142 (****)

Evaluate the following expression

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1}.$$

SP, 2

• REVERSE THE SUMMATION AS FOLLOWS

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1} = \sum_{k=1}^{\infty} \left[\frac{1}{\frac{k(k+1)}{2}} \right]$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{4+3} + \frac{1}{9+4} + \dots$$

• INTRODUCE A LIMIT (UNTIL THE SUMMATION, SAY N)

$$= \lim_{n \rightarrow \infty} \left[1 + \frac{1}{2} + \frac{1}{4+3} + \frac{1}{9+4} + \dots + \frac{1}{(n-1)+n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k^2 + (k+1)} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k(k+1)} \right]$$

• SPLIT INTO TWO FRACTIONS BY INSPECTION

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k} - \frac{1}{k+1} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right]$$

$$= 2$$

Question 143 (*****)

The first three terms of a series S are

$$S = 7 + 9x + 8x^2 + \dots$$

The n^{th} term of S is given by

$$A\left(\frac{3}{4}x\right)^n + B\left(\frac{1}{3}x\right)^n,$$

where A and B are non zero constants.

Given that the sum to infinity of S is 19, determine the value of x .

$$\boxed{x = \frac{12}{19}}$$

The image shows a handwritten solution for Question 143, divided into two parts. The left part finds constants A and B, and the right part finds the value of x.

Left part (Finding A and B):

Given $S = 7 + 9x + 8x^2 + \dots$ and the n^{th} term is $A\left(\frac{3}{4}x\right)^n + B\left(\frac{1}{3}x\right)^n$.

For $n=0$: $(A+B)x^0 = 7 \Rightarrow A+B=7$

For $n=1$: $\left(\frac{3}{4}A + \frac{1}{3}B\right)x = 9 \Rightarrow \frac{3}{4}A + \frac{1}{3}B = 9$

Solving the system:

$$\begin{aligned} A+B &= 7 & \times 9 & \Rightarrow 9A+9B=63 \\ \frac{3}{4}A+\frac{1}{3}B &= 9 & \times 12 & \Rightarrow 9A+4B=108 \\ \hline & & & \Rightarrow 5B = -45 \\ & & & \Rightarrow B = -9 \\ & & & \Rightarrow A = 16 \end{aligned}$$

Right part (Finding x):

Now the series is:

$$S = \sum_{n=0}^{\infty} \left[16\left(\frac{3}{4}\right)^n - 9\left(\frac{1}{3}\right)^n \right] x^n$$

$$S = 16 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n x^n - 9 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n x^n$$

$$S = 16 \sum_{n=0}^{\infty} \left(\frac{3}{4}x\right)^n - 9 \sum_{n=0}^{\infty} \left(\frac{1}{3}x\right)^n$$

$$S = 16 \left[1 + \frac{3}{4}x + \left(\frac{3}{4}x\right)^2 + \dots \right] - 9 \left[1 + \frac{1}{3}x + \left(\frac{1}{3}x\right)^2 + \dots \right]$$

$$S = 16 \times \left(\frac{1}{1-\frac{3}{4}x} \right) - 9 \left(\frac{1}{1-\frac{1}{3}x} \right)$$

Now the sum to infinity is 19:

$$\Rightarrow \frac{16}{1-\frac{3}{4}x} - \frac{9}{1-\frac{1}{3}x} = 19$$

$$\Rightarrow \frac{64}{4-3x} - \frac{27}{3-x} = 19$$

$$\Rightarrow 64(3-x) - 27(4-3x) = 19(3-x)(3-x)$$

$$\Rightarrow 192 - 64x - 108 + 81x = 19(9 - 6x + x^2)$$

$$\Rightarrow 84 + 17x = 171 - 114x + 19x^2$$

$$\Rightarrow 0 = 19x^2 - 247x + 84$$

$$\Rightarrow 0 = 19x^2 - 247x + 84$$

$$\Rightarrow 0 = 19x^2 - 247x + 84$$

$$\Rightarrow 0 = (19x - 12)(x - 4)$$

$$\Rightarrow x = \frac{12}{19} \text{ or } x = 4$$

BUT IN ORDER TO CONVERGE $\left| \frac{3}{4}x \right| < 1$ and $\left| \frac{1}{3}x \right| < 1$

$\therefore x = \frac{12}{19}$ (as 4 is greater than $\frac{4}{3}$ or 3)

Question 144 (****)

Find a simplified expression for the following sum

$$\frac{1}{100!} + \frac{1}{99!} + \frac{1}{2!98!} + \frac{1}{3!97!} + \frac{1}{4!96!} + \dots + \frac{1}{2!98!} + \frac{1}{99!} + \frac{1}{100!}$$

SPY, $\frac{2^{100}}{100!}$

Let $S = \frac{1}{100!} + \frac{1}{99!} + \frac{1}{2!98!} + \frac{1}{3!97!} + \frac{1}{4!96!} + \dots + \frac{1}{99!} + \frac{1}{100!}$

$S = \frac{1}{100!} + \frac{1}{99!} + \frac{1}{2!98!} + \frac{1}{3!97!} + \dots + \frac{1}{99!} + \frac{1}{100!}$

$S = \sum_{r=0}^{100} \frac{1}{r!(100-r)!}$

$S = \frac{1}{100!} \sum_{r=0}^{100} \frac{100!}{r!(100-r)!}$

$S = \frac{1}{100!} \sum_{r=0}^{100} \binom{100}{r}$

$S = \frac{1}{100!} \sum_{r=0}^{100} \left[\binom{100}{r} 1^r 1^{100-r} \right]$

$S = \frac{1}{100!} \times (1+1)^{100}$

$S = \frac{2^{100}}{100!}$

NOTE: $(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$

Question 145 (****)

Show by detailed workings that

$$\int_0^{\infty} \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$$

proof

• STARTING BY MULTIPLYING TOP & BOTTOM OF THE INTEGRAND BY e^{-x}

$$\int_0^{\infty} \frac{x}{e^x - 1} dx = \int_0^{\infty} \frac{x e^{-x}}{1 - e^{-x}} dx$$

• NOW FOR $|e^{-x}| < 1$ WE CAN WRITE $\frac{1}{1 - e^{-x}}$ (OR SUM TO INFINITY OF A GEOMETRIC PROGRESSION) AS

$$\frac{1}{1 - e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$$

• THIS WE HAVE

$$\begin{aligned} \dots &= \int_0^{\infty} x e^{-x} [1 + e^{-x} + e^{-2x} + e^{-3x} + \dots] dx \\ &= \int_0^{\infty} x [e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \dots] dx \\ &= \int_0^{\infty} x \sum_{k=1}^{\infty} e^{-kx} dx \end{aligned}$$

