

PARAMETERIZATION

Question 1

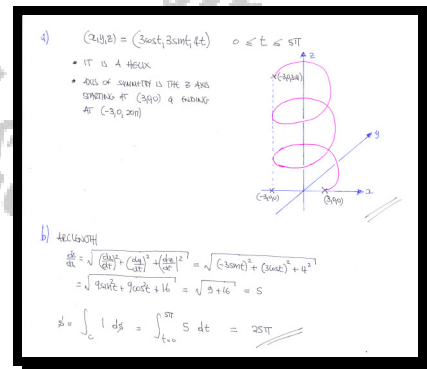
A curve C is defined parametrically

$$(x, y, z) = (3 \cos t, 3 \sin t, 4t), \quad 0 \leq t \leq 5\pi.$$

where t is a parameter.

- Sketch the graph of C .
- Find the length of C .

25π



Question 2

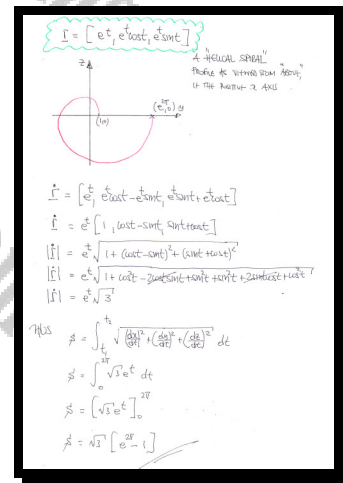
A curve C is defined parametrically

$$(x, y, z) = (e^t, e^t \cos t, e^t \sin t), \quad 0 \leq t \leq 2\pi.$$

where t is a parameter.

Describe the graph of C and find its length.

$$\text{arclength} = \sqrt{3} [e^{2\pi} - 1]$$



Question 3

The position vector of a curve C is given by

$$\mathbf{r}(t) = \cos(\cosh t)\mathbf{i} + \sin(\cosh t)\mathbf{j} + t\mathbf{k},$$

where t is a scalar parameter with $0 \leq t \leq a$, $a \in \mathbb{R}$.

Determine the length of C .

$$\text{arclength} = \sinh a$$

Handwritten solution for Question 3:

$$\mathbf{r}(t) = [\cos(\cosh t), \sin(\cosh t), t] \quad 0 \leq t \leq a$$

$$s = \int_0^a \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$$

$$\dot{x} = -\sin(\cosh t) \cdot \sinh t \Rightarrow \dot{x}^2 = \sin^2(\cosh t) \sinh^2 t$$

$$\dot{y} = \cos(\cosh t) \cdot \sinh t \Rightarrow \dot{y}^2 = \cos^2(\cosh t) \sinh^2 t$$

$$\dot{z} = 1$$

Thus

$$s = \int_0^a \sqrt{\sin^2(\cosh t) \sinh^2 t + \cos^2(\cosh t) \sinh^2 t + 1} dt$$

$$s = \int_0^a \sqrt{\sinh^2 t [\sin^2(\cosh t) + \cos^2(\cosh t)] + 1} dt$$

$$s = \int_0^a \sqrt{\sinh^2 t + 1} dt = \int_0^a \cosh t dt$$

$$s = [\sinh t]_0^a = \sinh a - \sinh 0 = \sinh a$$

Question 4

Evaluate the integral

$$\int_{(1,0)}^{(3,3)} (y+x) dx + (y-x) dy,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1.$$

$$\frac{28}{3}$$

Handwritten solution for Question 4:

$\int_{(1,0)}^{(3,3)} (y+x) dx + (y-x) dy = \dots$ Switch into parametric
 $x = 2t^2 - 3t + 1 \quad dx = (4t-3) dt$
 $y = t^2 - 1 \quad dy = 2t dt$
 By inspection

$\int_{t=0}^{t=2} (3t^2 - 3t)(4t-3) dt + (-t^2 + 3t - 1)(2t) dt$
 $= \int_0^2 (12t^3 - 21t^2 + 9t - t^3 + 3t^2 - 2) dt$
 $= \int_0^2 (12t^3 - 22t^2 + 12t - 2) dt$
 $= \left[3t^4 - \frac{22}{3}t^3 + 6t^2 - 2t \right]_0^2$
 $= \frac{48}{3} - \frac{22}{3} + 24 - 4$
 $= \frac{28}{3}$

