

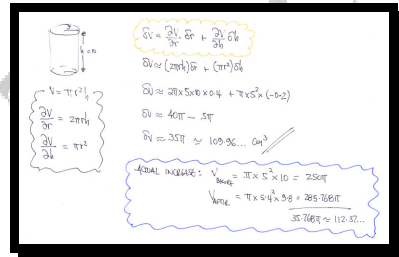
PARTIAL DIFFERENTIATION INTRODUCTION

Question 1 ()**

A right circular cylinder has radius 5 cm and height 10 cm.

Use a differential approximation to find an approximate increase in the volume of this cylinder if the radius increases by 0.4 cm and its height decreases by 0.2 cm.

$$35\pi \approx 109.96... \text{cm}^3$$

**Question 2 (**)**

$$y = \frac{xz^3}{w^4}, \quad w \neq 0.$$

Determine an approximate percentage increase in y , if x decreases by 5%, z increases by 2% and w decreases by 10%.

$$\approx 41\%$$

Handwritten solution for Question 2:

Given: $y = \frac{xz^3}{w^4}$

Take $x = z = w = 100$ (100%)

Differentiating with respect to x , z , and w :

$$\frac{\partial y}{\partial x} = \frac{z^3}{w^4}$$

$$\frac{\partial y}{\partial z} = \frac{3xz^2}{w^4}$$

$$\frac{\partial y}{\partial w} = -\frac{4xz^3}{w^5}$$

Approximate change in y :

$$\Delta y \approx \frac{\partial y}{\partial x} \Delta x + \frac{\partial y}{\partial z} \Delta z + \frac{\partial y}{\partial w} \Delta w$$

$$\Delta y \approx \frac{100^3}{100^4} (-5) + \frac{3 \times 100 \times 100^2}{100^4} (2) + \frac{-4 \times 100 \times 100^3}{100^5} (-10)$$

$$\Delta y \approx -\frac{5}{100} + \frac{6}{100} + \frac{40}{100}$$

$$\Delta y \approx \frac{41}{100}$$

Percentage increase: $\approx 41\%$

Question 3 ()**

The function f depends on u and v so that

$$f[u(x, y, z), v(x, y, z)] = uv, \quad u = x + 2y + z^2 \quad \text{and} \quad v = xyz.$$

Find simplified expressions for $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, in terms of x , y and z .

$$\frac{\partial f}{\partial x} = 2xyz + 2y^2z + yz^3, \quad \frac{\partial f}{\partial y} = 4xyz + x^2z + xz^3, \quad \frac{\partial f}{\partial z} = 3xyz^2 + x^2y + 2xy^2$$

Handwritten solution for Question 3:

$$f(u, v) = uv, \quad u = x + 2y + z^2, \quad v = xyz$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = v \cdot 1 + u \frac{\partial v}{\partial x} = 2yz + (x + 2y + z^2)yz = 2xyz + x^2yz + 2y^2yz + yz^3$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = v \cdot 2 + u \frac{\partial v}{\partial y} = 2v + u \cdot xz = 2xyz + x(x + 2y + z^2)z = 2xyz + x^2z + 2xy^2z + xz^3$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} = v \cdot 2z + u \frac{\partial v}{\partial z} = 2vz + u \cdot xy = 2xyz + (x + 2y + z^2)xy = 2xyz + x^2y + 2xy^2 + xyz^2$$

Question 4 ()**

A surface S is defined by the Cartesian equation

$$x^2 + y^2 = 25.$$

- Draw a sketch of S , and describe it geometrically.
- Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(3, 4, 5)$.

$$3x + 4y = 25$$

Handwritten solution for Question 4:

IT REPRESENTS THE CYLINDER SURFACE OF CIRCULAR BASED AROUND THE Z-AXIS AND HAS RADIUS 5

Now let $f(x, y, z) = x^2 + y^2 - 25$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, 0)$$

$$\nabla f_{(3,4,5)} = (6, 8, 0) \leftarrow \text{NORMAL VECTOR}$$

\therefore PERPENDICULAR TO THE CIRCULAR AT $(3, 4, 5)$ IS $\vec{n} = (6, 8, 0)$

EQUATION OF THE PLANE IS

$$3x + 4y + 0z = \text{constant}$$

$$1 \times (3, 4, 5) \quad 3 \times 3 + 4 \times 4 = \text{constant}$$

$$25 = \text{constant}$$

$\therefore 3x + 4y = 25$

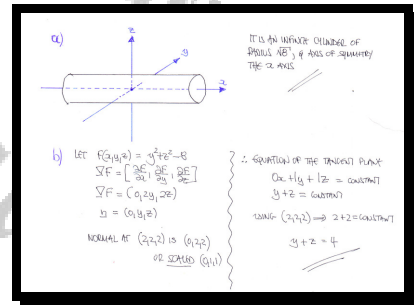
Question 5 (**)

A surface S is defined by the Cartesian equation

$$y^2 + z^2 = 8.$$

- a) Draw a sketch of S , and describe it geometrically.
- b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(2, 2, 2)$.

$$y + z = 4$$



Question 6 (**)

The function ϕ depends on u , v and w so that

$$\phi[u(x, y, z), v(x, y, z), w(x, y, z)] = uv + w.$$

It is further given that

$$u = x + 2y, \quad v = xyz \quad \text{and} \quad w = z^2.$$

By using the chain rule for partial differentiation find simplified expressions for $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$, in terms of x , y and z .

$$\boxed{\frac{\partial \phi}{\partial x} = 2yz(x+y)}, \quad \boxed{\frac{\partial \phi}{\partial y} = xz(x+4y)}, \quad \boxed{\frac{\partial \phi}{\partial z} = x^2y + 2xy^2 + 2z}$$

$\phi(u, v, w) = uv + w$

$u = x + 2y$
 $v = xyz$
 $w = z^2$

- $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial x}$

$$= v \times 1 + u \times yz + 1 \times 0$$

$$= yz + (x+2y)yz$$

$$= yz + xyz + 2y^2z$$

$$= yz(x+4y)$$
- $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial y}$

$$= v \times 2 + u \times xz + 1 \times 0$$

$$= 2yz + (x+2y)xz$$

$$= 2xyz + x^2z + 2xy^2z$$

$$= xz(2y + x + 2y^2) = xz(x+4y)$$
- $\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial z}$

$$= v \times 0 + u \times xy + 1 \times 2z$$

$$= (x+2y)xy + 2z$$

$$= x^2y + 2xy^2 + 2z$$

ALTERNATIVE APPROACH: MULTIPLY CHAIN RULE

$\phi(x, y, z) = uv + w = (x+2y)yz + z^2$
 $= xyz + 2y^2z + z^2$

- $\frac{\partial \phi}{\partial x} = yz + 2y^2z = yz(x+4y)$
- $\frac{\partial \phi}{\partial y} = xz + 4yz = xz(x+4y)$
- $\frac{\partial \phi}{\partial z} = xy + 2xy^2 + 2z$

Question 7 ()**

The point $P(1, y)$ lies on the contour with equation $x^2y + y^2x - 6 = 0$.

Determine the possible normal vectors at P

$$8\mathbf{i} + 5\mathbf{j}, \quad 3\mathbf{i} - 5\mathbf{j}$$

Handwritten solution for Question 7:

$$x^2y + y^2x - 6 = 0$$

Let $f(x, y) = x^2y + y^2x - 6$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2xy + y^2, x^2 + 2xy)$$

$$\nabla f|_{(1, y)} = (2(1)y + y^2, 1^2 + 2(1)y) = (y^2 + 2y, 1 + 2y)$$

$$\nabla f|_{(1, -5)} = (2(1)(-5) + (-5)^2, 1^2 + 2(1)(-5)) = (-5, -9)$$

$\therefore 8\mathbf{i} + 5\mathbf{j}$ or $3\mathbf{i} - 5\mathbf{j}$

Question 8 ()**

The radius of a right circular cylinder is increasing at the constant rate of 0.2 cm s^{-1} and its height is decreasing at the constant rate of 0.2 cm s^{-1} .

Determine the rate at which the volume of this cylinder is increasing when the radius is 5 cm and its height is 16 cm.

$$27\pi \approx 84.82 \dots \text{cm}^3 \text{s}^{-1}$$

Handwritten solution for Question 8:

$$V = \pi r^2 h$$

$$\frac{\partial V}{\partial r} = 2\pi h$$

$$\frac{\partial V}{\partial h} = \pi r^2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt}$$

$$\frac{dV}{dt} = (2\pi h) \frac{dr}{dt} + (\pi r^2) \frac{dh}{dt}$$

$$\frac{dV}{dt} = (2\pi \times 16 \times 0.2) + (\pi \times 5^2 \times (-0.2))$$

$$\frac{dV}{dt} = 32\pi - 5\pi$$

$$\frac{dV}{dt} = 27\pi$$

Question 9 ()**

A curve has implicit equation

$$x^2 + 2xy + y^3 = 8.$$

Use partial differentiation to find an expression for $\frac{dy}{dx}$.*No credit will be given for obtaining the answer with alternative methods*

$$\frac{dy}{dx} = -\frac{2x+2y}{2x+3y^2}$$

Let $f(x,y) = x^2 + 2xy + y^3$ and also $f(x,y) = 8$

$$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

Now as $f=8$, $\frac{df}{dx} = 0$

$$\Rightarrow 0 = (2x+2) + (2x+3y^2) \frac{dy}{dx}$$

$$\Rightarrow (2x+3y^2) \frac{dy}{dx} = -(2x+2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x+2}{2x+3y^2}$$

Question 10 (**)

A surface S is defined by the Cartesian equation

$$z = xy(x + y).$$

Find an equation of the tangent plane on S at the point $(1, 2, 6)$.

$$\boxed{}, \boxed{8x + 5y - z = 12}$$

METHOD A

$$z(x, y) = xy(x + y) = x^2y + xy^2$$

$$\frac{\partial z}{\partial x} = 2xy + y^2 \quad \frac{\partial z}{\partial x} \bigg|_{(1, 2, 6)} = 2 \times 1 \times 2 + 2^2 = 8$$

$$\frac{\partial z}{\partial y} = x^2 + 2xy \quad \frac{\partial z}{\partial y} \bigg|_{(1, 2, 6)} = 1^2 + 2 \times 1 \times 2 = 5$$

EQUATION OF THE TANGENT PLANE AT (x_0, y_0, z_0) HERE $(1, 2, 6)$

$$z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

$$z - 6 = 8(x - 1) + 5(y - 2)$$

$$z - 6 = 8x - 8 + 5y - 10$$

$$12 = 8x + 5y - z$$

METHOD B

$$\text{Let } f(x, y, z) = z - xy(x + y) = z - x^2y - xy^2$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (-2xy - y^2, -x^2 - 2xy, 1)$$

$$\nabla = \nabla f \bigg|_{(1, 2, 6)} = (-8, -5, 1)$$

EQUATION OF PLANE WITH ∇

$$-8x - 5y + z = \text{constant}$$

Using $P(1, 2, 6)$

$$-8(1) - 5(2) + 6 = \text{constant}$$

$$\text{constant} = -12$$

$$\Rightarrow -8x - 5y + z = -12$$

$$\Rightarrow 8x + 5y - z = 12$$

Question 11 (**)

A curve has implicit equation

$$e^{xy} + x + y = 1.$$

Use partial differentiation to find the value of $\frac{dy}{dx}$ at $(0,0)$.*No credit will be given for obtaining the answer with alternative methods*

$$\left. \frac{dy}{dx} \right|_{(0,0)} = -1$$

Handwritten solution for Question 11:

$$e^{xy} + x + y = 1$$

Let $f(x,y) = e^{xy} + x + y$
 $f(x,y) = 1$

$$\frac{dy}{dx} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{dy}{dx} = \frac{\frac{\partial}{\partial x}(e^{xy} + x + y)}{\frac{\partial}{\partial y}(e^{xy} + x + y)}$$

$$0 = ye^{xy} + 1 + (xe^{xy} + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{ye^{xy} + 1}{xe^{xy} + 1}$$

$$\left. \frac{dy}{dx} \right|_{(0,0)} = -1$$

Question 12 (**+)

The function f is defined as

$$f(x, y, z) = 2x + y^2 + xz,$$

where $x = 2t$, $y = t^2$ and $z = 3$.