• INTERCHANGING SUMMATION & INTEGRATION

$$= \sum_{k=1}^{\infty} \int_0^{\infty} x e^{-kx} dx$$

• NOW BY SUBSTITUTION INTO A GRADUA FUNCTION, BY PARTS, DIFFERENTIATION (REMEMBER THE INTEGRAL SIGN OF GRADUA TENDS TO INFINITY)

$$\begin{aligned} &= \sum_{k=1}^{\infty} \int_0^{\infty} t e^{-kt} dt \quad \left\{ \begin{array}{l} x \rightarrow t \\ 1 \rightarrow k \end{array} \right\} \\ &= \sum_{k=1}^{\infty} \left[-\frac{1}{k^2} \right]_0^{\infty} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \end{aligned}$$

Question 146 (*****)

Consider the following convergent infinite series.

$$\sum_{r=0}^{\infty} \frac{2^{r+4}}{(r+3)r!}$$

Use appropriate techniques to show that the sum to infinity of the above series is

$$4(e^2 - 1)$$

 , proof

FROM THE ANSWER IT IS EVIDENT THAT THE EXPONENTIAL FUNCTION IS INVOLVED WITH SOME MODIFICATIONS

LET $S = \frac{2^1}{3 \times 0!} + \frac{2^2}{4 \times 1!} + \frac{2^3}{5 \times 2!} + \frac{2^4}{6 \times 3!} + \frac{2^5}{7 \times 4!} + \dots$

WITH $2 = 2$

THEN WE HAVE

$$\Rightarrow \frac{S}{2} = \frac{2^1}{3 \times 0!} + \frac{2^2}{4 \times 1!} + \frac{2^3}{5 \times 2!} + \frac{2^4}{6 \times 3!} + \frac{2^5}{7 \times 4!} + \dots$$

$$\Rightarrow \frac{d}{dx} \left(\frac{S}{2} \right) = \frac{2^1}{3!} + \frac{2^2}{4!} + \frac{2^3}{5!} + \frac{2^4}{6!} + \frac{2^5}{7!} + \dots$$

$$\Rightarrow \frac{d}{dx} \left(\frac{S}{2} \right) = 2^1 \left[\frac{1}{3!} + \frac{2^1}{4!} + \frac{2^2}{5!} + \frac{2^3}{6!} + \frac{2^4}{7!} + \dots \right]$$

$$\Rightarrow \frac{d}{dx} \left(\frac{S}{2} \right) = 2^2 e^2$$

$$\Rightarrow \frac{S}{2} = \int 2^2 e^2 dx$$

DOUBLE INTEGRATION BY PARTS OR DIFFERENTIATION ALONG THE INTEGRAL SIGN

$$\text{e.g. } \int x^2 e^{2x} dx = \frac{d}{dx} \left[\frac{1}{2} x^2 \right] = \frac{d}{dx} \left[\frac{1}{2} x^2 \right]$$

$$= \frac{d}{dx} \left[\frac{1}{2} x^2 \right] = \frac{1}{2} \frac{d}{dx} x^2 = \frac{1}{2} \cdot 2x = x$$

$$\therefore \int x^2 e^{2x} dx = 2e^2 - 2e^0 - 2e^0 + 2e^0 = (2^2 - 2 \times 2)e^2 + C$$

THUS WE NOW HAVE

$$\Rightarrow \frac{S}{2} = (2^2 - 2 \times 2)e^2 + C$$

TO FIND THE CONSTANT LET $2 = 0$

$$\frac{S}{2} \Big|_{2=0} = 0 \quad \text{SINCE } \frac{S}{2} = \frac{2^1}{3 \times 0!} + \frac{2^2}{4 \times 1!} + \frac{2^3}{5 \times 2!} + \frac{2^4}{6 \times 3!} + \dots$$

$$\therefore 0 = 2 + C$$

$$C = -2$$

THUS WE OBTAIN

$$\Rightarrow \frac{S}{2} = (2^2 - 2 \times 2)e^2 - 2$$

$$\Rightarrow S = (2^2 - 2 \times 2)e^2 - 2 \times 2$$

$$\Rightarrow \frac{2^5}{3!0!} + \frac{2^6}{4!1!} + \frac{2^7}{5!2!} + \frac{2^8}{6!3!} + \dots = (2^2 - 2 \times 2)e^2 - 2 \times 2$$

$$\Rightarrow \frac{2^1}{3!0!} + \frac{2^2}{4!1!} + \frac{2^3}{5!2!} + \frac{2^4}{6!3!} + \dots = (2^2 - 2 \times 2)e^2 - 2 \times 2$$

THIS

$$\sum_{r=0}^{\infty} \left[\frac{2^{r+4}}{(r+3)r!} \right] = 4e^2 - 4 = 4(e^2 - 1)$$

Question 147 (****)

A family of infinite geometric series S_k , has first term $\frac{k-1}{k!}$ and common ratio $\frac{1}{k}$, where $k = 3, 4, 5, 6, \dots, 99, 100$.

Find the value of

$$\frac{10^4}{100!} + \sum_{k=3}^{100} \left[[(k-1)(k-2)-1] S_k \right].$$

,

• WRITE THE FIRST FEW TERMS OF THIS GEOMETRIC PROGRESSION

$$\Rightarrow S_3 = \frac{2}{3!} + \frac{2}{3!} \cdot \frac{1}{3} + \frac{2}{3!} \cdot \frac{1}{3^2} + \dots$$

$$\Rightarrow S_k = \frac{k-1}{k!} \left[1 + \frac{1}{k} + \frac{1}{k^2} + \dots \right]$$