Question 5

It is given that

$$\mathbf{F}(x, y, z) = \mathbf{j} \wedge \mathbf{r},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t) = (t - t^2)\mathbf{i} + (2t - 2t^2)\mathbf{j} + (t^2 - t^3)\mathbf{k}, \quad 0 \leq t \leq 1.$$

| |
|----|
| 1 |
| 30 |

$$\begin{aligned} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \end{vmatrix} = (z - 0)\mathbf{i} - (0 - z)\mathbf{j} + (0 - z)\mathbf{k} = (z)\mathbf{i} - (z)\mathbf{j} \\ \text{for } C: \quad \mathbf{r}(t) &= [t - t^2]\mathbf{i} + [2t - 2t^2]\mathbf{j} + [t^2 - t^3]\mathbf{k} \\ \frac{d\mathbf{r}}{dt} &= [1 - 2t]\mathbf{i} + [2 - 4t]\mathbf{j} + [2t - 3t^2]\mathbf{k} \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C (z)\mathbf{i} \cdot (1 - 2t)\mathbf{i} + (-z)\mathbf{j} \cdot (2 - 4t)\mathbf{j} + (-z)\mathbf{k} \cdot (2t - 3t^2)\mathbf{k} \\ &= \int_0^1 (t^2 - t^3)(1 - 2t) dt - \int_0^1 (2t - 2t^2)(2 - 4t) dt - \int_0^1 (t^2 - t^3)(2t - 3t^2) dt \\ &= \int_0^1 (t^2 - 2t^3 - t^3 + 2t^4) dt - \int_0^1 (4t - 8t^2 - 4t^2 + 8t^3) dt - \int_0^1 (2t^3 - 3t^4 - 2t^4 + 3t^5) dt \\ &= \int_0^1 (t^2 - 2t^3 - t^3 + 2t^4) dt - \int_0^1 (4t - 8t^2 - 4t^2 + 8t^3) dt - \int_0^1 (2t^3 - 3t^4 - 2t^4 + 3t^5) dt \\ &= \left[\frac{1}{3}t^3 - \frac{2}{4}t^4 - \frac{1}{4}t^4 + \frac{2}{5}t^5 \right]_0^1 - \left[2t^2 - 4t^3 - 2t^3 + 2t^4 \right]_0^1 - \left[\frac{1}{2}t^4 - \frac{3}{5}t^5 - \frac{2}{5}t^5 + \frac{3}{6}t^6 \right]_0^1 \\ &= \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2} \right) - (0) = \frac{10 + 6 - 15}{30} = \frac{1}{30} \end{aligned}$$

Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y) dx + (y+z) dy + (1-z^2) dz,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

| |
|----------------|
| $\frac{32}{3}$ |
|----------------|

Vector $\vec{a} = (1,1,0)$ & $\vec{b} = (5,3,4)$
 $\vec{AB} = \vec{b} - \vec{a} = (5,3,4) - (1,1,0) = (4,2,4)$
 PARAMETERISE THE LINE: $\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a})$ $\begin{matrix} x = 1 + 4t \\ y = 1 + 2t \\ z = 0 + 4t \end{matrix} \Rightarrow \begin{matrix} dx = 4 dt \\ dy = 2 dt \\ dz = 4 dt \end{matrix}$
 $0 \leq t \leq 1$
 Thus $\int_{(1,1,0)}^{(5,3,4)} (3x-2y) dx + (y+z) dy + (1-z^2) dz$
 $= \int_{t=0}^1 [3(1+4t) - 2(1+2t)](4 dt) + [(1+2t) + 4t](2 dt) + [1 - (4t)^2](4 dt)$
 $= \int_0^1 (12t + 4) + 2(6t + 4t) + 4(1 - 16t^2) dt$
 $= \int_0^1 (12t^2 + 44t + 10) dt$
 $= \left[\frac{12}{3}t^3 + 22t^2 + 10t \right]_0^1 = \left(\frac{12}{3} + 22 + 10 \right) - (0) = \frac{32}{3}$

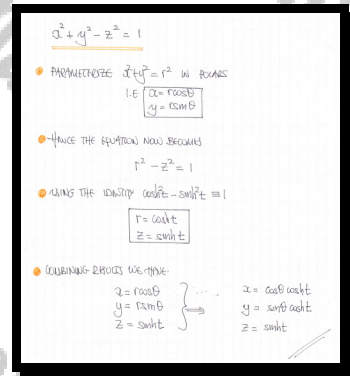
Question 7

A surface S has equation

$$x^2 + y^2 - z^2 = 1.$$

Find a suitable parameterization of S .

$$x = \cos \theta \cosh t, \quad y = \sin \theta \cosh t, \quad z = \sinh t$$



Question 8

The position vector of a curve C is given by

$$\mathbf{r}(t) = \left(\frac{2}{1+t^2} - 1 \right) \mathbf{i} + \left(\frac{2t}{1+t^2} \right) \mathbf{j},$$

where t is a scalar parameter with $t \in \mathbb{R}$.