- a) Use partial differentiation to find an expression for $\frac{df}{dt}$, in terms of t .
- b) Verify the answer obtained in part (a) by a method **not** involving partial differentiation.

$$\frac{df}{dt} = 4t^3 + 10$$

Q12

$f(x, y, z) = 2x + y^2 + xz$
 $x = 2t$
 $y = t^2$
 $z = 3$

a) Use partial differentiation to find an expression for $\frac{df}{dt}$, in terms of t .

$\frac{df}{dx} = 2 + z = 5$
 $\frac{df}{dy} = 2y = 2t^2$
 $\frac{df}{dz} = x = 2t$

$\frac{dx}{dt} = 2$
 $\frac{dy}{dt} = 2t$
 $\frac{dz}{dt} = 0$

$\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt}$
 $\frac{df}{dt} = 5 \times 2 + 2t^2 \times 2t + 2t \times 0$
 $\frac{df}{dt} = 10 + 4t^3$

b) Verify the answer obtained in part (a) by a method **not** involving partial differentiation.

$f(t) = 2(2t) + (t^2)^2 + 2t(3)$
 $f(t) = 4t + t^4 + 6t$
 $f(t) = 10t + t^4$
 $\frac{df}{dt} = 10 + 4t^3$

Question 13 (+)**

The function φ is defined as

$$\varphi(x, y, z) \equiv x^2 + y^2 + tz + t, \quad t \neq 0,$$

where $x = 3t$, $y = t^2$ and $z = \frac{1}{t}$.

- a) Use partial differentiation to find an expression for $\frac{d\varphi}{dt}$, in terms of t .
- b) Verify the answer obtained in part (a) by a method not involving partial differentiation.

$$\frac{d\varphi}{dt} = 4t^3 + 18t + 1$$

a) $\varphi(x, y, z) = x^2y^2 + tz + t$
 $x = 3t$
 $y = t^2$
 $z = \frac{1}{t}$

$\frac{d\varphi}{dt} = 2x \frac{dy}{dt} + 2y \frac{dx}{dt} + \left(\frac{dz}{dt}\right) + \left(\frac{d}{dt}\right)t$
 $\frac{d\varphi}{dt} = 6t + 4t^3 - \frac{1}{t^2} + 1$
 $\frac{d\varphi}{dt} = 4t^3 + 18t + 1$

b) CHECK: SUBSTITUTION
 $\varphi(x, y, z) = x^2y^2 + tz + t$
 $\varphi(t) = (3t)^2(t^2) + \left(\frac{1}{t}\right)t + t$
 $\varphi(t) = 9t^4 + 1 + t$
 $\frac{d\varphi}{dt} = 36t^3 + 1 = 4t^3 + 18t + 1$

Question 14 (+)**

Plane Cartesian coordinates (x, y) are related to plane polar coordinates (r, θ) by the transformation equations

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

a) Find simplified expressions for $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$, in terms of r and θ .

b) Deduce simplified expressions for $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$, in terms of x and y .

$$\frac{\partial r}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

Handwritten student work for Question 14. The work is organized into two parts, (a) and (b), each with a set of equations for r and θ in terms of x and y .

Part (a) uses polar coordinates (r, θ) . The student starts with the transformation equations $x = r \cos \theta$ and $y = r \sin \theta$. They then use implicit differentiation to find the partial derivatives of r and θ with respect to x and y . The results are:

- $\frac{\partial r}{\partial x} = \cos \theta$, $\frac{\partial r}{\partial y} = \sin \theta$
- $\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}$, $\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$

Part (b) uses Cartesian coordinates (x, y) . The student starts with the identity $r^2 = x^2 + y^2$. They then use implicit differentiation to find the partial derivatives of r and θ with respect to x and y . The results are:

- $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$, $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$
- $\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}$, $\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$

Question 15 (**+)

The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3.$$

At the point P , the two surfaces intersect each other at an angle θ .

Given further that P lies in the first octant, determine the exact value of $\cos \theta$.

$$\cos \theta = \frac{8}{3\sqrt{21}}$$

Handwritten solution for Question 15:

Given surfaces: $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$.

Point $P(1, y_0, z_0)$ lies on both surfaces.

Substituting $x=1$ into the second equation: $z = 1 + y^2 - 3 = y^2 - 2$.

Substituting $x=1$ and $z = y^2 - 2$ into the first equation: $1 + y^2 + (y^2 - 2)^2 = 9$.

Solving for y : $1 + y^2 + y^4 - 4y^2 + 4 = 9 \Rightarrow y^4 - 3y^2 + 5 = 9 \Rightarrow y^4 - 3y^2 - 4 = 0$.

Let $u = y^2$: $u^2 - 3u - 4 = 0 \Rightarrow (u-4)(u+1) = 0 \Rightarrow u = 4$ (since $u = -1$ is not possible for y^2).

Thus $y^2 = 4 \Rightarrow y = \pm 2$.

Since P is in the first octant, $y = 2$.

Substituting $x=1$ and $y=2$ into $z = x^2 + y^2 - 3$: $z = 1 + 4 - 3 = 2$.

Thus $P(1, 2, 2)$.

Let $f(x, y, z) = x^2 + y^2 + z^2 - 9$ and $g(x, y, z) = x^2 + y^2 - z - 3$.

Gradients at P :

$\nabla f = (2x, 2y, 2z) = (2, 4, 4)$

$\nabla g = (2x, 2y, -1) = (2, 4, -1)$

At P :

$\nabla f = (2, 4, 4)$

$\nabla g = (2, 4, -1)$

Dot product:

$(2, 4, 4) \cdot (2, 4, -1) = 4 + 16 - 4 = 16$

Magnitudes:

$|\nabla f| = \sqrt{2^2 + 4^2 + 4^2} = \sqrt{36} = 6$

$|\nabla g| = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{21}$

Thus $\cos \theta = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$.

Question 16 (+)**

The point $P(1,1,2)$ lies on both surfaces with Cartesian equations

$$z(z-1) = x^2 + xy \quad \text{and} \quad z = x^2y + xy^2.$$

At the point P , the two surfaces intersect each other at an angle θ .

Determine the exact value of θ .

$$\theta = \arccos\left(\frac{15}{19}\right)$$

Handwritten solution for Question 16:

Let $f(x,y,z) = x^2 + xy - z^2 + z$
 $\nabla f = (2x + y, x, -2z + 1)$
 $\nabla f|_{(1,1,2)} = (3, 1, -3)$

Let $g(x,y,z) = x^2y + xy^2 - z$
 $\nabla g = (2xy + y^2, x^2 + 2xy, -1)$
 $\nabla g|_{(1,1,2)} = (3, 3, -1)$

BY THE DOT PRODUCT
 $(3, 1, -3) \cdot (3, 3, -1) = |3, 1, -3| |3, 3, -1| \cos \theta$
 $9 + 3 + 3 = \sqrt{3^2 + 1^2 + 9} \sqrt{3^2 + 3^2 + 1} \cos \theta$
 $15 = 19 \cos \theta$
 $\cos \theta = \frac{15}{19}$
 $\theta = \arccos\left(\frac{15}{19}\right)$

Question 17 (*)**

The point $P(-1, 1, 3)$ lies on both surfaces with Cartesian equations

$$z(z-2) = x^2 - 2xy \quad \text{and} \quad z = xy(Ax + By),$$

where A and B are non zero constants.

The two surfaces intersect each other orthogonally at the point P .

Determine the value of A and the value of B .

$$\boxed{A = -14}, \quad \boxed{B = -17}$$

$\vec{r}(z=2) = 2^2 - 2xy \quad \vec{r} = xy \quad (A_2 + B_4) \quad \vec{r}(-1,1,3)$

$\bullet \quad z = 2^2 - 2xy$
 $2^2 - 2xy + 2x - 2z = 0$
 $\vec{f}(x,y,z) = 2^2 - 2xy + 2x - 2z$
 $\nabla \vec{f} = \left(\frac{\partial \vec{f}}{\partial x}, \frac{\partial \vec{f}}{\partial y}, \frac{\partial \vec{f}}{\partial z} \right)$
 $\nabla \vec{f} = (2 - 2y, -2x, 2 - 2z)$
 $\nabla \vec{f} = (-4, -2, -4)$
 $\vec{u}_{(1)}$
 TABLE NORMAL $\vec{r}(1,2)$

$\bullet \quad z = Ax^2y + Bxy^2$
 $Ax^2y + Bxy^2 - z = 0$
 $\vec{f}(x,y,z) = Ax^2y + Bxy^2 - z$
 $\nabla \vec{f} = \left(\frac{\partial \vec{f}}{\partial x}, \frac{\partial \vec{f}}{\partial y}, \frac{\partial \vec{f}}{\partial z} \right)$
 $\nabla \vec{f} = (2Ax + By^2, Ax^2 + 2By, -1)$
 $\nabla \vec{f} = (2+3, 4-2B, -1)$
 $\vec{u}_{(2)}$
 NORMAL $(-2B, 4, 2B, -1)$

Now $(-1,3)$ MUST SATISFY $\vec{f}(x,y,z)$
 $3 = A - B$
 $1E \quad A - B = 3$

SURFACES ARE ORTHOGONAL AT $(-1,1,3)$, IF NORMALS DOT = ZERO
 $(-2, -1, 2) \cdot (-2B, 4, 2B, -1) = 0$
 $-4 + 2B - 4B - 2 = 0$
 $-5A + 4B = 2$

SOLVING BY SUBSTITUTION $A = B + 3$
 $-5(B+3) + 4B = 2$
 $-5B - 15 + 4B = 2$
 $-B = 17$
 $B = -17$
 $\vec{u} \quad A = -14$

Question 18 (*)**

The function f depends on u , v and t so that

$$f\{u[x(t), y(t), z(t)], v[x(t), y(t), z(t)], t\} = u^2 + v + 2t.$$

It is further given that

$$u = x + y - 2z, \quad v = 4x - 2y - z \quad \text{and} \quad x = 2t, \quad y = t^2, \quad z = 5.$$

- a) Find simplified expressions for $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, in terms of x , y and z .
- b) Determine an expression for $\frac{df}{dt}$, in terms of t .

$$\frac{\partial f}{\partial x} = 2x + 2y - 4z + 5, \quad \frac{\partial f}{\partial y} = 2x + 2y - 4z + \frac{1}{\sqrt{z}} - 2, \quad \frac{\partial f}{\partial z} = -x - 4y + 8z - 1,$$

$$\frac{df}{dt} = 4t^3 + 12t^2 - 36t - 30$$

Handwritten solution for Question 18:

a) $f(u, v, t) = u^2 + v + 2t$
 $u = x + y - 2z$
 $v = 4x - 2y - z$
 $x = 2t, y = t^2, z = 5$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial x}$
 $= 2u \times 1 + 1 \times 4 + 2 \times \frac{1}{2t}$
 $= 2u + 4 + \frac{1}{t}$
 $= 2(x + y - 2z) + 4 + \frac{1}{t}$
 $= 2x + 2y - 4z + 5 + \frac{1}{t}$

$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial y}$
 $= 2u \times 1 + 1 \times (-2) + 2 \times \frac{1}{2t^2}$
 $= 2u - 2 + \frac{1}{t^2}$
 $= 2(x + y - 2z) - 2 + \frac{1}{t^2}$
 $= 2x + 2y - 4z - 2 + \frac{1}{t^2}$

$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial z}$
 $= 2u \times (-2) + 1 \times (-1) + 2 \times 0$
 $= -4u - 1$
 $= -4(x + y - 2z) - 1$
 $= -4x - 4y + 8z - 1$

b) $\frac{df}{dt} = \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt} + \frac{\partial f}{\partial t} \frac{dt}{dt}$
 $= (2u \times 2) + (2u \times 2) + (1 \times 4) + (2 \times 2) + (2 \times 2t) + 2$
 $= 4u + 4t + 2 = 4(x + y - 2z) + 4t + 2$
 $= 4(2t + t^2 - 10) + 4t + 2 = 8t + 4t^2 - 40 + 2 = 4t^2 + 8t - 38$
 $= 4t^2 + 8t - 38$

Question 19 (***)