• SUMMING TO INFINITY USING $S_{\infty} = \frac{a}{1-r}$

$$\Rightarrow S_k = \frac{k-1}{k!} \cdot \frac{1}{1-\frac{1}{k}}$$

$$\Rightarrow S_k = \frac{k-1}{k!} \times \frac{k}{k-1}$$

$$\Rightarrow S_k = \frac{1}{k!}$$

• NEXT CONSIDER THE SUMMATION WITH THE ORIGINAL S_k FORM

$$\sum_{k=3}^{100} \left[S_k [(k-1)(k-2)-1] \right]$$

$$= \sum_{k=3}^{100} \left[\frac{(k-1)(k-2)-1}{(k-1)!} \right]$$

$$= \sum_{k=3}^{100} \left[\frac{(k-1)(k-2)}{(k-1)!} - \frac{1}{(k-1)!} \right]$$

$$= \sum_{k=3}^{100} \left[\frac{1}{(k-2)!} - \frac{1}{(k-1)!} \right]$$

• WRITING THE SUM EXPLICITLY IN A TABLE FORM

$\frac{1}{0!}$	$-\frac{1}{1!}$	$+\frac{1}{2!}$	$-\frac{1}{3!}$	$+\frac{1}{4!}$	$-\frac{1}{5!}$	$+\frac{1}{6!}$	$-\frac{1}{7!}$	$+\frac{1}{8!}$	$-\frac{1}{9!}$	$+\frac{1}{10!}$	$-\frac{1}{11!}$	$+\frac{1}{12!}$	$-\frac{1}{13!}$	$+\frac{1}{14!}$	$-\frac{1}{15!}$	$+\frac{1}{16!}$	$-\frac{1}{17!}$	$+\frac{1}{18!}$	$-\frac{1}{19!}$	$+\frac{1}{20!}$	$-\frac{1}{21!}$	$+\frac{1}{22!}$	$-\frac{1}{23!}$	$+\frac{1}{24!}$	$-\frac{1}{25!}$	$+\frac{1}{26!}$	$-\frac{1}{27!}$	$+\frac{1}{28!}$	$-\frac{1}{29!}$	$+\frac{1}{30!}$	$-\frac{1}{31!}$	$+\frac{1}{32!}$	$-\frac{1}{33!}$	$+\frac{1}{34!}$	$-\frac{1}{35!}$	$+\frac{1}{36!}$	$-\frac{1}{37!}$	$+\frac{1}{38!}$	$-\frac{1}{39!}$	$+\frac{1}{40!}$	$-\frac{1}{41!}$	$+\frac{1}{42!}$	$-\frac{1}{43!}$	$+\frac{1}{44!}$	$-\frac{1}{45!}$	$+\frac{1}{46!}$	$-\frac{1}{47!}$	$+\frac{1}{48!}$	$-\frac{1}{49!}$	$+\frac{1}{50!}$	$-\frac{1}{51!}$	$+\frac{1}{52!}$	$-\frac{1}{53!}$	$+\frac{1}{54!}$	$-\frac{1}{55!}$	$+\frac{1}{56!}$	$-\frac{1}{57!}$	$+\frac{1}{58!}$	$-\frac{1}{59!}$	$+\frac{1}{60!}$	$-\frac{1}{61!}$	$+\frac{1}{62!}$	$-\frac{1}{63!}$	$+\frac{1}{64!}$	$-\frac{1}{65!}$	$+\frac{1}{66!}$	$-\frac{1}{67!}$	$+\frac{1}{68!}$	$-\frac{1}{69!}$	$+\frac{1}{70!}$	$-\frac{1}{71!}$	$+\frac{1}{72!}$	$-\frac{1}{73!}$	$+\frac{1}{74!}$	$-\frac{1}{75!}$	$+\frac{1}{76!}$	$-\frac{1}{77!}$	$+\frac{1}{78!}$	$-\frac{1}{79!}$	$+\frac{1}{80!}$	$-\frac{1}{81!}$	$+\frac{1}{82!}$	$-\frac{1}{83!}$	$+\frac{1}{84!}$	$-\frac{1}{85!}$	$+\frac{1}{86!}$	$-\frac{1}{87!}$	$+\frac{1}{88!}$	$-\frac{1}{89!}$	$+\frac{1}{90!}$	$-\frac{1}{91!}$	$+\frac{1}{92!}$	$-\frac{1}{93!}$	$+\frac{1}{94!}$	$-\frac{1}{95!}$	$+\frac{1}{96!}$	$-\frac{1}{97!}$	$+\frac{1}{98!}$	$-\frac{1}{99!}$	$+\frac{1}{100!}$
----------------	-----------------	-----------------	-----------------	-----------------	-----------------	-----------------	-----------------	-----------------	-----------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	------------------	-------------------

$$= \frac{1}{0!} + \frac{1}{1!} - \left(\frac{1}{26!} + \frac{1}{99!} \right)$$

$$= 2 - \left(\frac{26}{99!} + 1 \right)$$

$$= 2 - \frac{100}{99!}$$

• FINALLY ADDING THE TERM AT THE FRONT OF THE SUMMATION

$$\frac{10^4}{100!} + \sum_{k=3}^{100} \left[[(k-1)(k-2)-1] S_k \right] = \frac{10^4}{100!} + 2 - \frac{100}{99!}$$

$$= \frac{100^4}{100!} + 2 - \frac{100}{99!}$$

$$= \frac{100}{99!} + 2 - \frac{100}{99!}$$

$$= 2$$

Question 148 (*****)

A discrete random variable X is geometrically distributed with parameter p .

Show that ...

a) ... $E(X) = \frac{1}{p}$.

b) ... $E(X) = \frac{1-p}{p^2}$.

 , proof

a) Let $X \sim \text{Geo}(p)$ $0 < p < 1$

x	1	2	3	4	5	...
$P(X=x)$	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	$(1-p)^4p$...

$$E(X) = p + 2p(1-p) + 3p(1-p)^2 + 4p(1-p)^3 + 5p(1-p)^4 + \dots$$

$$E(X) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + 5(1-p)^4 + \dots]$$

MULTIPLY THE ABOVE LINE BY $-(1-p)$

$$-(1-p)E(X) = p[-(1-p) - 2(1-p)^2 - 3(1-p)^3 - 4(1-p)^4 - \dots]$$

$$E(X) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + 5(1-p)^4 + \dots]$$

ADDING THE TWO LINES ABOVE (SEE ABOVE)

$$[-(1-p) + 1]E(X) = p[1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + \dots]$$

$$pE(X) = p[1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + \dots]$$

THIS IS A GEOMETRIC SERIES (with $a=1$)

$$E(X) = \frac{1}{1-(1-p)} = \frac{1}{p}$$

$E(X) = \frac{1}{p}$ Q.E.D.