Find an expression for the position vector of C , giving the answer in the form

$$\mathbf{r}(s) = f(s) \mathbf{i} + g(s) \mathbf{j},$$

where s is the arc length of a general point on C , measured from the point $(1,0)$.

$$\mathbf{r}(s) = (\cos s) \mathbf{i} + (\sin s) \mathbf{j}$$

Handwritten solution for Question 8:

$$\mathbf{r}(t) = \left(\frac{2}{1+t^2} - 1 \right) \mathbf{i} + \left(\frac{2t}{1+t^2} \right) \mathbf{j}$$

$$\bullet \ x = \frac{2}{1+t^2} - 1 \Rightarrow \dot{x} = \frac{d}{dt} \left(\frac{2}{1+t^2} - 1 \right) = - \frac{4t}{(1+t^2)^2}$$

$$\bullet \ y = \frac{2t}{1+t^2} \Rightarrow \dot{y} = \frac{d}{dt} \left(\frac{2t}{1+t^2} \right) = \frac{2(1+t^2) - 4t^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$(1,0) \Rightarrow t=0$$

$$s = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_0^t \sqrt{\frac{16t^2}{(1+t^2)^4} + \frac{4(1-t^2)^2}{(1+t^2)^4}} dt$$

$$= \int_0^t \sqrt{\frac{16t^2 + 4(1-t^2)^2}{(1+t^2)^4}} dt = \int_0^t \frac{\sqrt{4t^2 + 4(1-t^2)^2}}{(1+t^2)^2} dt$$

$$= \int_0^t \frac{\sqrt{4(t^2 + (1-t^2)^2)}}{(1+t^2)^2} dt = \int_0^t \frac{2\sqrt{t^2 + (1-t^2)^2}}{(1+t^2)^2} dt$$

$$= \int_0^t \frac{2}{1+t^2} dt = \left[2 \arctan t \right]_0^t = 2 \arctan t - \arctan 0$$

Thus $s = 2 \arctan t$
 $\frac{s}{2} = \arctan t$
 $\tan \frac{s}{2} = t$

$$\bullet \ x = \frac{2}{1+t^2} - 1 = \frac{2 - (1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2} = \cos s$$

$$\bullet \ y = \frac{2t}{1+t^2} = \sin s$$

These are the little bit logarithms

$$\therefore \mathbf{r}(s) = (\cos s) \mathbf{i} + (\sin s) \mathbf{j}$$

Question 9

Evaluate the line integral

$$\oint_C y^5 dx,$$

where C is a circle of radius 2, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

$$\boxed{-40\pi}$$

Handwritten solution for the line integral problem:

Method 1: Cartesian Coordinates

Circle equation: $x^2 + y^2 = 4$
 $y = \pm \sqrt{4 - x^2}$
 $dy = \frac{-x}{\sqrt{4 - x^2}} dx$

Line integral: $\oint_C y^5 dx$

Split into two parts: $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$

For $y = \sqrt{4 - x^2}$: $\int_{-2}^2 (\sqrt{4 - x^2})^5 dx$

For $y = -\sqrt{4 - x^2}$: $\int_{-2}^2 (-\sqrt{4 - x^2})^5 dx$

Result: -40π

Method 2: Polar Coordinates

Circle equation: $r^2 = 4$
 $r = 2$
 $dr = 0$

Line integral: $\oint_C y^5 dx$

Convert to polar: $y = r \sin \theta$, $dx = -r \sin \theta dr + r \cos \theta d\theta$

Result: -40π

Question 10

A curve C is defined by $\mathbf{r} = \mathbf{r}(t)$, $0 \leq t \leq 2\pi$ as

$$\mathbf{r}(t) = (x, y, z) = [2(t - \sin t), \sqrt{3} \cos t, 1 + \cos t].$$

Evaluate the integral

$$\int_C z \, ds,$$

where s is the arclength along C .

$$\frac{32}{3}$$

Handwritten solution for the integral of $z \, ds$ over the curve C :

Given: $x = 2(t - \sin t)$, $y = \sqrt{3} \cos t$, $z = 1 + \cos t$, $0 \leq t \leq 2\pi$

Firstly $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{4(1 - \cos t)^2 + 3 \sin^2 t + (-\sin t)^2} dt$

$= \sqrt{4(1 - \cos t)^2 + 3 \sin^2 t + \sin^2 t} dt = \sqrt{4 - 8 \cos t + 4 \cos^2 t + 4 \sin^2 t} dt$