The function z depends on u and v so that

$$z = (2x + 3y)^2, \quad u = x^2 + y^2 \quad \text{and} \quad v = x + 2y.$$

Find simplified expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, in terms of x and y .

$$\frac{\partial z}{\partial u} = \frac{2x + 3y}{2x - y}, \quad \frac{\partial z}{\partial v} = \frac{2(3x - 2y)(2x + 3y)}{2x - y}$$

Handwritten solution for Question 19:

Given: $z = (2x + 3y)^2$ and $v = x + 2y$

• First, $J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 1 & 2 \end{vmatrix} = 4x - 2y$

• Then, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{1}{\frac{\partial u}{\partial x}} = \frac{\partial z}{\partial x} \cdot \frac{1}{2x}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{1}{\frac{\partial v}{\partial x}} = \frac{\partial z}{\partial x} \cdot \frac{1}{1}$

Since $\frac{\partial z}{\partial x} = 4(2x + 3y)$, then:

$\frac{\partial z}{\partial u} = \frac{4(2x + 3y)}{2x} = \frac{2(2x + 3y)}{x}$

And $\frac{\partial z}{\partial v} = 4(2x + 3y)$

Therefore, $\frac{\partial z}{\partial u} = \frac{2(2x + 3y)}{x}$ and $\frac{\partial z}{\partial v} = 4(2x + 3y)$

Question 20 (***)

The function $w = \varphi[u(x, y), v(x, y)]$ satisfies

$$x = e^u \cos v \quad \text{and} \quad y = e^{-u} \sin v.$$

Determine simplified expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, in terms of u and v .

$$\boxed{}, \quad \frac{\partial u}{\partial x} = \frac{e^{-u} \cos v}{\cos 2v}, \quad \frac{\partial u}{\partial y} = \frac{e^u \sin v}{\cos 2v}, \quad \frac{\partial v}{\partial x} = \frac{e^{-u} \sin v}{\cos 2v}, \quad \frac{\partial v}{\partial y} = \frac{e^u \cos v}{\cos 2v}$$

METHOD A - BY DIRECT EVALUATION

$w = \varphi(u, v)$ $x = e^u \cos v$ $y = e^{-u} \sin v$

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \end{aligned} \quad \Rightarrow \quad \begin{aligned} dx &= e^u \cos v du - e^u \sin v dv \\ dy &= -e^u \sin v du + e^{-u} \cos v dv \end{aligned}$$

ELIMINATE du

$$\begin{aligned} e^u \cos v dx &= e^{2u} \cos^2 v du - e^{2u} \sin v \cos v dv \\ e^u \sin v dy &= -e^{2u} \sin v \cos v du + e^{2u} \sin^2 v dv \end{aligned} \quad \Rightarrow \quad \text{ADD}$$

$$\begin{aligned} e^u \cos v dx + e^u \sin v dy &= -\sin v \cos v du - \sin v \cos v dv \\ e^u \sin v dy &= -\sin v \cos v du + \cos^2 v dv \end{aligned} \quad \Rightarrow \quad \text{SUBTRACT}$$

$$\Rightarrow (e^u \cos v dx + e^u \sin v dy) - e^u \sin v dy = e^{2u} \cos^2 v du - e^{2u} \sin^2 v dv$$

$$\Rightarrow dx = \frac{e^{2u} \cos^2 v}{\cos 2v} du + \frac{e^{2u} \sin^2 v}{\cos 2v} dv \quad \leftarrow \text{dividing by } e^{2u}$$

$\therefore \frac{\partial x}{\partial u} = \frac{e^{2u} \cos^2 v}{\cos 2v}$ $\frac{\partial x}{\partial v} = \frac{e^{2u} \sin^2 v}{\cos 2v}$

IN A SIMILAR FASHION, ELIMINATE dv

$$\begin{aligned} e^u \cos v dx &= e^{2u} \cos^2 v du - e^{2u} \sin v \cos v dv \\ e^u \sin v dy &= -e^{2u} \sin v \cos v du + e^{2u} \sin^2 v dv \end{aligned} \quad \Rightarrow \quad \text{ADD}$$

$$\begin{aligned} e^u \cos v dx + e^u \sin v dy &= -\sin v \cos v du - \sin v \cos v dv \\ e^u \sin v dy &= -\sin v \cos v du + \cos^2 v dv \end{aligned} \quad \Rightarrow \quad \text{SUBTRACT}$$

METHOD B - USING JACOBIANS

IF $x = f(u, v)$ & $y = g(u, v)$ THEN

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\partial f}{\partial u} & \frac{\partial x}{\partial v} &= \frac{\partial f}{\partial v} \\ \frac{\partial y}{\partial u} &= \frac{\partial g}{\partial u} & \frac{\partial y}{\partial v} &= \frac{\partial g}{\partial v} \end{aligned} \quad \text{WHERE } J = \frac{\partial(x, y)}{\partial(u, v)}$$

• FIRST $x = e^u \cos v$ $y = e^{-u} \sin v$

• $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ -e^u \sin v & e^{-u} \cos v \end{vmatrix}$

$= e^{2u} \cos^2 v - (-\sin^2 v) = e^{2u} \cos^2 v + \sin^2 v$

• $\frac{\partial x}{\partial u} = \frac{\partial f}{\partial u} = \frac{e^{2u} \cos^2 v}{\cos 2v}$ • $\frac{\partial x}{\partial v} = \frac{\partial f}{\partial v} = \frac{e^{2u} \sin^2 v}{\cos 2v}$

• $\frac{\partial y}{\partial u} = \frac{\partial g}{\partial u} = \frac{e^{2u} \sin^2 v}{\cos 2v}$ • $\frac{\partial y}{\partial v} = \frac{\partial g}{\partial v} = \frac{e^{2u} \cos^2 v}{\cos 2v}$

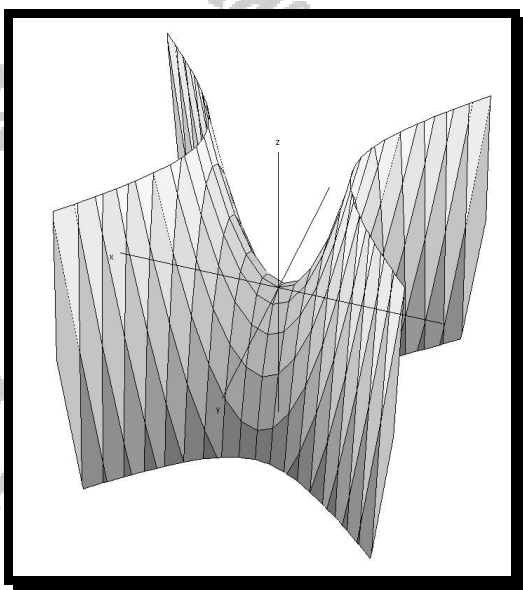
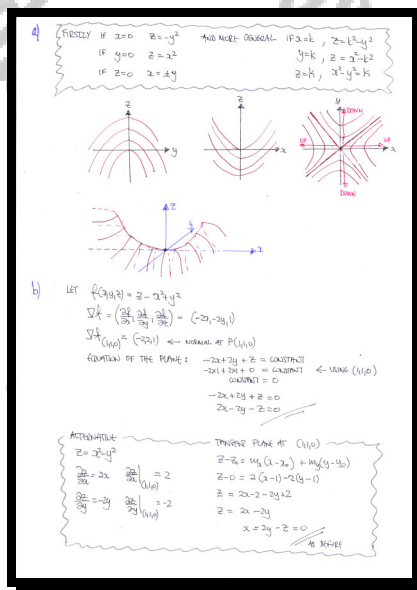
Question 21 (***)

A surface S has Cartesian equation

$$z = x^2 - y^2.$$

- a) Sketch profiles of S parallel to the y - z plane, parallel to the x - z plane, and parallel to the x - y plane.
- b) Find an equation of the tangent plane on S , at the point $P(1,1,0)$.

$$2x - 2y - z = 0$$



Question 22 (***)

A surface S is given parametrically by

$$x = at \cosh \theta, \quad y = bt \sinh \theta, \quad z = t^2,$$

where t and θ are real parameters, and a and b are non zero constants.

- a) Find a Cartesian equation for S .
- b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates (x_0, y_0, z_0) .

$$\boxed{}, \quad \boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \quad \boxed{2b^2 x x_0 - 2a^2 y y_0 = a^2 b^2 (z + z_0)}$$

a) ELIMINATE THE θ BY HYPERBOLIC IDENTITIES

$x = at \cosh \theta$ $y = bt \sinh \theta$ $z = t^2$

$\frac{x}{a} = t \cosh \theta$ $\frac{y}{b} = t \sinh \theta$

$\Rightarrow \cosh^2 \theta - \sinh^2 \theta = 1$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = z$

b) START WITH THE NORMAL

$\vec{r}(t, \theta) = \begin{pmatrix} at \cosh \theta \\ bt \sinh \theta \\ t^2 \end{pmatrix}$

$\vec{r}_t = \begin{pmatrix} a \cosh \theta \\ b \sinh \theta \\ 2t \end{pmatrix}$

$\vec{r}_\theta = \begin{pmatrix} at \sinh \theta \\ bt \cosh \theta \\ 0 \end{pmatrix}$

$\vec{n} = \vec{r}_t \times \vec{r}_\theta = \begin{pmatrix} -2abt \cosh \theta \sinh \theta \\ -a^2 t^2 \sinh^2 \theta - b^2 t^2 \cosh^2 \theta \\ abt^2 (\cosh^2 \theta - \sinh^2 \theta) \end{pmatrix}$

$\vec{n} = \begin{pmatrix} -2abz \cosh \theta \sinh \theta \\ -a^2 z \sinh^2 \theta - b^2 z \cosh^2 \theta \\ abz \end{pmatrix}$

EQUATION OF THE TANGENT PLANE

$\Rightarrow \left(-\frac{2ab}{a^2} \right) x + \left(\frac{2ab}{b^2} \right) y + z = \text{constant}$

$\Rightarrow \frac{2b}{a} x - \frac{2a}{b} y + z = \text{constant}$

LOAD THE POINT (x_0, y_0, z_0)

$\Rightarrow \left(\frac{2b}{a} \right) x_0 - \left(\frac{2a}{b} \right) y_0 + z_0 = \text{constant}$

$\Rightarrow \frac{2b}{a} x_0 - \frac{2a}{b} y_0 + z_0 = \frac{2b}{a} x - \frac{2a}{b} y + z$

$\Rightarrow 2b^2 x x_0 - 2a^2 y y_0 = a^2 b^2 (z + z_0)$

Question 23 (***)

The function z depends on x and y so that

$$z^2(x, y) = \frac{y - x^3 - xy^2}{x}, \quad x \neq 0.$$

Show that

$$xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = z^2 - \frac{y}{x}.$$

proof

Handwritten proof showing the derivation of the identity:

$$z^2 = \frac{y - x^3 - xy^2}{x} = \frac{y}{x} - x^2 - y^2$$

$$\begin{aligned} \bullet \quad xz \frac{\partial z}{\partial x} &= -\frac{y}{x} - 2x & \bullet \quad yz \frac{\partial z}{\partial y} &= \frac{y}{x} - 2y \\ \frac{\partial z}{\partial x} &= -\frac{y}{x^2} - 2x & \frac{\partial z}{\partial y} &= \frac{1}{x} - 2y \end{aligned}$$

$$\rightarrow xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = \cancel{-\frac{y}{x}} - 2x^2 + \cancel{\frac{y}{x}} - 2y^2$$

$$\rightarrow xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = -2(x^2 + y^2)$$

$$\rightarrow xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = -2(x^2 + y^2)$$

BUT $-(x^2 + y^2) = z^2 - \frac{y}{x}$ FROM ORIGINAL EXPRESSION

$$xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = z^2 - \frac{y}{x}$$

Q.E.D.