ALTERNATIVE METHOD

FINDING FORMER AN EXPRESSION FOR EXPECTATION AS ZERO

$$E(X) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \dots]$$

Let $q = 1-p$

$$\Rightarrow E(X) = (1-q)[1 + 2q + 3q^2 + 4q^3 + \dots]$$

$$\Rightarrow E(X) = (1-q) \frac{d}{dq} [q + q^2 + q^3 + q^4 + \dots]$$

CONSIDERING G.P. with $a=q$
 $r=q$
 $S_{\infty} = \frac{a}{1-r}$

$$\Rightarrow E(X) = (1-q) \frac{d}{dq} \left[\frac{q}{1-q} \right]$$

$$\Rightarrow E(X) = (1-q) \times \frac{(1-q)(1) - q(-1)}{(1-q)^2}$$

$$\Rightarrow E(X) = (1-q) \times \frac{1-q+q}{(1-q)^2}$$

$$\Rightarrow E(X) = \frac{1}{1-q}$$

$$\Rightarrow E(X) = \frac{1}{p}$$

b) NEXT THE VARIANCE

$$E(X^2) = [1^2 \times p] + [2^2 \times p(1-p)] + [3^2 \times p(1-p)^2] + [4^2 \times p(1-p)^3] + \dots$$

$$E(X^2) = p[1 + 4(1-p) + 9(1-p)^2 + 16(1-p)^3 + 25(1-p)^4 + \dots]$$

MULTIPLY THE EXPRESSION BY $-(1-p)$

$$-(1-p)E(X^2) = p[-(1-p) - 4(1-p)^2 - 9(1-p)^3 - 16(1-p)^4 - 25(1-p)^5 - \dots]$$

$$E(X^2) = p[1 + 4(1-p) + 9(1-p)^2 + 16(1-p)^3 + 25(1-p)^4 + \dots]$$

ADDING THE TWO LINES ABOVE

$$[-(1-p) + 1]E(X^2) = p[1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots]$$

$$pE(X^2) = p[1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots]$$

$$E(X^2) = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots$$

REARRANGING THE ABOVE LINE (SEE ABOVE)

$$-(1-p)E(X^2) = -(1-p) - 3(1-p)^2 - 5(1-p)^3 - 7(1-p)^4 - 9(1-p)^5 - \dots$$

$$E(X^2) = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots$$

ADDING AGAIN THE TWO LINES ABOVE

$$[-(1-p) + 1]E(X^2) = p[1 + 2(1-p) + 2(1-p)^2 + 2(1-p)^3 + 2(1-p)^4 + \dots]$$

$$pE(X^2) = p[1 + 2(1-p) + 2(1-p)^2 + 2(1-p)^3 + 2(1-p)^4 + \dots]$$

THIS IS A GEOMETRIC G.P.

$$a=1-p$$

$$r=1-p$$

$$S_{\infty} = \frac{a}{1-r}$$

$$\therefore E(X^2) = 1 + 2 \times \frac{1-p}{1-(1-p)}$$

FINALLY THE VARIANCE

#FINALLY FINISHING UP

$$pE(X^2) = 1 + \frac{2(1-p)}{p}$$

$$p^2E(X^2) = p + 2 - 2p$$

$$p^2E(X^2) = 2 - p$$

$$E(X^2) = \frac{2-p}{p^2}$$

FINALLY THE VARIANCE

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2}$$

Question 149 (****)

Find the sum to infinity of the following convergent series

$$1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \frac{5^3}{5!} + \frac{6^3}{6!} + \dots$$

5e

$$1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \frac{5^3}{5!} + \dots = ?$$

$$S = 1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \frac{5^3}{5!} + \dots = \sum_{n=1}^{\infty} \frac{n^3}{n!}$$

Now expand the numerator - split the fraction - cancel down and separate the dummy variable

$$\Rightarrow S = \sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n!} = \left[\sum_{n=1}^{\infty} \frac{n^2}{n!} \right] + \left[2 \sum_{n=1}^{\infty} \frac{n}{n!} \right] + \left[\sum_{n=1}^{\infty} \frac{1}{n!} \right]$$

$$\Rightarrow S = \left[\sum_{n=1}^{\infty} \frac{n}{(n-1)!} \right] + \left[2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \right] + e$$

$$\Rightarrow S = \left[\sum_{n=0}^{\infty} \frac{n+1}{n!} \right] + 2 \left[\sum_{n=0}^{\infty} \frac{1}{n!} \right] + e$$

Repeat the process with the first sum

$$\Rightarrow S = \left[\sum_{n=0}^{\infty} \frac{n+1}{n!} \right] + \left[\sum_{n=0}^{\infty} \frac{1}{n!} \right] + 2e + e$$

$$\Rightarrow S = \left[\sum_{n=0}^{\infty} \frac{n+1}{n!} \right] + 4e$$

$$\Rightarrow S = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} + 4e$$

$$\Rightarrow S = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + 4e$$

$$\Rightarrow S = \sum_{n=0}^{\infty} \frac{1}{n!} + 4e$$

$$\Rightarrow S = e + 4e = 5e$$

Question 150 (****)

The function f is defined as

$$f(n, x) \equiv \sum_{r=0}^n \binom{n}{r} r x^r (1-x)^{n-r},$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$.

Show that $f(n, x) \equiv nx$.

☐, ☐ proof

$f(x) = \sum_{r=0}^n \left[\binom{n}{r} r x^r (1-x)^{n-r} \right]$

$\sum_{r=0}^n \left[\binom{n}{r} r x^r (1-x)^{n-r} \right] = \sum_{r=0}^n \left[\frac{n!}{r!(n-r)!} \times r \times x^r (1-x)^{n-r} \right]$

• THE TERM IS ACTUALLY ZERO

$= \sum_{r=0}^n \left[\frac{n!}{r!(n-r)!} \times r \times x^r (1-x)^{n-r} \right]$

• FACTORISE OUT OF THE SUMMATION THE GIVEN

$= nx \sum_{r=1}^n \frac{(n-1)!}{r!(n-r)!} \times r \times x^{r-1} (1-x)^{n-r}$

$= nx \sum_{r=1}^n \frac{(n-1)!}{(r-1)!(n-r)!} \times x^{r-1} (1-x)^{n-r}$

• ADJUST THE SUMMATION SO IT STARTS FROM $r=0$

$$\begin{matrix} r = N+1 \\ r = R+1 \end{matrix} \quad \begin{matrix} r = N \\ r = 1 \end{matrix} \quad \begin{matrix} R=N \\ R=N+1 \end{matrix} \quad \begin{matrix} R=N \\ R=0 \end{matrix}$$

$= nx \sum_{r=0}^n \frac{((N+1)-1)! \times x^{r-1} (1-x)^{N-(r-1)}}{((R+1)-1)!(N-(R+1))!}$

$= nx \sum_{r=0}^N \frac{N! x^r (1-x)^{N-r}}{R!(N-R)!}$

$= nx \sum_{r=0}^N \left[\frac{N!}{R!(N-R)!} \times x^r (1-x)^{N-r} \right]$

$= nx \sum_{r=0}^N \left[\binom{N}{r} x^r (1-x)^{N-r} \right]$

• BUT THIS IS THE DEFINITION OF A BINOMIAL EXPANSION

$= nx [x + (1-x)]^N$

$= nx [1]^N$

$= nx$

✓ AS REQUIRED

Question 151 (**)**

The binomial probability distribution $X \sim B(n, p)$ satisfies

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r},$$

where $r = 0, 1, 2, 3, \dots, n$ and $0 < p < 1$.