$= \sqrt{4 - 8 \cos t + 4} dt = \sqrt{8 - 8 \cos t} dt = \sqrt{8(1 - \cos t)} dt = 4 \sin \frac{t}{2} dt$

$\int_C z \, ds = \int_{t=0}^{2\pi} (1 + \cos t) (4 \sin \frac{t}{2}) dt$

$= \int_0^{2\pi} [1 + (2 \cos \frac{t}{2} - 1)] [4 \sin \frac{t}{2}] dt$

$= \int_0^{2\pi} 8 \cos \frac{t}{2} \sin \frac{t}{2} dt$

$= \left[\frac{8}{3} \times (-2 \cos^2 \frac{t}{2}) \right]_0^{2\pi}$

$= \left[-\frac{16}{3} \cos^2 \frac{t}{2} \right]_0^{2\pi}$

$= -\frac{16}{3} - \left(-\frac{16}{3} \right) = \frac{32}{3}$

Question 11

$$V(x, y, z) = 60xyz^2.$$

Evaluate the following integral along C , from $(3, 1, 1)$ to $(4, 3, 2)$,

$$\int_C V \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = t + 2, \quad y = 2t - 1, \quad z = t.$$

$$1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}$$

$V(x,y,z) = 60xyz^2$ $x = t+2$ $dx = dt$
 $y = 2t-1$ $dy = 2dt$
 $z = t$ $dz = dt$

$$\begin{aligned}
 \text{The } \int_C V \, d\mathbf{r} &= \int_{(3,1,1)}^{(4,3,2)} 60xyz^2 (dx, dy, dz)^T = \int_{t=1}^{t=2} 60(t+2)(2t-1)t^2 (dt, 2dt, dt)^T \\
 &= \int_{t=1}^{t=2} 60(t^4 + 3t^3 - 2t^2) dt \\
 &= 60 \left(\frac{t^5}{5} + \frac{3t^4}{4} - \frac{2t^3}{3} \right) \Big|_1^2 \\
 &= 60 \left(\frac{32}{5} + 12 - \frac{16}{3} \right) - \left(\frac{6}{5} + \frac{3}{4} - \frac{2}{3} \right) \\
 &= 1139 \left(\frac{1}{5} + \frac{2}{3} + \frac{1}{4} \right) \\
 &= 1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}
 \end{aligned}$$

Question 12

$$\phi(x, y, z) \equiv 3x + 2y + z.$$

Evaluate the following integral along C , from $(1, 0, 0)$ to $(2, 2, 1)$,

$$\int_C \phi \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = t + 1, \quad y = 2t, \quad z = t^2.$$

$$\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}$$

Handwritten solution for the line integral problem:

$$\begin{aligned} \phi(x, y, z) &= 3x + 2y + z & x = t+1 &\Rightarrow dx = dt \\ & & y = 2t &\Rightarrow dy = 2dt \\ & & z = t^2 &\Rightarrow dz = 2t dt \end{aligned}$$

$$\begin{aligned} \text{Thus } \int_C \phi \, d\mathbf{r} &= \int_{t=0}^{t=1} (3x + 2y + z) (dx \, dy \, dz) = \int_{t=0}^{t=1} [3(t+1) + 2(2t) + t^2] (dt \cdot 2dt \cdot 2t \, dt) \\ &= \int_{t=0}^{t=1} (3t + 3 + 4t + t^2) (1, 2, 2t) \, dt \\ &= \int_{t=0}^{t=1} (3t^2 + 7t + 3) (1, 2, 2t) \, dt \\ &= \int_0^1 [12t^3 + 14t^2 + 6t, 6t^3 + 14t^2 + 4t, 6t^4 + 14t^3 + 3t^2] \, dt \\ &= \left[\frac{12}{4}t^4 + \frac{14}{3}t^3 + \frac{6}{2}t^2, \frac{6}{4}t^4 + \frac{14}{3}t^3 + \frac{4}{2}t^2, \frac{6}{5}t^5 + \frac{14}{4}t^4 + \frac{3}{3}t^3 \right]_0^1 \\ &= \left(\frac{12}{4} + \frac{14}{3} + 3, \frac{6}{4} + \frac{14}{3} + 2, \frac{6}{5} + \frac{14}{4} + 1 \right) = \left(\frac{41}{3}, \frac{41}{6}, \frac{49}{6} \right) \\ &\quad \text{i.e. } \frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k} \end{aligned}$$

Question 13

$$F(x, y, z) = xyz.$$

Evaluate the following integral along C , from $(1,0,0)$ to $(0,1,4)$,

$$\int_C F \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = \frac{8t}{\pi}.$$

$$\frac{16-12\pi}{9\pi} \mathbf{i} + \frac{16}{9\pi} \mathbf{j} + \frac{8}{\pi} \mathbf{k}$$