Question 24 (***)

The function f depends on u and v where

$$u = 2xy \quad \text{and} \quad v = x^2 - y^2.$$

Assuming $x \neq y$, $x \neq 0$ and $y \neq 0$, show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2 \left[u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} \right].$$

, proof

It is given that $z = f(u, v)$ where $u(x, y) = 2xy$ & $v(x, y) = x^2 - y^2$

• DIFFERENTIATE USING THE CHAIN RULE

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} (2y) + \frac{\partial z}{\partial v} (2x)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} (2x) + \frac{\partial z}{\partial v} (-2y)$$

• TIDING UP

$$\frac{\partial z}{\partial x} = 2y \frac{\partial z}{\partial u} + 2x \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = 2x \frac{\partial z}{\partial u} - 2y \frac{\partial z}{\partial v}$$

• ADDING THE EQUATIONS

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2y \frac{\partial z}{\partial u} + 2x \frac{\partial z}{\partial v} + 2x \frac{\partial z}{\partial u} - 2y \frac{\partial z}{\partial v}$$

$$= 2x \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + 2y \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

$$= 2 \left[x \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + y \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \right]$$

$$= 2 \left[u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} \right]$$

Question 25 (***)

The functions F and G satisfy

$$G(x, y) \equiv F[u(x, y), v(x, y)],$$

where u and v satisfy the following transformation equations.

$$u = x \cos y, \quad v = x \sin y.$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial x} \right]^2 + \left[\frac{1}{x} \frac{\partial G}{\partial y} \right]^2 = \left[\frac{\partial F}{\partial u} \right]^2 + \left[\frac{\partial F}{\partial v} \right]^2.$$

 , proof

Handwritten solution for the proof:

$G(x, y) = F(u(x, y), v(x, y))$ $u = x \cos y$
 $v = x \sin y$

START BY OBTAINING SOME BASIC PARTIAL DERIVATIVES

- $\frac{\partial u}{\partial x} = \cos y$ $\frac{\partial v}{\partial x} = \sin y$
- $\frac{\partial u}{\partial y} = -x \sin y$ $\frac{\partial v}{\partial y} = x \cos y$

BY THE CHAIN RULE WE HAVE

$$\frac{\partial G}{\partial x} = \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y$$

$$\frac{\partial G}{\partial y} = \frac{\partial F}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial y} = -\frac{\partial F}{\partial u} x \sin y + \frac{\partial F}{\partial v} x \cos y$$

FINALLY WE OBTAIN

$$\left(\frac{\partial G}{\partial x} \right)^2 + \left(\frac{1}{x} \frac{\partial G}{\partial y} \right)^2 = \left[\frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \right]^2 + \left[\frac{\partial F}{\partial u} (-\sin y) + \frac{\partial F}{\partial v} \cos y \right]^2$$

$$= \left[\frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \right]^2 + \left[\frac{\partial F}{\partial u} \sin y - \frac{\partial F}{\partial v} \cos y \right]^2$$

$$= \left(\frac{\partial F}{\partial u} \right)^2 \cos^2 y + 2 \frac{\partial F}{\partial u} \frac{\partial F}{\partial v} \cos y \sin y + \left(\frac{\partial F}{\partial v} \right)^2 \sin^2 y$$

$$+ \left(\frac{\partial F}{\partial u} \right)^2 \sin^2 y - 2 \frac{\partial F}{\partial u} \frac{\partial F}{\partial v} \sin y \cos y + \left(\frac{\partial F}{\partial v} \right)^2 \cos^2 y$$

$$= \left(\frac{\partial F}{\partial u} \right)^2 (\cos^2 y + \sin^2 y) + \left(\frac{\partial F}{\partial v} \right)^2 (\sin^2 y + \cos^2 y)$$

$$= \left(\frac{\partial F}{\partial u} \right)^2 + \left(\frac{\partial F}{\partial v} \right)^2$$

Question 26 (*)**

The function f is defined as

$$f(x, y, z) = x^3 - 75x + 3z(y-1)^2 + z^3.$$

The point Q lies on f .

The derivatives at Q in the directions $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + \mathbf{j} - \mathbf{k}$, are equal.

- a) Show that Q must lie on the surface of a sphere S .

The point $P(1, 3, a)$ lies on S .

- b) Find a vector equation of the normal line to S at P .

A sphere T is concentric to S and has radius three times as large as that of S .

The normal line to S at P intersects the surface of T at the points A and B .

- c) Determine the coordinates of A and B .

$$(x, y, z) = \left[\lambda + 1, 2\lambda + 3, 2(\lambda + 1)\sqrt{5} \right], \quad A(3, 7, 6\sqrt{5}) \quad B(-3, -5, -6\sqrt{5})$$

a) $\nabla f = (3x^2 - 75, 6z(y-1), 3z^2) = (0, 0, 0)$
 $\Rightarrow x^2 = 25, y = 1, z = 0$
 $\Rightarrow x = \pm 5, y = 1, z = 0$
 $\Rightarrow S: x^2 + (y-1)^2 + z^2 = 25$

b) $P(1, 3, a)$ lies on S
 $1^2 + (3-1)^2 + a^2 = 25 \Rightarrow a^2 = 20 \Rightarrow a = \pm 2\sqrt{5}$
 $\nabla f(P) = (3 - 75, 6a, 3a^2) = (-72, 12\sqrt{5}, 60)$
 $\Rightarrow \text{Normal line: } (x-1)/(-72) = (y-3)/(12\sqrt{5}) = (z-a)/60$

c) Sphere T is concentric to S and has radius three times as large as that of S .
 $T: x^2 + (y-1)^2 + z^2 = 225$
 $\Rightarrow \text{Intersection of normal line and } T$
 $\Rightarrow A(3, 7, 6\sqrt{5}), B(-3, -5, -6\sqrt{5})$

Question 27 (***)

The function z depends on x and y so that

$$z = r^2 \tan \theta, \quad x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

- a) Express r and θ in terms of x and y and hence determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, in terms of x and y .

Give each of the answers as a single simplified fraction

- b) Verify the answer to part (a) by implicit differentiation using Jacobians

$$\frac{\partial z}{\partial x} = \frac{y(x^2 - y^2)}{x^2}, \quad \frac{\partial z}{\partial y} = \frac{x^2 + 3y^2}{x}$$

a) USING STANDARD FORMS THE DIRECT IS TOWARD

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \Rightarrow \begin{aligned} r^2 &= x^2 + y^2 \\ \tan \theta &= \frac{y}{x} \end{aligned}$$

Hence we have

$$z = r^2 \tan \theta = (x^2 + y^2) \left(\frac{y}{x} \right) = xy + y^2 x^{-1}$$

- $\frac{\partial z}{\partial x} = y - xy^2 x^{-2} = y - \frac{xy^2}{x^2} = \frac{x^2 y - xy^2}{x^2} = \frac{y(x^2 - y^2)}{x^2}$
- $\frac{\partial z}{\partial y} = x + 2y x^{-1} = x + \frac{2y}{x} = \frac{x^2 + 2y^2}{x}$

b) FINDING THE JACOBIAN

$$J = \frac{\partial(z, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$$

$$J = \cos^2 \theta + \sin^2 \theta = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

Now we use the standard result

$$\begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial z}{\partial r} \\ \frac{\partial z}{\partial \theta} \end{bmatrix} \quad \text{NOTE THE MINUS SIGN!}$$

DEFINING WITH FINAL OF THE JACOBIAN

- $\frac{\partial z}{\partial x} = \frac{1}{J} \frac{\partial z}{\partial r} = \frac{1}{r^2} (r \cos \theta) = \cos \theta$
- $\frac{\partial z}{\partial y} = \frac{1}{J} \frac{\partial z}{\partial r} = \frac{1}{r^2} (-r \sin \theta) = -\sin \theta$
- $\frac{\partial z}{\partial x} = \frac{1}{J} \frac{\partial z}{\partial \theta} = \frac{1}{r^2} (r \sin \theta) = \frac{\sin \theta}{r}$
- $\frac{\partial z}{\partial y} = \frac{1}{J} \frac{\partial z}{\partial \theta} = \frac{1}{r^2} (r \cos \theta) = \frac{\cos \theta}{r}$

Now by the chain rule we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial x} = (r \cos \theta) \left(\frac{1}{r} \right) + (-r \sin \theta) \left(-\frac{\sin \theta}{r^2} \right) \\ &= \cos \theta + \sin^2 \theta \\ &= \cos \theta + \sin \theta \tan \theta \\ &= \cos \theta + \sin \theta \left(\frac{y}{x} \right) \\ &= \cos \theta + \frac{y \sin \theta}{x} = \frac{x \cos \theta + y \sin \theta}{x} = \frac{x^2 + y^2}{x^2} = \frac{x^2 + y^2}{x^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial y} = (r \cos \theta) \left(\frac{1}{r} \right) + (-r \sin \theta) \left(\frac{\cos \theta}{r^2} \right) \\ &= \cos \theta - \sin^2 \theta \\ &= \cos \theta - \sin \theta \tan \theta \\ &= \cos \theta - \frac{y \sin \theta}{x} = \frac{x \cos \theta - y \sin \theta}{x} = \frac{x^2 - y^2}{x^2} \end{aligned}$$

Question 28 (***)

The function ϕ depends on u and v so that

$$x = 2u + e^{2v} \quad \text{and} \quad y = 2v + e^{-2u}.$$

Without using standard results involving Jacobians, determine simplified expressions

for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, in terms of u and v .

$$\boxed{}, \quad \frac{\partial u}{\partial x} = \frac{1}{2 + 2e^{2(v-u)}}, \quad \frac{\partial u}{\partial y} = -\frac{e^{2v}}{2 + 2e^{2(v-u)}}, \quad \frac{\partial v}{\partial x} = \frac{e^{-2u}}{2 + 2e^{2(v-u)}}, \quad \frac{\partial v}{\partial y} = \frac{1}{2 + 2e^{2(v-u)}}$$

• START BY FINDING DIFFERENTIALS OF BOTH

$$x = 2u + e^{2v} \quad y = 2v + e^{-2u}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$dx = 2du + 2e^{2v} dv \quad dy = -2e^{-2u} du + 2dv$$

• ADDING THE EXPRESSIONS TO ELIMINATE THE "du" TERMS

$$\Rightarrow e^{-2u} dx + dy = [2 + 2e^{2(v-u)}] dv$$

$$\Rightarrow dv = \frac{e^{-2u}}{2 + 2e^{2(v-u)}} dx + \frac{1}{2 + 2e^{2(v-u)}} dy$$

• COMPARING WITH

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\therefore \frac{\partial v}{\partial x} = \frac{e^{-2u}}{2 + 2e^{2(v-u)}} = \frac{1}{2e^{2u} + 2e^{2v}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2 + 2e^{2(v-u)}} = \frac{e^{2u}}{2e^{2u} + 2e^{2v}}$$

• SIMILARLY STARTING FROM THE PREVIOUSLY DERIVED EXPRESSIONS

$$\left(\frac{dx}{dy} = 2 \frac{du}{dv} + 2e^{2v} \frac{dv}{dv} \right) \times e^{2u}$$

$$\left(\frac{dx}{dy} = 2 \frac{du}{dv} + 2e^{2v} \right) e^{2u} = 2e^{2u} \frac{du}{dv} + 2e^{2(v+u)}$$

• SUBSTITUTING TO ELIMINATE THE "dv" TERMS

$$\Rightarrow dx - 2e^{2u} dy = [2 + 2e^{2(v-u)}] du$$

$$\Rightarrow du = \frac{1}{2 + 2e^{2(v-u)}} dx + \frac{-e^{2u}}{2 + 2e^{2(v-u)}} dy$$

$$\frac{\partial u}{\partial x} = \frac{1}{2 + 2e^{2(v-u)}} = \frac{e^{-2u}}{2e^{-2u} + 2e^{2v}}$$

$$\frac{\partial u}{\partial y} = \frac{-e^{2u}}{2 + 2e^{2(v-u)}} = \frac{-e^{2(v+u)}}{2e^{2u} + 2e^{2v}}$$

Question 29 (***)

A hill is modelled by the equation

$$f(x, y) = 300e^{-(x^2+y^2)}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

A railway runs along the straight line with equation

$$y = x - 2.$$

Determine the steepest slope that the train needs to climb.