The expectation of X is defined as

$$E(X) \equiv \sum_{r=0}^n [r P(X = r)]$$

Show that

$$E(X) = np.$$

, proof

Handwritten proof for the expectation of a binomial distribution:

$$\begin{aligned}
 P(X=r) &= \binom{n}{r} p^r (1-p)^{n-r} \quad \text{where } X \sim B(n, p) \\
 E(X) &= \sum_{r=0}^n r P(X=r) = \sum_{r=0}^n r \binom{n}{r} p^r (1-p)^{n-r} \\
 &= \sum_{r=0}^n r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \\
 &= \sum_{r=1}^n r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \quad (\text{SINCE TERM FOR } r=0) \\
 &= np \sum_{r=1}^n \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r} \\
 &\quad \text{ADJUST THE SUMMATION'S "ONE INDEX" SO IT STARTS FROM ZERO} \\
 &= np \sum_{r=0}^{n-1} \frac{(n-1)!}{r!(n-1-r)!} p^r (1-p)^{(n-1)-r} \quad \left(\begin{matrix} r \rightarrow r+1 \\ r-1 \rightarrow r \end{matrix} \right) \\
 &= np \sum_{r=0}^{n-1} \frac{n!}{r!(n-r)!} p^r (1-p)^{(n-1)-r} \\
 &= np (p + (1-p))^n \\
 &= np \times 1^n \\
 &= np
 \end{aligned}$$

Question 152 (**)**

The function f is defined in terms of the real constants, a , b and c , by

$$f(x) = (a + bx + cx^2)(1-x)^{-3}, \quad x \in \mathbb{R}, \quad |x| < 1.$$

a) Show that

$$f(x) = a + (3a+b)x + \frac{1}{2} \sum_{n=2}^{\infty} \left[a(n+1)(n+2) + bn(n+1) + cn(n-1) \right] x^n.$$

b) Use the expression of part (a) to deduce the value of

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

6

a) SOMETHING WITH THE BINOMIAL EXPANSION OF $(1-x)^{-3}$

$$\rightarrow (1-x)^{-3} = 1 + \frac{3}{1}x + \frac{3 \times 4}{1 \times 2}x^2 + \frac{3 \times 4 \times 5}{1 \times 2 \times 3}x^3 + \frac{3 \times 4 \times 5 \times 6}{1 \times 2 \times 3 \times 4}x^4 + \dots$$

$$\rightarrow (1-x)^{-3} = 1 + 3x + \frac{3 \times 4}{1 \times 2}x^2 + \frac{3 \times 4 \times 5}{1 \times 2 \times 3}x^3 + \frac{3 \times 4 \times 5 \times 6}{1 \times 2 \times 3 \times 4}x^4 + \dots$$

• THIS THE COEFFICIENT OF x^n IS

$$\frac{1}{2} \left(\frac{2 \times 3 \times 4 \times \dots \times (n+2)}{n!} \right) = \frac{1}{2} \frac{(n+2)!}{n!} = \frac{1}{2} \frac{(n+2)(n+1)n!}{n!}$$

$$(1-x)^{-3} = \sum_{n=0}^{\infty} \frac{1}{2} (n+1)(n+2) x^n$$

• THIS LOOKING GOOD AT $f(x)$

$$\rightarrow f(x) = (a + bx + cx^2)(1-x)^{-3} = (a + bx + cx^2) \sum_{n=0}^{\infty} \frac{1}{2} (n+1)(n+2) x^n$$

$$\rightarrow f(x) = \frac{1}{2} a \sum_{n=0}^{\infty} (n+1)(n+2) x^n + \frac{1}{2} b \sum_{n=0}^{\infty} (n+1)(n+2) x^{n+1} + \frac{1}{2} c \sum_{n=0}^{\infty} (n+1)(n+2) x^{n+2}$$

$$\rightarrow f(x) = \frac{1}{2} a \times 2 \times 3 x^0 + \frac{1}{2} b \times 1 \times 2 \times 3 x^1 + \frac{1}{2} a \sum_{n=2}^{\infty} (n+1)(n+2) x^n + \frac{1}{2} b \sum_{n=1}^{\infty} (n+1)(n+2) x^n + \frac{1}{2} c \sum_{n=0}^{\infty} (n+1)(n+2) x^{n+2}$$

• $f(x) = (a + bx + cx^2)(1-x)^{-3}$

$$= a + (b+3a)x + \frac{1}{2} \sum_{n=2}^{\infty} (n+1)(n+2) x^n + \frac{1}{2} b \sum_{n=1}^{\infty} (n+1)(n+2) x^{n+1} + \frac{1}{2} c \sum_{n=0}^{\infty} (n+1)(n+2) x^{n+2}$$

• ANALISE THE SUMMATIONS SO THEY ALL START FROM $n=2$

$$\Rightarrow f(x) = a + (3a+b)x + \frac{1}{2} a \sum_{n=2}^{\infty} (n+1)(n+2) x^n + \frac{1}{2} b \sum_{n=2}^{\infty} n(n+1) x^n + \frac{1}{2} c \sum_{n=2}^{\infty} (n-1)n x^n$$

$$\Rightarrow f(x) = a + (3a+b)x + \frac{1}{2} \sum_{n=2}^{\infty} [a(n+1)(n+2) + bn(n+1) + cn(n-1)] x^n$$

b) NOW LOOKING AT THE COEFFICIENT OF x^n (MATCHING THE $\frac{1}{2}$ INTO THE SUM)

$$\left(\frac{1}{2} a n^2 + \frac{3}{2} a n + \frac{1}{2} b n^2 + \frac{1}{2} b n + \frac{1}{2} c n^2 - \frac{1}{2} c n \right) x^n$$

