VARIOUS P.D.E.s

Question 1

It is given that $\psi = \psi(r,\theta)$.

Find a general circularly symmetric solution to the partial differential equation

$$\nabla^4 \psi = 0 .$$

$$\boxed{\psi(r) = A + Br^2 + C\ln r + Dr^2 \ln r}$$

$\nabla^4 \psi = 0$ & LOOK FOR A SYMMETRIC SOLUTION IN POLARS $\Rightarrow$

$\psi(r,\theta) = \psi(r)$

$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

HERE

$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$

• THUS

$\nabla^4 \psi = \nabla^2(\nabla^2 \psi) = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr}\right)$

$= \frac{d^4\psi}{dr^4} + \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\psi}{dr}\right) + \frac{1}{r}\frac{d^3\psi}{dr^3} + \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{dr}\right)$

$= \frac{d^4\psi}{dr^4} + \frac{d}{dr}\left[-\frac{1}{r^2}\frac{d\psi}{dr} + \frac{1}{r}\frac{d^2\psi}{dr^2}\right] + \frac{1}{r}\frac{d^3\psi}{dr^3} + \frac{1}{r}\left[-\frac{1}{r^2}\frac{d\psi}{dr} + \frac{1}{r}\frac{d^2\psi}{dr^2}\right]$

$= \frac{d^4\psi}{dr^4} + \frac{d}{dr}\left[\frac{1}{r}\frac{d^2\psi}{dr^2} - \frac{1}{r^2}\frac{d\psi}{dr}\right] + \frac{1}{r}\frac{d^3\psi}{dr^3} - \frac{1}{r^3}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{dr^2}$

$= \frac{d^4\psi}{dr^4} \cancel{- \frac{1}{r^2}\frac{d^2\psi}{dr^2}} + \frac{1}{r}\frac{d^3\psi}{dr^3} + \frac{2}{r^3}\frac{d\psi}{dr} - \frac{1}{r^2}\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d^3\psi}{dr^3} - \frac{1}{r^3}\frac{d\psi}{dr} \cancel{+ \frac{1}{r^2}\frac{d^2\psi}{dr^2}}$

$= \frac{d^4\psi}{dr^4} + \frac{2}{r}\frac{d^3\psi}{dr^3} - \frac{1}{r^2}\frac{d^2\psi}{dr^2} + \frac{1}{r^3}\frac{d\psi}{dr}$

• RETURNING TO THE O.D.E

$\Rightarrow \frac{d^4\psi}{dr^4} + \frac{2}{r}\frac{d^3\psi}{dr^3} - \frac{1}{r^2}\frac{d^2\psi}{dr^2} + \frac{1}{r^3}\frac{d\psi}{dr} = 0$

$\Rightarrow r^3\frac{d^4\psi}{dr^4} + 2r^2\frac{d^3\psi}{dr^3} - r\frac{d^2\psi}{dr^2} + \frac{d\psi}{dr} = 0$

LET $y = \frac{d\psi}{dr}$, $\frac{dy}{dr} = \frac{d^2\psi}{dr^2}$, $\frac{d^2y}{dr^2} = \frac{d^3\psi}{dr^3}$, $\frac{d^3y}{dr^3} = \frac{d^4\psi}{dr^4}$

WE COULD OF COURSE GO STRAIGHT INTO A TRIAL SOLUTION

$\Rightarrow r^3\frac{d^3y}{dr^3} + 2r^2\frac{d^2y}{dr^2} - r\frac{dy}{dr} + y = 0$

TRY SOLUTION $y = r^\lambda$, $\frac{dy}{dr} = \lambda r^{\lambda-1}$, $\frac{d^2y}{dr^2} = \lambda(\lambda-1)r^{\lambda-2}$, $\frac{d^3y}{dr^3} = \lambda(\lambda-1)(\lambda-2)r^{\lambda-3}$

$\Rightarrow \lambda(\lambda-1)(\lambda-2)r^\lambda + 2\lambda(\lambda-1)r^\lambda - \lambda r^\lambda + r^\lambda = 0$

$\Rightarrow \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1) - \lambda + 1 = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 \cancel{+ 2\lambda} + 2\lambda^2 \cancel{- 2\lambda} - \lambda + 1 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\Rightarrow \lambda^2(\lambda-1) - (\lambda-1) = 0$

$\Rightarrow (\lambda-1)(\lambda^2-1) = 0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda+1) = 0$

$\Rightarrow \lambda = \begin{cases} -1 \\ 1 \end{cases}$ $\leftarrow$ REPEATED

THUS

REPEATED

$y = Ar^{-1} + Br^{1} + Cr^{1}\ln r$

$\frac{d\psi}{dr} = \frac{A}{r} + Br + Cr\ln r$

$\psi = \int \frac{A}{r} + Br + Cr\ln r \, dr$ BY PARTS

$\ln r$	$\frac{1}{r}$
$\frac{1}{2}r^2$	r

$\psi = A\ln r + Br^2 + Cr^2\ln r - C\int \frac{1}{2}r \, dr$

$\psi = A\ln r + Br^2 + Cr^2\ln r + Er^2 + D$

$\psi(r) = A\ln r + Br^2 + Cr^2\ln r + D$