$$\pm 300\sqrt{2}e^{-\frac{5}{2}}$$

$(x, y) = (x, x-2)$
 $\bullet \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left(-600xe^{-(x^2+y^2)}, -600ye^{-(x^2+y^2)} \right)$
 $= -600e^{-(x^2+y^2)} (x, y)$
 \bullet THE LINE $y = x-2$ HAS DIRECTION VECTOR $(1, 1)$ BY INSPECTION
 A UNIT VECTOR IN THAT DIRECTION IS $\frac{1}{\sqrt{2}}(1, 1)$
 \bullet THE DIRECTIONAL DERIVATIVE IN THE DIRECTION OF THIS UNIT IS
 $-600e^{-(x^2+y^2)} (x, y) \cdot \frac{1}{\sqrt{2}}(1, 1)$
 $= -300\sqrt{2}e^{-(x^2+y^2)} (x+y)$
 \bullet LET $g(x) = -300\sqrt{2}e^{-(x^2+(x-2)^2)} (x+x-2)$
 $g(x) = -300\sqrt{2}(2x-2)e^{-(2x^2-4x+4)}$
 $g'(x) = -600\sqrt{2}e^{-2x^2+4x-4} + 600\sqrt{2}(1-2)(-4x+4)e^{-2x^2+4x-4}$
 $g'(x) = 600\sqrt{2}e^{-2x^2+4x-4} [-1 + (1-2)(4-4x)]$
 $g'(x) = 600\sqrt{2}e^{-2x^2+4x-4} [4(1-2)^2 - 1]$
 $g'(x) = 600\sqrt{2}e^{-2x^2+4x-4} [2(1-2) + 1]$
 $g'(x) = 600\sqrt{2}e^{-2x^2+4x-4} (1-2)(3-2x)$
 \bullet SOLVING FOR ZERO YIELDS $2x = \frac{1}{2}$ $g'(x) < \frac{300\sqrt{2}}{-300\sqrt{2}}e^{-\frac{5}{2}}$

Question 30 (***)

The functions F and G satisfy

$$G(u, v) \equiv F[x(u, v), y(u, v)],$$

where x and y satisfy the following transformation equations.

$$x = uv, \quad y = \frac{u+v}{u-v}.$$

Use the chain rule for partial derivatives to show that

$$u \frac{\partial G}{\partial u} + v \frac{\partial G}{\partial v} = 2x \frac{\partial F}{\partial x} \quad \text{and} \quad \frac{u^2 - v^2}{2uv} \left[v \frac{\partial G}{\partial v} - u \frac{\partial G}{\partial u} \right] = 2y \frac{\partial F}{\partial y}.$$

, proof

$G(u, v) = F[x(u, v), y(u, v)]$

- $x = uv$
- $y = \frac{u+v}{u-v}$

START BY DEFINING $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$

- $\frac{\partial x}{\partial u} = v$
- $\frac{\partial x}{\partial v} = u$
- $\frac{\partial y}{\partial u} = \frac{(u-v) - (u+v)}{(u-v)^2} = -\frac{2v}{(u-v)^2}$
- $\frac{\partial y}{\partial v} = \frac{(u-v) - (u+v)}{(u-v)^2} = -\frac{2u}{(u-v)^2}$

BY THE CHAIN RULE

$$\frac{\partial G}{\partial u} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial u} = v \frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2} \frac{\partial F}{\partial y}$$

$$\frac{\partial G}{\partial v} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial v} = u \frac{\partial F}{\partial x} - \frac{2u}{(u-v)^2} \frac{\partial F}{\partial y}$$

THIS WE HAVE

$$u \frac{\partial G}{\partial u} + v \frac{\partial G}{\partial v} = u \left[v \frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2} \frac{\partial F}{\partial y} \right] + v \left[u \frac{\partial F}{\partial x} - \frac{2u}{(u-v)^2} \frac{\partial F}{\partial y} \right]$$

$$= uv \frac{\partial F}{\partial x} - \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} + uv \frac{\partial F}{\partial x} - \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y}$$

$$= 2uv \frac{\partial F}{\partial x}$$

$$= 2x \frac{\partial F}{\partial x}$$

AND IN SIMILAR FASHION

$$\frac{u^2 - v^2}{2uv} \left[v \frac{\partial G}{\partial v} - u \frac{\partial G}{\partial u} \right] = \frac{u^2 - v^2}{2uv} \left[v \left[u \frac{\partial F}{\partial x} - \frac{2u}{(u-v)^2} \frac{\partial F}{\partial y} \right] - u \left[v \frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2} \frac{\partial F}{\partial y} \right] \right]$$

$$= \frac{u^2 - v^2}{2uv} \left[uv \frac{\partial F}{\partial x} - \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} - uv \frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} \right]$$

$$= \frac{u^2 - v^2}{2uv} \times \frac{4uv}{(u-v)^2} \frac{\partial F}{\partial y}$$

$$= \frac{2(u^2 - v^2)}{(u-v)^2} \frac{\partial F}{\partial y}$$

$$= \frac{2(u-v)(u+v)}{(u-v)^2} \frac{\partial F}{\partial y}$$

$$= 2 \frac{u+v}{u-v} \frac{\partial F}{\partial y}$$

$$= 2y \frac{\partial F}{\partial y}$$

Question 31 (**)**

The function z depends on x and y so that

$$z = (u + v)^2, \quad x = u^2 - v^2 \quad \text{and} \quad y = uv.$$

Show clearly that ...

i. ... $\frac{\partial z}{\partial x} = \frac{x}{z - 2y}.$

ii. ... $\frac{\partial z}{\partial y} = \frac{2z}{z - 2y}.$

, proof

$z = (u+v)^2$ $x = u^2 - v^2$ $y = uv$

START WITH THE JACOBIAN

$$J = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{vmatrix} 2u & 2v \\ 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$$

NOW WE HAVE

$$\frac{\partial z}{\partial x} = \frac{1}{2u^2 + 2v^2} \quad \frac{\partial z}{\partial y} = \frac{1}{2u^2 + 2v^2}$$

THIS WE COULD

$$\frac{\partial z}{\partial x} = \frac{1}{2u^2 + 2v^2} \quad \frac{\partial z}{\partial y} = \frac{1}{2u^2 + 2v^2}$$

NOW BY THE CHAIN RULE

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = 2(u+v) \times \frac{u}{2u^2 + 2v^2} + 2(u+v) \times \frac{-v}{2u^2 + 2v^2}$$

$$= \frac{2u(u+v) - 2v(u+v)}{2u^2 + 2v^2}$$

$$= \frac{u(u+v) - v(u+v)}{u^2 + v^2}$$

$$= \frac{(u-v)(u+v)}{u^2 + v^2}$$

$$= \frac{u^2 - v^2}{u^2 + v^2}$$

$$= \frac{x}{(u^2 + v^2) - 2uv}$$

$$= \frac{x}{z - 2y}$$

AND IN AN ALTERNATIVE METHOD

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = 2(u+v) \times \frac{u}{2u^2 + 2v^2} + 2(u+v) \times \frac{-v}{2u^2 + 2v^2}$$

$$= \frac{2u(u+v) - 2v(u+v)}{2u^2 + 2v^2}$$

$$= \frac{2u(u+v) - 2v(u+v)}{2(u^2 + v^2)}$$

$$= \frac{2(u-v)(u+v)}{2(u^2 + v^2)}$$

$$= \frac{(u-v)(u+v)}{u^2 + v^2}$$

$$= \frac{x}{z - 2y}$$

QED

Question 32 (****)

The function z depends on x and y so that

$$z = f(u, v), \quad u = x + y \quad \text{and} \quad v = 2x - 2y.$$

Show clearly that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4 \frac{\partial^2 z}{\partial v^2}.$$

proof

Handwritten proof showing the derivation of the identity:

$$z = f(u, v) \quad u = x + y \quad v = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4 \frac{\partial^2 z}{\partial v^2}$$

First:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \times 1 + \frac{\partial z}{\partial v} \times 2 = \frac{\partial z}{\partial u} + 2 \frac{\partial z}{\partial v}$$

Now differentiate w.r.t. y :

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left[\frac{\partial z}{\partial u} + 2 \frac{\partial z}{\partial v} \right] \quad \text{chain rule}$$

$$= \frac{\partial^2 z}{\partial u^2} \times \frac{\partial u}{\partial y} + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} \times 1 + 2 \frac{\partial^2 z}{\partial u \partial v} (-2)$$

$$= \frac{\partial^2 z}{\partial u^2} - 4 \frac{\partial^2 z}{\partial v^2} \quad \text{Q.E.D.}$$

Question 33 (**)**

The function z depends on x and y so that

$$z = \frac{x}{1+x^2} \quad \text{where} \quad f = f\left(\frac{1}{y} - \frac{1}{x}\right).$$

Show clearly that

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2.$$

proof

Handwritten proof:

$$z(x,y) = \frac{x}{(1+x^2)(y-\frac{1}{x})} = \frac{x}{(1+x^2)}$$

$$\bullet \frac{\partial z}{\partial x} = \frac{(1+x^2) \cdot 1 - x(0+1 \cdot 2x + x^2 \cdot \frac{1}{x})}{(1+x^2)^2} = \frac{(1+x^2) - 2x^2}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$$\bullet \frac{\partial z}{\partial y} = \frac{(1+x^2) \cdot x \cdot (-1) - x(0+x^2 \cdot 1 + 1 \cdot \frac{1}{y^2})}{(1+x^2)^2} = \frac{\frac{-x^2}{y^2} - \frac{x^3}{y^2}}{(1+x^2)^2} = \frac{-x^2}{y^2(1+x^2)^2}$$

Thus

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = x^2 \left[\frac{1-x^2}{(1+x^2)^2} \right] + y^2 \left[\frac{-x^2}{y^2(1+x^2)^2} \right]$$

$$= \frac{x^2 - x^4}{(1+x^2)^2} + \frac{-x^2}{(1+x^2)^2}$$

$$= \frac{x^2 - x^4 - x^2}{(1+x^2)^2} = \frac{-x^4}{(1+x^2)^2} = -\frac{x^4}{(1+x^2)^2} \quad \text{As required}$$

Question 34 (****)

The surface S has equation

$$z = y f\left(\frac{x}{y}\right),$$

where the function $f\left(\frac{x}{y}\right)$ is differentiable.

Show that the tangent plane at any point on S passes through the origin O

proof

$$z = y f\left(\frac{x}{y}\right)$$

$$\text{Let } g(x, y, z) = y f\left(\frac{x}{y}\right) - z$$

$$\Rightarrow \nabla g = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right] = \left[y f'\left(\frac{x}{y}\right) \times \frac{1}{y}, f\left(\frac{x}{y}\right) + y f'\left(\frac{x}{y}\right) \left(-\frac{x}{y^2}\right), -1 \right]$$

$$\Rightarrow \nabla g = \left[f'\left(\frac{x}{y}\right), f\left(\frac{x}{y}\right) - \frac{x}{y} f'\left(\frac{x}{y}\right), -1 \right]$$

• Take a random point on the surface, $P(x_0, y_0, z_0)$
 And note that $z_0 = y_0 f\left(\frac{x_0}{y_0}\right)$

$$\text{Then } \nabla g|_P = \left[f'\left(\frac{x_0}{y_0}\right), f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0} f'\left(\frac{x_0}{y_0}\right), -1 \right]$$

• Equation of a plane is therefore

$$f'\left(\frac{x_0}{y_0}\right)x + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0} f'\left(\frac{x_0}{y_0}\right) \right]y - z = \text{constant}$$
 But (x_0, y_0, z_0) lies on the plane so it must satisfy the above equation

$$\therefore \text{constant} = f'\left(\frac{x_0}{y_0}\right)x_0 + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0} f'\left(\frac{x_0}{y_0}\right) \right]y_0 - z_0$$

$$\text{constant} = x_0 f'\left(\frac{x_0}{y_0}\right) + y_0 f\left(\frac{x_0}{y_0}\right) - x_0 f'\left(\frac{x_0}{y_0}\right) - y_0 f\left(\frac{x_0}{y_0}\right)$$

$$\text{constant} = 0$$

• Plane is of the form $Ax + By + Cz = 0$
 so it passes through the origin

Question 35 (****)

The functions $u = u(x, y)$ and $v = v(x, y)$ satisfy

$$u + 3v^3 = 3x + y^2 \quad \text{and} \quad v - 2u^3 = x^3 - 2y.$$

Determine the value of $\frac{\partial(u, v)}{\partial(x, y)}$ at $(x, y) = (0, 0)$.

$$\frac{\partial(u, v)}{\partial(x, y)} \bigg|_{(0,0)} = -6$$

Handwritten solution for Question 35:

Given equations:

$$u + 3v^3 = 3x + y^2$$

$$v - 2u^3 = x^3 - 2y$$

Partial derivatives for the first equation:

$$\frac{\partial}{\partial x} (u + 3v^3) = \frac{\partial}{\partial x} (3x + y^2)$$

$$\frac{\partial u}{\partial x} + 9v^2 \frac{\partial v}{\partial x} = 3$$

At $x=0, y=0$, $u=v=0$:

$$\frac{\partial u}{\partial x} = 3, \quad \frac{\partial v}{\partial x} = 0$$

Partial derivatives for the second equation:

$$\frac{\partial}{\partial y} (v - 2u^3) = \frac{\partial}{\partial y} (x^3 - 2y)$$

$$\frac{\partial v}{\partial y} - 6u^2 \frac{\partial u}{\partial y} = -2$$

At $x=0, y=0$, $u=v=0$:

$$\frac{\partial v}{\partial y} = -2, \quad \frac{\partial u}{\partial y} = 0$$

Jacobian determinant at $(0,0)$:

$$\frac{\partial(u, v)}{\partial(x, y)} \bigg|_{(0,0)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & -2 \end{vmatrix} = -6$$

Question 36 (****)

A surface S has equation $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + 2xy - 4x + 2y^2 + 2yz - 8y - z^2 + 4z.$$

- a) Show that there is no point on S where the normal to S is parallel to the z axis and hence state the geometric significance of this result with reference to the stationary points of S .