WORK FOR THIS TO RESOLVE TO $n^2 - 2n$

$$\therefore \begin{cases} a=0 \\ b=0 \end{cases} \quad \frac{1}{2} b - \frac{1}{2} c = 0 \quad \frac{1}{2} b + \frac{1}{2} c = 1$$

$$\begin{cases} b=0 \\ c=1 \end{cases}$$

$\Rightarrow f(x) = (a + bx + cx^2)(1-x)^{-3} = a + (3a+b)x + \frac{1}{2} \sum_{n=2}^{\infty} [a(n+1)(n+2) + bn(n+1) + cn(n-1)] x^n$

• LET $a=0, b=1, c=1$

$$\Rightarrow f(x) = (x + x^2)(1-x)^{-3} = x + \sum_{n=2}^{\infty} n^2 x^n$$

$$\Rightarrow (1-x)^{-3} = \left(\frac{1}{1-x} \right)^3 = \frac{1}{1-x} + \sum_{n=2}^{\infty} n^2 x^{n-1}$$

$$\Rightarrow \frac{1}{1-x} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n^2}{2^n} x^{n-1}$$

$$\Rightarrow \frac{1}{1-x} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{n^2}{2^n} x^{n-1}$$

$$\Rightarrow \frac{1}{1-x} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{n^2}{2^n} x^{n-1}$$

$$\therefore \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

Question 153 (****)

The function f is defined by

$$f(x) \equiv \sum_{n=1}^{\infty} [nx^n], \quad x \in \mathbb{R}, \quad |x| < 1.$$

Use the above function to find the sum to infinity of the following series.

$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \dots$$

$$\boxed{}, \quad \boxed{\frac{3}{4}}$$

Handwritten solution for the sum to infinity of the series $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \dots$.

Recognise as sum notation for $f(x)$.

$$\sum_{n=1}^{\infty} [nx^n] = \sum_{n=1}^{\infty} [n \cdot x^n] = \sum_{n=1}^{\infty} [n \cdot x^n] = \sum_{n=1}^{\infty} [n \cdot \left(\frac{1}{3}\right)^n]$$

The sum expression looks like a 'differentiation'.

Let $f(x) = \sum_{n=1}^{\infty} [n \cdot x^n]$ (if $x = \frac{1}{3}$ we get our sum)

$$f(x) = \sum_{n=1}^{\infty} [n \cdot x^n] = x \sum_{n=1}^{\infty} [n \cdot x^{n-1}]$$

$$f(x) = x \cdot \frac{d}{dx} \left[\sum_{n=1}^{\infty} x^n \right]$$

$$f(x) = x \cdot \frac{d}{dx} [x + x^2 + x^3 + x^4 + \dots]$$

Convergent GP with $a = x$ and $r = x$. $\frac{a}{1-r} = \frac{x}{1-x}$

$$f(x) = x \cdot \frac{d}{dx} \left[\frac{x}{1-x} \right]$$

$$f(x) = x \cdot \frac{[(1-x) \cdot 1 - x(-1)]}{(1-x)^2} = x \cdot \frac{(1-x) + x}{(1-x)^2} = \frac{x}{(1-x)^2}$$

Now substitute $x = \frac{1}{3}$

$$f\left(\frac{1}{3}\right) = \frac{\frac{1}{3}}{\left(1 - \frac{1}{3}\right)^2} = \frac{\frac{1}{3}}{\left(\frac{2}{3}\right)^2} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{1}{3} \cdot \frac{9}{4} = \frac{3}{4}$$

Question 154 (****)

Find the value of $x \in \mathbb{R}$ in the following equation

$$\sum_{n=0}^{\infty} \left[\frac{n(n-1)(n-2)(n-3)}{2^{n+k}} \right] = 3.$$

$$\boxed{}, \quad \boxed{k=4}$$

Let us note that the first 4 terms of the series are zero, so

$$\Rightarrow \sum_{n=4}^{\infty} \left[\frac{n(n-1)(n-2)(n-3)}{2^{n+k}} \right] = 3$$

$$\Rightarrow \frac{1}{2^k} \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) \left(\frac{1}{2}\right)^n] = 3$$

$$\Rightarrow \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) \left(\frac{1}{2}\right)^n] = 3 \times 2^k$$

Let $f(x) = \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) x^n]$

$$\Rightarrow f(x) = \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) x^{n-4} \times x^4]$$

$$\Rightarrow f(x) = x^4 \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) x^{n-4}]$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[\sum_{n=4}^{\infty} x^n \right]$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} [x + x^2 + x^3 + x^4 + \dots] \quad |x| < 1$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[\frac{1}{1-x} \right] \quad \left\{ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \right\}$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[\frac{(-1)^4 (1-x)^{-1}}{1-x} \right] \quad \left\{ \text{MANIPULATE F.R. FIRST DERIVATIVE} \right\}$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[-1 + \frac{1}{1-x} \right]$$

Let us note that the first 4 terms of the series are zero, so

$$f(x) = x^4 \frac{d^4}{dx^4} \left[-1 + (1-x)^{-1} \right]$$

$$f(x) = x^4 \left[2 \times 3 \times 4 \times (-1-x)^{-1} \right]$$

$$f(x) = \frac{24x^4}{(1-x)^2}$$

Hence we have

$$f\left(\frac{1}{2}\right) = \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) \left(\frac{1}{2}\right)^n] = \frac{24 \times \left(\frac{1}{2}\right)^4}{\left(1-\frac{1}{2}\right)^2}$$

$$\sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) \left(\frac{1}{2}\right)^n] = \frac{24 \times \frac{1}{16}}{\frac{1}{4}} = \frac{24}{4} = 48$$

$$\sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) \left(\frac{1}{2}\right)^n] = 48$$

From the L.H.S.

$$48 = 3 \times 2^k$$

$$16 = 2^k$$

$$k = 4$$

Question 155 (****)

Evaluate the following expression

$$\sum_{n=0}^{\infty} \sum_{m=0}^n \left[\frac{1}{2^{m+n}} \right].$$

Detailed workings must be shown.