Handwritten solution for the line integral problem. The solution starts with the vector field $F(x,y,z) = xyz$ and the differential vector $d\mathbf{r} = (dx, dy, dz)^T$. The curve C is defined by the parametric equations $x = \cos t$, $y = \sin t$, and $z = \frac{8t}{\pi}$. The integral is set up as $\int_C F \, d\mathbf{r} = \int_0^{\frac{\pi}{2}} (x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt}) dt$. The integrand is then simplified to $\frac{8}{\pi} t \cos t \sin t$. The integral is evaluated using integration by parts, resulting in the final answer $\frac{16-12\pi}{9\pi} \mathbf{i} + \frac{16}{9\pi} \mathbf{j} + \frac{8}{\pi} \mathbf{k}$.

Question 14

A curve C has equation

$$x^2 + xy + y^2 = 1, \quad 0 \leq x \leq 3.$$

Find a suitable parameterization of C in the form

$$x = A \cos \theta + B \sin \theta \quad \text{and} \quad y = A \cos \theta - B \sin \theta,$$

where A and B are suitable constants.

$$x = \frac{1}{\sqrt{3}} \cos \theta + \sin \theta, \quad y = \frac{1}{\sqrt{3}} \cos \theta - \sin \theta$$

Handwritten solution for Question 14:

- Given: $x^2 + xy + y^2 = 1$
- Let $x = a \cos \theta + b \sin \theta$
- Let $y = a \cos \theta - b \sin \theta$
- Then we have:

$$(a \cos \theta + b \sin \theta + a \cos \theta - b \sin \theta)^2 - (a \cos \theta + b \sin \theta)(a \cos \theta - b \sin \theta) = 1$$

$$= (2a \cos \theta)^2 - (a^2 \cos^2 \theta - b^2 \sin^2 \theta)$$

$$= 4a^2 \cos^2 \theta - a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

$$= 3a^2 \cos^2 \theta + b^2 \sin^2 \theta$$
- Thus we get to $\frac{1}{3}$ i.e. $a = \frac{1}{3} \cos \theta + \sin \theta$
- $b = \frac{1}{3} \cos \theta - \sin \theta$
- Check:

$$\left(\frac{1}{3} \cos \theta + \sin \theta\right)^2 + \left(\frac{1}{3} \cos \theta + \sin \theta\right)\left(\frac{1}{3} \cos \theta - \sin \theta\right) + \left(\frac{1}{3} \cos \theta - \sin \theta\right)^2 = 1$$

$$= \left(\frac{1}{9} \cos^2 \theta + \frac{2}{3} \cos \theta \sin \theta + \sin^2 \theta\right) + \left(\frac{1}{9} \cos^2 \theta - \sin^2 \theta\right) + \left(\frac{1}{9} \cos^2 \theta - \frac{2}{3} \cos \theta \sin \theta + \sin^2 \theta\right)$$

$$= \frac{1}{3} \cos^2 \theta + \sin^2 \theta = 1$$

Question 15

A surface S is given parametrically by

$$x = at \cosh \theta, \quad y = bt \sinh \theta, \quad z = t^2,$$

where t and θ are real parameters, and a and b are non zero constants.

- Find a Cartesian equation for S .
- Determine an equation of the tangent plane on S at the point with Cartesian coordinates (x_0, y_0, z_0) .

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad 2b^2 x x_0 + 2a^2 y y_0 - a^2 b^2 z = 0$$

$x = at \cosh \theta$
 $y = bt \sinh \theta$
 $z = t^2$

$\frac{x}{a} = t \cosh \theta \rightarrow \frac{x^2}{a^2} = t^2 \cosh^2 \theta$
 $\frac{y}{b} = t \sinh \theta \rightarrow \frac{y^2}{b^2} = t^2 \sinh^2 \theta$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2 (\cosh^2 \theta - \sinh^2 \theta)$
 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2 (1)$
 $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Now let $f(x, y, z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$
 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{2x}{a^2}, -\frac{2y}{b^2}, -1 \right)$
 At (x_0, y_0, z_0)

Equation of tangent plane
 $\frac{2x_0}{a^2}(x - x_0) - \frac{2y_0}{b^2}(y - y_0) - (z - z_0) = 0$

Using (x_0, y_0, z_0) to find constant
 constant = $\frac{2x_0^2}{a^2} - \frac{2y_0^2}{b^2} - z_0$
 But this must be zero (equation of surface)

\therefore Tangent plane
 $\frac{2x_0}{a^2}(x - x_0) - \frac{2y_0}{b^2}(y - y_0) - (z - z_0) = 0$
 or $2b^2 x_0 x - 2a^2 y_0 y - a^2 b^2 z = 0$

The surface of a torus has parametric equations

where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq 2\pi$.