S is translated to give a new surface T with equation

$$f(x, y, z) = -56.$$

The plane with equation $x + y + z = k$, where k is a constant, is a tangent plane to T .

- b) Determine the two possible values of k .

$$\boxed{}, \quad k = 2 \cup k = 6$$

Handwritten work for Question 36:

Part a)

Given $f(x, y, z) = x^2 + 2xy - 4x + 2y^2 + 2yz - 8y - z^2 + 4z$

Partial derivatives:

$$\frac{\partial f}{\partial x} = 2x + 2y - 4$$

$$\frac{\partial f}{\partial y} = 2x + 4y + 2z - 8$$

$$\frac{\partial f}{\partial z} = 2y + 2z - 8$$

Gradient vector:

$$\nabla f = \begin{pmatrix} 2x + 2y - 4 \\ 2x + 4y + 2z - 8 \\ 2y + 2z - 8 \end{pmatrix}$$

For the normal to be parallel to the z -axis, ∇f must be a scalar multiple of $(0, 0, 1)$.

Setting $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$:

$$2x + 2y - 4 = 0 \Rightarrow x + y = 2$$

$$2x + 4y + 2z - 8 = 0 \Rightarrow x + 2y + z = 4$$

$$2y + 2z - 8 = 0 \Rightarrow y + z = 4$$

Substituting $x = 2 - y$ into the second equation:

$$(2 - y) + 2y + z = 4 \Rightarrow 2 + y + z = 4 \Rightarrow y + z = 2$$

Comparing $y + z = 4$ and $y + z = 2$, we see a contradiction. Therefore, there is no point on S where the normal is parallel to the z -axis.

Part b)

The surface S is translated to T with equation $f(x, y, z) = -56$.

The plane $x + y + z = k$ is a tangent plane to T .

The normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The normal to the surface T is $\nabla f = \begin{pmatrix} 2x + 2y - 4 \\ 2x + 4y + 2z - 8 \\ 2y + 2z - 8 \end{pmatrix}$.

For the plane to be tangent to T , the normals must be parallel:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2x + 2y - 4 \\ 2x + 4y + 2z - 8 \\ 2y + 2z - 8 \end{pmatrix}$$

Equating components:

$$1 = \lambda(2x + 2y - 4)$$

$$1 = \lambda(2x + 4y + 2z - 8)$$

$$1 = \lambda(2y + 2z - 8)$$

Dividing the first equation by the third:

$$\frac{1}{1} = \frac{2x + 2y - 4}{2y + 2z - 8} \Rightarrow 2y + 2z - 8 = 2x + 2y - 4 \Rightarrow 2z - 8 = 2x - 4 \Rightarrow z - 4 = x - 2 \Rightarrow z = x + 2$$

Substituting $z = x + 2$ into the second equation:

$$1 = \lambda(2x + 4y + 2(x + 2) - 8) = \lambda(4x + 4y - 4) = 4\lambda(x + y - 1)$$

From the first equation, $\lambda = \frac{1}{2x + 2y - 4}$.

Substituting λ into the second equation:

$$1 = \frac{4(x + y - 1)}{2x + 2y - 4} \Rightarrow 2x + 2y - 4 = 4(x + y - 1) \Rightarrow 2x + 2y - 4 = 4x + 4y - 4 \Rightarrow 0 = 2x + 2y \Rightarrow x + y = 0$$

Substituting $x + y = 0$ into $z = x + 2$:

$$z = x + 2 = -y + 2$$

Substituting $x + y = 0$ and $z = -y + 2$ into the third equation:

$$1 = \lambda(2y + 2(-y + 2) - 8) = \lambda(2y - 2y + 4 - 8) = \lambda(-4) \Rightarrow \lambda = -\frac{1}{4}$$

Substituting $\lambda = -\frac{1}{4}$ into the first equation:

$$1 = -\frac{1}{4}(2x + 2y - 4) \Rightarrow 4 = -(2x + 2y - 4) \Rightarrow 4 = -2x - 2y + 4 \Rightarrow 0 = -2x - 2y \Rightarrow x + y = 0$$

Substituting $x + y = 0$ into $z = x + 2$:

$$z = x + 2 = -y + 2$$

Substituting $x + y = 0$ and $z = -y + 2$ into the equation $x + y + z = k$:

$$x + y + z = k \Rightarrow x + y + (-y + 2) = k \Rightarrow x + 2 = k \Rightarrow x = k - 2$$

Substituting $x = k - 2$ into $x + y = 0$:

$$k - 2 + y = 0 \Rightarrow y = 2 - k$$

Substituting $x = k - 2$ and $y = 2 - k$ into $z = x + 2$:

$$z = (k - 2) + 2 = k$$

Therefore, the possible values of k are $k = 2$ and $k = 6$.

Question 37 (****)

A surface S has equation $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + 3y^2 + 2z^2 + 2yz + 6xz - 4xy - 24.$$

Show that the plane with equation

$$10x - y + 2z = 6,$$

is a tangent plane to S , and find the coordinates of the point of tangency.

$$(-2, -6, 10)$$

Handwritten solution for Question 37:

Given $f(x, y, z) = x^2 + 3y^2 + 2z^2 + 2yz + 6xz - 4xy - 24$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x + 6z - 4y, 6y + 2z - 4x, 2y + 2z + 6x)$

Now, solving the gradient vector

$\begin{cases} 2x - 4y + 6z = 10k \\ -4x + 6y + 2z = -k \\ 6x + 2y + 2z = 2k \end{cases} \Rightarrow \begin{pmatrix} 1 & -2 & 3 & 10k \\ -2 & 3 & 1 & -k \\ 3 & 1 & 2 & 2k \end{pmatrix} \xrightarrow{R_2 + 2R_1, R_3 - 3R_1} \begin{pmatrix} 1 & -2 & 3 & 10k \\ 0 & -1 & 7 & 19k \\ 0 & 7 & -7 & -28k \end{pmatrix} \xrightarrow{R_3 + 7R_2} \begin{pmatrix} 1 & -2 & 3 & 10k \\ 0 & -1 & 7 & 19k \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore \begin{cases} x - 2y + 3z = 10k \\ -y + 7z = -19k \end{cases} \Rightarrow \begin{cases} x = 2y - 3z + 10k \\ y = 7z - 19k \end{cases}$

$\therefore \begin{cases} x = 2(7z - 19k) - 3z + 10k = 11z - 28k \\ y = 7z - 19k \end{cases}$

Substituting into the plane equation $10x - y + 2z = 6$

$10(11z - 28k) - (7z - 19k) + 2z = 6$

$110z - 280k - 7z + 19k + 2z = 6$

$105z - 261k = 6$

$105z = 261k + 6$

$z = \frac{261k + 6}{105}$

Substituting z back into the equations for x and y

$x = 11\left(\frac{261k + 6}{105}\right) - 28k = \frac{2871k + 66}{105} - 28k = \frac{2871k + 66 - 2940k}{105} = \frac{-69k + 66}{105}$

$y = 7\left(\frac{261k + 6}{105}\right) - 19k = \frac{1827k + 42}{105} - 19k = \frac{1827k + 42 - 1995k}{105} = \frac{-168k + 42}{105}$

For the point of tangency, $k = 0$

$x = \frac{66}{105} = \frac{22}{35}$

$y = \frac{42}{105} = \frac{2}{5}$

$z = \frac{6}{105} = \frac{2}{35}$

Check if this point satisfies the plane equation $10x - y + 2z = 6$

$10\left(\frac{22}{35}\right) - \frac{2}{5} + 2\left(\frac{2}{35}\right) = \frac{220}{35} - \frac{14}{35} + \frac{4}{35} = \frac{210}{35} = 6$

\therefore The point of tangency is $\left(\frac{22}{35}, \frac{2}{5}, \frac{2}{35}\right)$

Question 38 (**)**

It is given that g is a twice differentiable function of one variable, with domain all real numbers.

It is further given that for $x > 0$

$$f(x, y) = g(y \ln x).$$

Show that

$$x^2 \ln x \frac{\partial^2 f}{\partial x^2} - xy \frac{\partial^2 f}{\partial x \partial y} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0.$$

proof

$f(x, y) = g(y \ln x)$ g is a function of one variable

let $u = y \ln x \Rightarrow \frac{\partial u}{\partial x} = \frac{y}{x}$
 $\Rightarrow \frac{\partial u}{\partial y} = \ln x$

next obtain the required derivatives of f

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(g(u)) = \frac{dg}{du} \frac{\partial u}{\partial x} = g'(u) \times \frac{y}{x} = \frac{y}{x} g'(u)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(g(u)) = \frac{dg}{du} \frac{\partial u}{\partial y} = g'(u) \times \ln x = g'(u) \ln x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{y}{x} g'(u) \right] = -\frac{y}{x^2} g'(u) + \frac{y}{x} \times g'(u) \frac{\partial \ln x}{\partial x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left[\frac{y}{x} g'(u) \right] = g'(u) \frac{\partial}{\partial y} \left(\frac{y}{x} \right) + g''(u) \times \frac{y}{x} \times \frac{\partial \ln x}{\partial y}$$

$$= \frac{1}{x} g'(u) + \frac{y \ln x}{x} g''(u)$$

check the partial differential equation

$$x^2 \ln x \frac{\partial^2 f}{\partial x^2} - xy \frac{\partial^2 f}{\partial x \partial y} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

$$= x^2 \ln x \left[-\frac{y}{x^2} g'(u) + \frac{y}{x} g'(u) \frac{\partial \ln x}{\partial x} \right] - xy \left[\frac{1}{x} g'(u) + \frac{y \ln x}{x} g''(u) \right] + x \left[\frac{y}{x} g'(u) \right] + y \left[g'(u) \ln x \right]$$

$$= -y \ln x g'(u) + y \ln x g'(u) - y g'(u) - y^2 \ln x g''(u) + y g'(u) + y \ln x g'(u)$$

$$= 0$$

Question 39 (**)**

The function w depends on x and y so that

$$w = f(u), \quad \text{and} \quad u = (x - x_0)(y - y_0),$$

where x_0 and y_0 are constants.

Show clearly that

$$\frac{\partial^2 w}{\partial x \partial y} = u \frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial u}.$$

proof

Handwritten proof showing the derivation of the second partial derivative $\frac{\partial^2 w}{\partial x \partial y}$ using the chain rule. The proof starts with $w = f(u)$ and $u = (x - x_0)(y - y_0)$. It then shows the first partial derivative $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x}$ and the second partial derivative $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial u} \frac{\partial u}{\partial x} \right)$. The final result is $\frac{\partial^2 w}{\partial x \partial y} = u \frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial u}$.