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Work as follows

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^n \left[\frac{1}{2^{m+n}} \right] &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^n \left[\frac{1}{2^m} \times \frac{1}{2^n} \right] \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \sum_{m=0}^n \left(\frac{1}{2^m} \right) \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} \right) \right] \end{aligned}$$

G.P. with $a=1$
 $r=\frac{1}{2}$
n terms

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

$$S_n = \frac{1(1-\frac{1}{2^{n+1}})}{1-\frac{1}{2}}$$

Thus we simplify to

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \times \frac{1(1-\frac{1}{2^{n+1}})}{1-\frac{1}{2}} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \times 2 \cdot \left(1 - \frac{1}{2^{n+1}} \right) \right] \\ &= 2 \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \left(1 - \frac{1}{2^{n+1}} \right) \right] \\ &= 2 \sum_{n=0}^{\infty} \left[\frac{1}{2^n} - \frac{1}{2^{n+1}} \right] \end{aligned}$$

Work the geometric progression explicitly

$$\begin{aligned} \dots &= 2 \sum_{n=0}^{\infty} \frac{1}{2^n} - 2 \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \\ &= 2 \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right] - 2 \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right] \end{aligned}$$

\uparrow
 $\frac{1}{2^n}$
 \uparrow
 $\frac{1}{2^{n+1}}$

Using $S_{\infty} = \frac{a}{1-r}$ in each case

$$\begin{aligned} &= 2 \times \frac{1}{1-\frac{1}{2}} - 2 \times \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &= 2 \times \frac{1}{\frac{1}{2}} - 2 \times \frac{\frac{1}{2}}{\frac{1}{2}} \\ &= 2 \times 2 - 2 \times 1 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

$\frac{1}{2}$

Question 156 (****)

It is given that for $x \in \mathbb{R}$, $-\frac{1}{k} < x < \frac{1}{k}$, $k > 0$,

$$f(x, k) \equiv \frac{k+1}{(1-x)(1+kx)}.$$

Given further that

$$f(x, k) \equiv \sum_{r=0}^{\infty} [a_r x^r],$$

where a_r are functions of k , show that

$$\sum_{r=0}^{\infty} [a_r^2 x^r] = \frac{(1-kx)(1+k)^2}{(1-x)(1+kx)(1-k^2x)}.$$

You may assume that $\sum_{r=0}^{\infty} [a_r^2 x^r]$ converges.

 , proof

$f(x, k) = \frac{k+1}{(1-x)(1+kx)}$ $-\frac{1}{k} < x < \frac{1}{k}$, $k > 0$

• CONVERT A GIVEN FUNCTION INTO A POWER SERIES

$\Rightarrow f(x, k) = (k+1)(1-x)^{-1}(1+kx)^{-1} = (k+1) \left[1 + x + x^2 + x^3 + x^4 + \dots \right] \left[1 - kx + k^2x^2 - k^3x^3 + k^4x^4 + \dots \right]$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$
 $(1+kx)^{-1} = 1 - kx + k^2x^2 - k^3x^3 + k^4x^4 + \dots$

• FINDING THE EXPRESSION

$\Rightarrow f(x, k) = (k+1) \begin{bmatrix} 1 - kx + k^2x^2 - k^3x^3 + k^4x^4 - \dots \\ x - kx^2 + k^2x^3 - k^3x^4 + \dots \\ x^2 - kx^3 + k^2x^4 - \dots \\ x^3 - kx^4 + \dots \\ x^4 - \dots \end{bmatrix}$

$\Rightarrow f(x, k) = (k+1) \left[1 + (1-k)x + (1-k^2)x^2 + (-k+k^2)x^3 + (-k^2+k^3)x^4 + \dots \right]$

• CHECKS THE IDENTITY $a^n - b^n \equiv (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})(a-b)$

$\Rightarrow f(x, k) = (k+1) \left[1 + (1-k)x + (1-k^2)x^2 + (1-k^3)x^3 + (1-k^4)x^4 + \dots \right]$

$\Rightarrow f(x, k) = \sum_{r=0}^{\infty} \left[(1+k(-k)^r)x^r \right]$

$\Rightarrow f(x, k) = \sum_{r=0}^{\infty} \left[(1+k(-k)^r)x^r \right]$

• NEXT CONSIDER THE DEGREE SERIES

$\Rightarrow g(x, k) = \sum_{r=0}^{\infty} \left[(1+k(-k)^r)x^r \right] = \sum_{r=0}^{\infty} \left[(1+k(-k)^r)x^r \right]$

$\Rightarrow g(x, k) = \sum_{r=0}^{\infty} [x^r] + 2k \sum_{r=0}^{\infty} [(-k)^r]x^r + k^2 \sum_{r=0}^{\infty} [(-k)^{2r}]x^r$

$\Rightarrow g(x, k) = \sum_{r=0}^{\infty} [x^r] + 2k \sum_{r=0}^{\infty} [(-k)^r]x^r + k^2 \sum_{r=0}^{\infty} [(-k)^{2r}]x^r$

• NEXT RECALLING THE GEOMETRIC SERIES, WE ALSO KNOW

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots = \sum_{r=0}^{\infty} x^r$
 $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_{r=0}^{\infty} (-x)^r$

$\Rightarrow g(x, k) = \frac{1}{1-x} + \frac{2k}{1-kx} + \frac{k^2}{1-k^2x}$

$\Rightarrow g(x, k) = \frac{(1+k)(1-k^2x) + 2k(1-x)(1-kx) + k^2(1-x)(1+kx)}{(1-x)(1+kx)(1-k^2x)}$

$\Rightarrow g(x, k) = \frac{\begin{bmatrix} 1 + kx - k^2x - k^2x^2 \\ 2k - 2kx - 2k^2x + 2k^2x^2 \\ k^2 + k^2x - k^2x^2 \end{bmatrix}}{(1-x)(1+kx)(1-k^2x)}$

$\Rightarrow g(x, k) = \frac{(k^2+2k+1) - (k^2-2k^2-1)x}{(1-x)(1+kx)(1-k^2x)}$

$\Rightarrow g(x, k) = \frac{(k^2+2k+1) - (k^2-2k^2-1)x}{(1-x)(1+kx)(1-k^2x)}$

$\Rightarrow g(x, k) = \frac{(k^2+2k+1)(1-kx)}{(1-x)(1+kx)(1-k^2x)}$

$\Rightarrow g(x, k) = \frac{(k+1)^2(1-kx)}{(1-x)(1+kx)(1-k^2x)}$

Question 157 (****)

It is given that

◆ $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{1}{4}\pi$

◆ $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \frac{1}{12} \pi^2$

◆ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x) \, dx.$$

[you may assume that integration and summation commute]

$$\boxed{}, \frac{1}{48}[\pi^2 - 12\pi + 24\ln 2]$$

[illegible]

Question 158 (**)**

Given that p and q are positive, shown that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right].$$

You may find the series expansion of $\operatorname{artanh}(x^2)$ useful in this question.