- A torus T has Cartesian equation

b) Use a suitable parameterization of T to find its surface area.

$$\boxed{}, \quad \boxed{z^2 + \left(R - \sqrt{x^2 + y^2}\right)^2 = r^2}, \quad \boxed{\text{area} = (2\pi r)(2\pi R) = 16\pi^2}$$

1) EXAMPLE: NOO CIRCLES AS PROFILES

$x(\phi) = (R + r \cos \phi) \cos \phi$
 $y(\phi) = (R + r \cos \phi) \sin \phi$
 $z(\phi) = 12 \sin \phi$

$0 \leq \phi \leq 2\pi$
 $0 \leq \phi \leq \pi$

$\vec{r}'(\phi) = (R + r \cos \phi) \cos \phi + (R + r \cos \phi) \sin \phi$
 $= (R + r \cos \phi) [\cos \phi + \sin \phi]$
 $= (R + r \cos \phi)^2$

$\vec{r}''(\phi) = -r \sin \phi = -z$
 $-r \cos \phi = -z^2$
 $\vec{r}''(\phi) = (-z, -z^2, 1)^T$

$\vec{r}'(\phi) \cdot \vec{r}''(\phi) = (R + r \cos \phi)^2 \cdot (-z, -z^2, 1)^T$
 $\vec{r}'(\phi) \cdot \vec{r}''(\phi) = (R + r \cos \phi)^2 \cdot (-z, -z^2, 1)^T$
 $\vec{r}'(\phi) \cdot \vec{r}''(\phi) = (R + r \cos \phi)^2 \cdot (-z, -z^2, 1)^T$

2) EXERCISE: THE SPHERE OF RADIUS 1

$(R + r \cos \phi)^2 = 1 + r^2 \cos^2 \phi$
 $(R + r \cos \phi)^2 = 1 + r^2 \cos^2 \phi$

$0 \leq \phi \leq 2\pi$

Question 17

A spiral ramp is modelled by the surface S defined by the vector function

$$\mathbf{r}(u, v) = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (u \cos v)\mathbf{i} + (u \sin v)\mathbf{j} + v\mathbf{k},$$

where $0 \leq u \leq 1$, $0 \leq v \leq 3\pi$.

Determine the value of

$$\int_S \sqrt{x^2 + y^2} \, dS$$

$$\pi \left[\sqrt{8} - 1 \right]$$

$\mathbf{r}(u, v) = [u \cos v, u \sin v, v]$ $0 \leq u \leq 1$
 $0 \leq v \leq 3\pi$

• FIRSTLY WE CALCULATE THE dS ELEMENT

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos v, \sin v, 0)$$

$$\frac{\partial \mathbf{r}}{\partial v} = (-u \sin v, u \cos v, 1)$$

$$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = [\sin v, -\cos v, u \cos^2 v + u \sin^2 v] = [\sin v, -\cos v, u]$$

$$\left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| = [\sin v, -\cos v, u] = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

$$\therefore dS = \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

$$dS = \sqrt{1 + u^2} du dv$$

• $\int_S \sqrt{x^2 + y^2} \, dS = \int_{v=0}^{3\pi} \int_{u=0}^1 \sqrt{u^2 \cos^2 v + u^2 \sin^2 v} \sqrt{1 + u^2} \, du dv$

$$= \int_{v=0}^{3\pi} \int_{u=0}^1 u (1 + u^2)^{\frac{3}{2}} \, du dv$$

$$= \left[\int_{v=0}^{3\pi} 1 \, dv \right] \left[\int_{u=0}^1 u (1 + u^2)^{\frac{3}{2}} \, du \right]$$

$$= 3\pi \times \left[\frac{1}{5} (u^2 + 1)^{\frac{5}{2}} \right]_0^1$$

$$= \pi \left[2^{\frac{5}{2}} - 1 \right]$$

$$= \pi \left[2\sqrt{2} - 1 \right]$$

Question 18

The surface S is defined by the vector equation

$$\mathbf{F}(u, v) = \left[u \cos v, u \sin v, \frac{1}{u} \right]^T, \quad u \neq 0.$$

Find the area of S lying above the region in the uv plane bounded by the curves

$$v = u^4, \quad v = 2u^4,$$

and the straight lines with equations $u = 3^{\frac{1}{4}}$ and $u = 8^{\frac{1}{4}}$.