Question 40 (**)**

The functions f and G satisfy

$$G(r, \theta, \varphi) \equiv f[x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi)],$$

where x , y and z satisfy the standard Spherical Polar Coordinates transformation relationships

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial r} \right]^2 + \left[\frac{1}{r} \frac{\partial G}{\partial \theta} \right]^2 + \left[\frac{1}{r \sin \theta} \frac{\partial G}{\partial \varphi} \right]^2 = \left[\frac{\partial f}{\partial x} \right]^2 + \left[\frac{\partial f}{\partial y} \right]^2 + \left[\frac{\partial f}{\partial z} \right]^2.$$

 , proof

$G(r, \theta, \varphi) = f[x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi)]$
 $x = r \sin \theta \cos \varphi$
 $y = r \sin \theta \sin \varphi$
 $z = r \cos \theta$

START BY OBTAINING SOME BASIC PARTIAL DERIVATIVES

- $\frac{\partial x}{\partial r} = \sin \theta \cos \varphi$
- $\frac{\partial x}{\partial \theta} = r \cos \theta \cos \varphi$
- $\frac{\partial x}{\partial \varphi} = -r \sin \theta \sin \varphi$
- $\frac{\partial y}{\partial r} = \sin \theta \sin \varphi$
- $\frac{\partial y}{\partial \theta} = r \cos \theta \sin \varphi$
- $\frac{\partial y}{\partial \varphi} = r \sin \theta \cos \varphi$
- $\frac{\partial z}{\partial r} = \cos \theta$
- $\frac{\partial z}{\partial \theta} = -r \sin \theta$
- $\frac{\partial z}{\partial \varphi} = 0$

BY THE CHAIN RULE WE HAVE

$$\frac{\partial G}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r}$$

$$= \frac{\partial f}{\partial x} \sin \theta \cos \varphi + \frac{\partial f}{\partial y} \sin \theta \sin \varphi + \frac{\partial f}{\partial z} \cos \theta$$

$$\frac{\partial G}{\partial \theta} = \frac{\partial f}{\partial x} r \cos \theta \cos \varphi + \frac{\partial f}{\partial y} r \cos \theta \sin \varphi + \frac{\partial f}{\partial z} (-r \sin \theta)$$

$$= r \left[\frac{\partial f}{\partial x} \cos \theta \cos \varphi + \frac{\partial f}{\partial y} \cos \theta \sin \varphi - \frac{\partial f}{\partial z} \sin \theta \right]$$

$$\frac{\partial G}{\partial \varphi} = \frac{\partial f}{\partial x} (-r \sin \theta \sin \varphi) + \frac{\partial f}{\partial y} (r \sin \theta \cos \varphi) + \frac{\partial f}{\partial z} (0)$$

$$= r \left[-\frac{\partial f}{\partial x} \sin \theta \sin \varphi + \frac{\partial f}{\partial y} \sin \theta \cos \varphi \right]$$

HENCE WE OBTAIN

$$\left(\frac{\partial G}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial G}{\partial \theta} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial G}{\partial \varphi} \right)^2$$

$$= \left[\frac{\partial f}{\partial x} \sin \theta \cos \varphi + \frac{\partial f}{\partial y} \sin \theta \sin \varphi + \frac{\partial f}{\partial z} \cos \theta \right]^2$$

$$+ \left[\frac{\partial f}{\partial x} \cos \theta \cos \varphi + \frac{\partial f}{\partial y} \cos \theta \sin \varphi - \frac{\partial f}{\partial z} \sin \theta \right]^2$$

$$+ \left[-\frac{\partial f}{\partial x} \sin \theta \sin \varphi + \frac{\partial f}{\partial y} \sin \theta \cos \varphi \right]^2$$

REARRANGING AFTER THE CANCELLATIONS

$$= \frac{\partial f^2}{\partial x} [\sin^2 \theta \cos^2 \varphi + \cos^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi]$$

$$+ \frac{\partial f^2}{\partial y} [\sin^2 \theta \sin^2 \varphi + \cos^2 \theta \sin^2 \varphi + \sin^2 \theta \cos^2 \varphi]$$

$$+ 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} [\sin^2 \theta \cos \varphi \sin \varphi + \cos^2 \theta \cos \varphi \sin \varphi - \sin \theta \cos \theta \sin \varphi \cos \varphi]$$

$$+ 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} [\sin^2 \theta \cos \varphi \cos \theta - \cos^2 \theta \cos \varphi \sin \theta - \sin^2 \theta \sin \varphi \cos \theta]$$

$$+ 2 \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} [\sin^2 \theta \sin \varphi \cos \theta - \cos^2 \theta \sin \varphi \sin \theta - \sin^2 \theta \cos \varphi \cos \theta]$$

$$= \frac{\partial f^2}{\partial x} [\sin^2 \theta \cos^2 \varphi + \cos^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi]$$

$$+ \frac{\partial f^2}{\partial y} [\sin^2 \theta \sin^2 \varphi + \cos^2 \theta \sin^2 \varphi + \sin^2 \theta \cos^2 \varphi]$$

$$+ 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} [\sin^2 \theta \cos \varphi \sin \varphi + \cos^2 \theta \cos \varphi \sin \varphi - \sin \theta \cos \theta \sin \varphi \cos \varphi]$$

$$+ 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} [\sin^2 \theta \cos \varphi \cos \theta - \cos^2 \theta \cos \varphi \sin \theta - \sin^2 \theta \sin \varphi \cos \theta]$$

$$+ 2 \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} [\sin^2 \theta \sin \varphi \cos \theta - \cos^2 \theta \sin \varphi \sin \theta - \sin^2 \theta \cos \varphi \cos \theta]$$

$$= \frac{\partial f^2}{\partial x} + \frac{\partial f^2}{\partial y} + \frac{\partial f^2}{\partial z}$$

Question 41 (****+)

It is given that

$$z(x, y) = f(u, v),$$

so that

$$u = x^3 + y^3 \quad \text{and} \quad v = \frac{y}{x}.$$

a) Use the chain rule to show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3u \frac{\partial f}{\partial u} \quad \text{and} \quad yx^3 \frac{\partial z}{\partial y} - xy^3 \frac{\partial z}{\partial x} = uv \frac{\partial f}{\partial v}.$$

b) Hence show further that

$$v \frac{\partial x}{\partial u} = \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial x}{\partial v} = -v^2 \frac{\partial y}{\partial v}.$$

proof

a) $u = x^3 + y^3$ $z(x, y) = f(u, v)$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \\ &= \frac{\partial f}{\partial u} (3x^2) + \frac{\partial f}{\partial v} \left(-\frac{y}{x^2}\right) \\ &= 3x^2 \frac{\partial f}{\partial u} - \frac{y}{x^2} \frac{\partial f}{\partial v} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \\ &= \frac{\partial f}{\partial u} (3y^2) + \frac{\partial f}{\partial v} \left(\frac{1}{x}\right) \\ &= 3y^2 \frac{\partial f}{\partial u} + \frac{1}{x} \frac{\partial f}{\partial v} \end{aligned}$$

$$\begin{aligned} x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= x \left(3x^2 \frac{\partial f}{\partial u} - \frac{y}{x^2} \frac{\partial f}{\partial v} \right) + y \left(3y^2 \frac{\partial f}{\partial u} + \frac{1}{x} \frac{\partial f}{\partial v} \right) \\ &= 3x^3 \frac{\partial f}{\partial u} - \frac{xy}{x^2} \frac{\partial f}{\partial v} + 3y^3 \frac{\partial f}{\partial u} + \frac{y}{x} \frac{\partial f}{\partial v} \\ &= 3(x^3 + y^3) \frac{\partial f}{\partial u} + \left(-\frac{y}{x} + \frac{y}{x} \right) \frac{\partial f}{\partial v} \\ &= 3u \frac{\partial f}{\partial u} \end{aligned}$$

b) $u = x^3 + y^3$ $v = \frac{y}{x}$

$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial (x^3 + y^3)} = \frac{1}{3x^2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial y}{\partial (x^3 + y^3)} = \frac{1}{3y^2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial x}{\partial \left(\frac{y}{x}\right)} = -\frac{y}{x^2}$$

$$\frac{\partial y}{\partial v} = \frac{\partial y}{\partial \left(\frac{y}{x}\right)} = \frac{1}{x}$$

$$v \frac{\partial x}{\partial u} = \frac{y}{x} \cdot \frac{1}{3x^2} = \frac{y}{3x^3}$$

$$\frac{\partial y}{\partial u} = \frac{1}{3y^2}$$

$$\frac{y}{3x^3} = \frac{1}{3y^2} \Rightarrow y^3 = x^3 \Rightarrow y = x$$

Question 42 (****+)

It is given that f and g are differentiable functions of one variable, with domain all real numbers.

It is further given that for $x > 0$

$$F(x, y) = f[x^2 + y^2 + g(3x - 2y)].$$

If the function $y = y(x)$ is a rearrangement of $F(x, y) = 0$, show that

$$\frac{dy}{dx} = \frac{3\frac{dg}{du} + 2x}{2\frac{dg}{du} - 2y},$$

where $u = 3x - 2y$

proof

$F(x,y) = f(x^2+y^2+g(2x-2y))$ f is a function of 1 variable
 g is a function of 1 variable

Let $F(x,y) = f(u)$, where $v = x^2+y^2+g(u)$
where $u = 2x-2y$

$\frac{\partial F}{\partial x} = \frac{df}{dv} \frac{\partial v}{\partial x} = \frac{df}{dv} \left[2x + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} \right] = \frac{df}{dv} \left[2x + \frac{\partial g}{\partial u} \right]$

$\frac{\partial F}{\partial y} = \frac{df}{dv} \frac{\partial v}{\partial y} = \frac{df}{dv} \left[2y + \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} \right] = \frac{df}{dv} \left[2y - 2 \frac{\partial g}{\partial u} \right]$

Now if $F(x,y) = 0$, then $\frac{d}{dx} = \frac{dF}{dF} \frac{dx}{dy}$

$\frac{dy}{dx} = - \frac{\frac{df}{dv} \left[2x + \frac{\partial g}{\partial u} \right]}{\frac{df}{dv} \left[2y - 2 \frac{\partial g}{\partial u} \right]}$

$\frac{dy}{dx} = \frac{- \frac{2x}{\frac{\partial g}{\partial u}} + \frac{2x}{\frac{\partial g}{\partial u}}}{\frac{2y}{\frac{\partial g}{\partial u}} - 2y}$

Question 43 (****+)

The surface S has Cartesian equation

$$z = f(x, y).$$

The tangent plane at any point on S passes through the point $(0, 0, -1)$.

Show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z = 1.$$

proof

Handwritten proof for the question:

- $z = f(x, y)$
- $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$
- $\nabla g = \left(\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right), \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) \right) = \left(\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y} \right)$
- Normal at a general point $(x, y, z) = \left[\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y), -1 \right]$
- Equation of the tangent plane

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$
- But the tangent plane passes through $(0, 0, -1)$

$$-1 - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(-y_0)$$
- Rearranging gives us a relation $z = f(x, y)$

$$-1 - z = \frac{\partial f}{\partial x}(x, y)(-x) + \frac{\partial f}{\partial y}(x, y)(-y)$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z = 1$$

Q.E.D.