, proof

STARTING FROM THE SERIES EXPANSION OF $\operatorname{artanh}(x)$ IN LOG FORM

$$\Rightarrow \operatorname{artanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \frac{1}{2} [\ln(1+x) - \ln(1-x)]$$

$$\Rightarrow \operatorname{artanh}(x) = \frac{1}{2} \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots \right]$$

$$\Rightarrow \operatorname{artanh}(x^2) = \frac{1}{2} \left[2x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \dots \right]$$

$$\Rightarrow \operatorname{artanh}(x^2) = x^2 + \frac{1}{3}x^6 + \frac{1}{5}x^{10} + \frac{1}{7}x^{14} + \dots$$

$$\Rightarrow \operatorname{artanh}(x^2) = \sum_{r=1}^{\infty} \left[\frac{x^{4r-2}}{2r-1} \right] = \frac{1}{2} \ln \left(\frac{1+x^2}{1-x^2} \right)$$

NEXT LET $x = \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$ IN THE ARGUMENT OF THE LOG ABOVE

$$\Rightarrow \frac{1+x^2}{1-x^2} = \frac{1 + \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^2}{1 - \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^2}$$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $(\sqrt{p}+\sqrt{q})^2$

$$\frac{1+x^2}{1-x^2} = \frac{(\sqrt{p}+\sqrt{q})^2 + (\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2 - (\sqrt{p}-\sqrt{q})^2}$$

$$\frac{1+x^2}{1-x^2} = \frac{p + 2\sqrt{pq} + q + p - 2\sqrt{pq} + q}{p + 2\sqrt{pq} + q - p + 2\sqrt{pq} - q}$$

$$\frac{1+x^2}{1-x^2} = \frac{2p + 2q}{4\sqrt{pq}} = \frac{p+q}{2\sqrt{pq}}$$

PUTTING ALL THE RESULTS TOGETHER

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right] = \frac{1}{2} \ln \left[\frac{1+x^2}{1-x^2} \right]$$

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right] = \frac{1}{2} \ln \left(\frac{p+q}{2\sqrt{pq}} \right)$$

$$2 \sum_{r=1}^{\infty} \left[\frac{1}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right] = \ln \left[\frac{p+q}{2\sqrt{pq}} \right]$$

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right] = \ln \left(\frac{p+q}{2} \right) - \ln \sqrt{pq}$$

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right] = \ln \left(\frac{p+q}{2} \right) - \frac{1}{2} \ln(pq)$$

THIS WE FINALLY HAVE THE DESIRED RESULT

$$\ln \left(\frac{p+q}{2} \right) - \frac{1}{2} \ln(pq) = \sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \right)^{4r-2} \right]$$

Question 159 (**)**

Show that

$$1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \frac{x^{12}}{12!} + \frac{x^{15}}{15!} + \dots = \frac{1}{3} \left[e^x + 2e^{\frac{1}{2}x} \cos\left(\frac{1}{2}\sqrt{3}x\right) \right].$$

You may find useful in this question the fact that if $z = e^{\frac{i2\pi}{3}}$ then $1 + z + z^2 = 0$.

 , **proof**

Left Page:

- Let $z = e^{\frac{i2\pi}{3}}$
- Then $z^3 = e^{i2\pi} = 1$, $1 + z + z^2 = 0$
- $z^2 = z^{-1}$, $z^4 = z$, $z^5 = z^2$, $z^6 = 1$, etc.
- AS THEY ARE THE 3RD ROOTS OF UNITY
- Next consider these exponential expansions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$$

$$e^{xz} = 1 + xz + \frac{(xz)^2}{2!} + \frac{(xz)^3}{3!} + \frac{(xz)^4}{4!} + \frac{(xz)^5}{5!} + \frac{(xz)^6}{6!} + \dots$$

$$e^{xz^2} = 1 + xz^2 + \frac{(xz^2)^2}{2!} + \frac{(xz^2)^3}{3!} + \frac{(xz^2)^4}{4!} + \frac{(xz^2)^5}{5!} + \frac{(xz^2)^6}{6!} + \dots$$
- We can write this as:

$$e^x + e^{xz} + e^{xz^2} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$$

$$= 1 + xz + \frac{(xz)^2}{2!} + \frac{(xz)^3}{3!} + \frac{(xz)^4}{4!} + \frac{(xz)^5}{5!} + \frac{(xz)^6}{6!} + \dots$$

$$= 1 + xz^2 + \frac{(xz^2)^2}{2!} + \frac{(xz^2)^3}{3!} + \frac{(xz^2)^4}{4!} + \frac{(xz^2)^5}{5!} + \frac{(xz^2)^6}{6!} + \dots$$

Right Page:

- TIDYING UP THE EXPANSIONS
- $e^x + e^{xz} + e^{xz^2} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$

$$= 1 + xz + \frac{(xz)^2}{2!} + \frac{(xz)^3}{3!} + \frac{(xz)^4}{4!} + \frac{(xz)^5}{5!} + \frac{(xz)^6}{6!} + \dots$$

$$= 1 + xz^2 + \frac{(xz^2)^2}{2!} + \frac{(xz^2)^3}{3!} + \frac{(xz^2)^4}{4!} + \frac{(xz^2)^5}{5!} + \frac{(xz^2)^6}{6!} + \dots$$
- Using Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) = \frac{1}{2} \left[e^{\frac{x}{2} + i\frac{x\sqrt{3}}{2}} + e^{\frac{x}{2} - i\frac{x\sqrt{3}}{2}} \right]$$

$$= \frac{1}{2} \left[e^{\frac{x}{2}} e^{i\frac{x\sqrt{3}}{2}} + e^{\frac{x}{2}} e^{-i\frac{x\sqrt{3}}{2}} \right]$$

$$= \frac{1}{2} \left[e^{\frac{x}{2}} \left(\cos\left(\frac{x\sqrt{3}}{2}\right) + i\sin\left(\frac{x\sqrt{3}}{2}\right) \right) + e^{\frac{x}{2}} \left(\cos\left(\frac{x\sqrt{3}}{2}\right) - i\sin\left(\frac{x\sqrt{3}}{2}\right) \right) \right]$$

$$= e^{\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)$$
- Final result: $\frac{1}{3} \left[e^x + 2e^{\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) \right]$