$$\frac{19}{6}$$

Handwritten solution for Question 18:

$$\mathbf{F}(u, v) = \left[u \cos v, u \sin v, \frac{1}{u} \right]^T$$

Partial derivatives:

$$\frac{\partial \mathbf{F}}{\partial u} = \left[\cos v, \sin v, -\frac{1}{u^2} \right]^T, \quad \frac{\partial \mathbf{F}}{\partial v} = \left[-u \sin v, u \cos v, 0 \right]^T$$

Cross product:

$$\frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & -\frac{1}{u^2} \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \left[\frac{1}{u} \cos v, \frac{1}{u} \sin v, u \cos^2 v + u \sin^2 v \right] = \left[\frac{1}{u} \cos v, \frac{1}{u} \sin v, u \right]^T$$

Magnitude of the cross product:

$$\left| \frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} \right| = \sqrt{\left(\frac{1}{u} \cos v \right)^2 + \left(\frac{1}{u} \sin v \right)^2 + u^2} = \sqrt{\frac{1}{u^2} + u^2} = \frac{\sqrt{1+u^4}}{u}$$

Area element:

$$dS = \left| \frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} \right| du dv = \frac{\sqrt{1+u^4}}{u} du dv$$

Integration limits: u from $3^{\frac{1}{4}}$ to $8^{\frac{1}{4}}$, v from u^4 to $2u^4$.

$$S = \int_{u=3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \int_{v=u^4}^{2u^4} \frac{\sqrt{1+u^4}}{u} dv du$$

$$= \int_{u=3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \left[\frac{1}{u} \sqrt{1+u^4} \right]_{v=u^4}^{2u^4} du = \int_{u=3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \frac{2\sqrt{1+u^4}}{u} du$$

$$= \int_{u=3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \frac{2}{u} \sqrt{1+u^4} du = \left[\frac{2}{3} (1+u^4)^{\frac{3}{2}} \right]_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} = \frac{2}{3} \left[27 - 8 \right] = \frac{38}{3}$$

Final answer: $\frac{19}{6}$

Question 19

The surface S is defined by the parametric equations

$$x = t \cosh \theta, \quad y = t \sinh \theta, \quad z = \frac{1}{2}(1 - t^2),$$

where t and θ are real parameters such that $0 \leq t \leq 1$ and $0 \leq \theta \leq 1$.

Find, in exact form, the value of

$$\int_S xy \, dS.$$

$$\frac{1}{30} \left[\frac{(\cosh 2 + 1)^{\frac{5}{2}} - 1}{\cosh 2} + 1 - 4\sqrt{2} \right] \approx 0.274397...$$

$\vec{r}(t, \theta) = (t \cosh \theta, t \sinh \theta, \frac{1}{2}(1 - t^2))$
 $\frac{\partial \vec{r}}{\partial t} = (\cosh \theta, \sinh \theta, -t)$
 $\frac{\partial \vec{r}}{\partial \theta} = (t \sinh \theta, t \cosh \theta, 0)$
 $\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \cosh \theta & \sinh \theta & -t \\ t \sinh \theta & t \cosh \theta & 0 \end{vmatrix} = \begin{vmatrix} t \cosh \theta & -t \sinh \theta & t \cosh \theta - t \sinh \theta \\ t \sinh \theta & t \cosh \theta & t \end{vmatrix}$
 $= \begin{vmatrix} t \cosh \theta & -t \sinh \theta & t \\ t \sinh \theta & t \cosh \theta & t \end{vmatrix}$ (since $\cosh^2 \theta - \sinh^2 \theta = 1$)
 $\therefore \left| \frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{t^2 \cosh^2 \theta + t^2 \sinh^2 \theta + t^2} = |t| \sqrt{\cosh^2 \theta + \sinh^2 \theta + 1}$
 $= |t| \sqrt{t^2 (\cosh^2 \theta + \sinh^2 \theta) + 1} = |t| \sqrt{t^2 \cosh 2\theta + 1}$
 $\therefore dS = |t| \sqrt{t^2 \cosh 2\theta + 1} \, dt \, d\theta$
 $\int_S xy \, dS = \int_0^1 \int_0^1 (t \cosh \theta)(t \sinh \theta) |t| \sqrt{t^2 \cosh 2\theta + 1} \, dt \, d\theta$
 $= \int_0^1 \int_0^1 \frac{1}{2} t^3 \sinh 2\theta (\cosh 2\theta + 1)^{\frac{1}{2}} \, dt \, d\theta$
 $= \int_0^1 \left[\frac{1}{4} t^4 \sinh 2\theta (\cosh 2\theta + 1)^{\frac{1}{2}} \right]_{t=0}^{t=1} d\theta$