Question 44 (***)**

It is given that the function f depends on x and y , and the function g depends on u and v , so that

$$f(x, y) = g(u, v), \quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

a) Show that

$$\frac{\partial^2 f}{\partial x^2} = 2 \frac{\partial g}{\partial u} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2},$$

and find a similar expression for $\frac{\partial^2 f}{\partial y^2}$.

b) Deduce that if $f(x, y) = x + y$

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$

proof

a) $f(x, y) = g(u, v)$ with $u = x^2 - y^2$ and $v = 2xy$

- START BY CALCULATING FIRST DERIVATIVES
- $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial x} = 2x \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v}$
- $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial y} = -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v}$
- WRITE THESE ALSO IN OPERATOR FORM, AS WE WILL NEED THEM FOR THE SECOND DERIVATIVES
- $\frac{\partial}{\partial x} = 2x \frac{\partial}{\partial u} + 2y \frac{\partial}{\partial v}$
- $\frac{\partial}{\partial y} = -2y \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}$
- NOT THE SECOND DERIVATIVES
- $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(2x \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v} \right)$
- $= 2 \frac{\partial g}{\partial u} + 2x \frac{\partial^2 g}{\partial u^2} + 2y \frac{\partial^2 g}{\partial u \partial v} + 2y \frac{\partial^2 g}{\partial v \partial u} + 2x \frac{\partial^2 g}{\partial v^2}$
- $= 2 \frac{\partial g}{\partial u} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2}$
- $= 2 \frac{\partial g}{\partial u} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2}$

b) $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}$

- $= 2 \frac{\partial g}{\partial u} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2} - 2 \frac{\partial g}{\partial u} + 4y^2 \frac{\partial^2 g}{\partial u^2} - 8xy \frac{\partial^2 g}{\partial u \partial v} + 4x^2 \frac{\partial^2 g}{\partial v^2}$
- $= (4x^2 + 4y^2) \frac{\partial^2 g}{\partial u^2} + (4x^2 + 4y^2) \frac{\partial^2 g}{\partial v^2}$
- $= (4x^2 + 4y^2) \left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right)$
- NOW IF $f(x, y) = x + y \Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$
- HENCE $\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = (4x^2 + 4y^2) \left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right)$
- $\Rightarrow 0 = (4x^2 + 4y^2) \left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right)$
- $\Rightarrow \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0$ AS $4x^2 + 4y^2 \neq 0$ ONLY IF $x=y=0$

Question 45 (*****)

The function z depends on x and y so that

$$z = f(u, v), \quad u = x - 2\sqrt{y} \quad \text{and} \quad v = x + 2\sqrt{y}.$$

Show that the partial differential equation

$$2 \frac{\partial^2 z}{\partial x^2} - 2y \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0,$$

can be simplified to

$$\frac{\partial^2 z}{\partial u \partial v} = 0.$$

, proof

START BY DEFINING ALL THE REQUIRED EXPRESSIONS IN ORDER TO SUBSTITUTION INTO THE GIVEN P.D.E

• $u = x - 2\sqrt{y}$
• $v = x + 2\sqrt{y}$

FORMAL AND SUBSTITUTION

$$2x = u + v \quad 4\sqrt{y} = v - u$$

$$x = \frac{1}{2}(u + v) \quad \sqrt{y} = \frac{1}{4}(v - u)$$

$$y = \frac{1}{16}(v - u)^2$$

SIMPLIFY THE FIRST ORDER DERIVATIVES

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot 1 = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \cdot (-1/\sqrt{y}) + \frac{\partial z}{\partial v} \cdot (1/\sqrt{y}) = \frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right)$$

NEXT THE SECOND DERIVATIVES

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \right) \cdot 1 + \left(\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} \right) \cdot 1 = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

FINISH THE OTHER SECOND DERIVATIVE - OPERATOR TERNARY TO AVOID MISSING OTHER TERMS AND PRODUCTS

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) \right) = \frac{\partial}{\partial u} \left(\frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) \right) \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial u} \left(\frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) \right) \cdot (-1/\sqrt{y}) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) \right) \cdot (1/\sqrt{y})$$

$$= \frac{1}{y} \left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} \right)$$

THEN AND REFINISHING

$$2 \frac{\partial^2 z}{\partial x^2} - 2y \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 2 \left(\frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) - 2y \left(\frac{1}{y} \left(\frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \right) - \frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right)$$

$$= 2 \frac{\partial^2 z}{\partial u^2} + 4 \frac{\partial^2 z}{\partial u \partial v} + 2 \frac{\partial^2 z}{\partial v^2} - 2 \frac{\partial^2 z}{\partial u^2} + 4 \frac{\partial^2 z}{\partial u \partial v} - 2 \frac{\partial^2 z}{\partial v^2} - \frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right)$$

$$= 8 \frac{\partial^2 z}{\partial u \partial v} - \frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right)$$

NEXT SUBSTITUTION THE RESULT INTO THE P.D.E

$$8 \frac{\partial^2 z}{\partial u \partial v} - \frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) = 0$$

RETURNING TO THE FIRST ORDER DERIVATIVES OBTAINED AT THE VERY BEGINNING

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{y}} \left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \right) = 0$$

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0 \quad \text{and} \quad \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = 0$$

$$\Rightarrow \frac{\partial z}{\partial u} = 0 \quad \text{and} \quad \frac{\partial z}{\partial v} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = 0$$

Question 46 (**)**

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0.$$

The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} = 0.$$

proof

$\nabla \cdot \phi = 0$
 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$z = r e^{i\theta}$
 $\bar{z} = r e^{-i\theta}$
 $r^2 = z \bar{z} \Rightarrow r = \sqrt{z \bar{z}}$
 $\frac{1}{z} = \frac{1}{r} e^{-i\theta} = \frac{\bar{z}}{r^2} = \frac{\bar{z}}{z \bar{z}} = \frac{1}{z}$

$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial z}$
 $\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \bar{z}}$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x} \left(\frac{1}{2} \sqrt{\frac{z}{\bar{z}}} \right) + \frac{\partial \phi}{\partial y} \left(\frac{i}{2} \sqrt{\frac{z}{\bar{z}}} \right) = \left(\frac{\partial \phi}{\partial x} \frac{1}{2} - \frac{\partial \phi}{\partial y} \frac{i}{2} \right) \sqrt{\frac{z}{\bar{z}}}$$

$$= \frac{1}{2} \sqrt{\frac{z}{\bar{z}}} \frac{\partial \phi}{\partial x} - \frac{i}{2} \sqrt{\frac{z}{\bar{z}}} \frac{\partial \phi}{\partial y} = \frac{1}{2} \sqrt{\frac{z}{\bar{z}}} \frac{\partial \phi}{\partial x} - \frac{i \partial \phi}{2 \sqrt{\bar{z} z}}$$

$\frac{\partial \phi}{\partial z} = \cos \theta \frac{\partial \phi}{\partial r} - \frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta}$

Similarly

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial \phi}{\partial x} \left(\frac{1}{2} \sqrt{\frac{\bar{z}}{z}} \right) + \frac{\partial \phi}{\partial y} \left(\frac{1}{2} \sqrt{\frac{\bar{z}}{z}} \right) = \frac{1}{2} \sqrt{\frac{\bar{z}}{z}} \frac{\partial \phi}{\partial x} + \frac{1}{2} \sqrt{\frac{\bar{z}}{z}} \frac{\partial \phi}{\partial y}$$

$$= \frac{1}{2} \sqrt{\frac{\bar{z}}{z}} \frac{\partial \phi}{\partial x} + \frac{1}{2} \sqrt{\frac{\bar{z}}{z}} \frac{\partial \phi}{\partial y} = \frac{1}{r} \cos \theta \frac{\partial \phi}{\partial r} + \frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta}$$

$\frac{\partial \phi}{\partial \bar{z}} = \sin \theta \frac{\partial \phi}{\partial r} + \frac{\cos \theta}{r} \frac{\partial \phi}{\partial \theta}$

or as observed $\left[\frac{\partial \phi}{\partial z} = \cos \theta \frac{\partial \phi}{\partial r} - \frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right]$

Now the second iteration

$$\bullet \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right) = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial \phi}{\partial r} - \frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right)$$

$$= \cos^2 \theta \left(\cos \theta \frac{\partial}{\partial r} \right) \left(\frac{\partial \phi}{\partial r} \right) + \cos \theta \frac{\partial}{\partial r} \left(\frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial \phi}{\partial r} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right)$$

$$= \cos^3 \theta \frac{\partial^2 \phi}{\partial r^2} - \cos \sin \theta \frac{\partial}{\partial r} \left(\frac{\sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial \phi}{\partial r} \right) + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

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 Product Rule

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 Product Rule

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 Product Rule

$$\begin{aligned}
&= (\omega_2^2 \frac{\partial^2}{\partial x^2} - \omega_2^2 \omega_1^2) \left[-\frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right] - \frac{\omega_2^2}{r^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial t^2} + \omega_2 \frac{\partial^2}{\partial t \partial r} \right) \\
&\quad + \frac{\omega_2^2}{r^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial r^2} + \omega_2 \omega_1^2 \frac{\partial^2}{\partial r \partial t} \right) \\
&= (\omega_2^2 \frac{\partial^2}{\partial t^2} + \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r^2} - \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t \partial r} - \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r^2} \\
&\quad + \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r^2} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial r \partial t}) \\
&= (\omega_2^2 \frac{\partial^2}{\partial t^2} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial r^2} + \frac{2\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} - \frac{2\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t^2} \\
&= (\omega_2^2 \frac{\partial^2}{\partial t^2} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial r^2} + \frac{2\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} - \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t^2} \\
&\quad + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t^2}) \\
&\quad \text{Now} \\
&\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \\
&= \omega_2^2 \frac{\partial^2}{\partial t^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial r^2} + \omega_2 \omega_1^2 \frac{\partial^2}{\partial r \partial t} + \omega_2 \frac{\partial^2}{\partial t^2} \right) + \omega_2^2 \frac{\partial^2}{\partial t^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial r^2} + \omega_2 \omega_1^2 \frac{\partial^2}{\partial r \partial t} + \omega_2 \frac{\partial^2}{\partial t^2} \right) \\
&= \omega_2^2 \frac{\partial^2}{\partial t^2} + \omega_2 \omega_1^2 \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial r \partial t} + \frac{\partial^2}{\partial t^2} \right) + \omega_2^2 \frac{\partial^2}{\partial t^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial r^2} + \omega_2 \omega_1^2 \frac{\partial^2}{\partial r \partial t} + \omega_2 \frac{\partial^2}{\partial t^2} \right) \\
&\quad \text{Product Rule} \\
&= \omega_2^2 \frac{\partial^2}{\partial t^2} + \omega_2 \omega_1^2 \left[-\frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right] + \frac{\omega_2^2}{r^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial t^2} + \omega_2 \frac{\partial^2}{\partial t \partial r} \right) \\
&\quad + \frac{\omega_2^2}{r^2} \left(\omega_2 \omega_1^2 \frac{\partial^2}{\partial r^2} + \omega_2 \omega_1^2 \frac{\partial^2}{\partial r \partial t} \right) \\
&= \omega_2^2 \frac{\partial^2}{\partial t^2} - \omega_2 \omega_1^2 \frac{\partial^2}{\partial r^2} + \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t^2} + \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} \\
&\quad - \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r^2} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t^2} \\
&= \omega_2^2 \frac{\partial^2}{\partial t^2} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial r^2} + \frac{2\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r \partial t} - \frac{\omega_2^2 \omega_1^2}{r^2} \frac{\partial^2}{\partial r^2} + \frac{\omega_2^2}{r^2} \frac{\partial^2}{\partial t^2}
\end{aligned}$$

$$= \sin^2 \theta \frac{\partial^2}{\partial x^2} + \cos^2 \theta \frac{\partial^2}{\partial y^2} + \frac{\sin 2\theta}{r} \frac{\partial^2}{\partial x \partial y} - \frac{\sin \theta}{r} \frac{\partial^2}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial^2}{\partial \phi}$$

Question 47 (****)

It is given that for $\phi = \phi(x, y)$ and $\psi = \psi(x, y)$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Show that the above pair of coupled partial differential equations transform in plane polar coordinates to

$$\frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad \frac{\partial \psi}{\partial r} = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}.$$

proof

The image shows two pages of handwritten mathematical work. The left page is titled 'STARTING FROM THE POLAR TRANSFORMATION EQUATIONS' and shows the derivation of the polar coordinate system from the Cartesian one. It starts with $x = r \cos \theta$ and $y = r \sin \theta$, leading to $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$. Then, it uses the chain rule to find the partial derivatives of x and y with respect to r and θ . The right page is titled 'REARRANGE' and shows the transformation of the Cauchy-Riemann equations. It starts with the equations $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$, and then uses the chain rule to express these in terms of r and θ . The final result is the transformed Cauchy-Riemann equations in polar coordinates: $\frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$ and $\frac{\partial \psi}{\partial r} = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}$.