$= \int_0^1 \left[\frac{1}{4} t^4 (\cosh 2\theta + 1)^{\frac{1}{2}} \right]_{t=0}^{t=1} d\theta$
 $= \int_0^1 \frac{1}{4} t^4 (\cosh 2\theta + 1)^{\frac{1}{2}} d\theta$
 $= \left[\frac{1}{30 \cosh 2} (\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{30} (\cosh 2)^{\frac{5}{2}} \right]_{\theta=0}^{\theta=1}$
 $= \left[\frac{1}{30 \cosh 2} (\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{30} \times 2^{\frac{5}{2}} \right] - \left[\frac{1}{30 \cosh 2} - \frac{1}{30} \right]$
 $= \frac{1}{30} \left[\frac{(\cosh 2 + 1)^{\frac{5}{2}}}{\cosh 2} - \frac{1}{\cosh 2} + 1 - 4\sqrt{2} \right]$

Question 20

$$\mathbf{F}(x, y, z) \equiv y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}.$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u+v)\mathbf{k}, \quad 0 \leq u \leq 1, \quad 1 \leq v \leq 4.$$

All integrations must be carried out in parametric.

$$\boxed{}, \quad \boxed{\frac{1}{2}}$$

$\mathbf{F}(u, v) = \begin{pmatrix} v \\ u^2 \\ u+v \end{pmatrix}$ $\mathbf{r}(u, v) = \begin{pmatrix} u \\ v \\ u+v \end{pmatrix} \quad \begin{matrix} 0 \leq u \leq 1 \\ 1 \leq v \leq 4 \end{matrix}$

Find an expression for the "area flux element" $d\mathbf{s}$

- $\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- $\frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
- NORMAL = $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (-1, -1, 1)$
- UNIT NORMAL $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$
 $\hat{\mathbf{n}} = \frac{(-1, -1, 1)}{\sqrt{3}}$

COLLECTING THESE RESULTS

$d\mathbf{s} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} du dv$

$\hat{\mathbf{n}} d\mathbf{s} = \hat{\mathbf{n}} \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| du dv$

$d\mathbf{s} = \frac{(-1, -1, 1)}{\sqrt{3}} \sqrt{3} du dv$

$d\mathbf{s} = (-1, -1, 1) du dv$

$d\mathbf{s} = (-1, -1, 1) du dv$

FIND THE FLUX BY EVALUATION

$$\begin{aligned} \text{FLUX} &= \int_S \mathbf{F} \cdot d\mathbf{s} = \int \mathbf{F}(u, v) \cdot \hat{\mathbf{n}} d\mathbf{s} \\ &= \int_1^4 \int_0^1 (v\mathbf{i} + u^2\mathbf{j} + (u+v)\mathbf{k}) \cdot (-1, -1, 1) du dv \\ &= \int_1^4 \int_0^1 (-v - u^2 + u + v) du dv \\ &= \int_1^4 \int_0^1 (u - u^2) du dv \\ &= \int_1^4 \left[\frac{1}{2}u^2 - \frac{1}{3}u^3 \right]_0^1 dv \\ &= \int_1^4 \left(\frac{1}{2} - \frac{1}{3} \right) dv \\ &= \int_1^4 \frac{1}{6} dv \\ &= \left[\frac{1}{6}v \right]_1^4 \\ &= \frac{4}{6} - \frac{1}{6} \\ &= \frac{3}{6} = \frac{1}{2} \end{aligned}$$

Question 21

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface represented parametrically by

$$\mathbf{r}(u, v) = \begin{bmatrix} u+v \\ u-v \\ u \end{bmatrix}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3,$$

and \mathbf{F} is the vector field

$$x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

All integrations must be carried out in parametric., 36

• PREPARE ALL THE NECESSARY ITEMS

• $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$

• $\mathbf{r}(u, v) = (u+v, u-v, u), 0 \leq u \leq 2, 0 \leq v \leq 3$
(THIS IS IN FACT A PLANE THROUGH O)

• $\frac{\partial \mathbf{r}}{\partial u} = (1, 1, 1), \frac{\partial \mathbf{r}}{\partial v} = (1, -1, 0)$

• $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1, 1, -2)$

• $\hat{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\|} = \frac{(1, 1, -2)}{\sqrt{6}}$

• $d\mathbf{S} = \left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\| du dv$

• HENCE WE NOW HAVE IN PARAMETRIC

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} dS$$

$$= \int_S \mathbf{F} \cdot \left(\frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\|} \right) \left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\| du dv$$

$$= \int_S \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv$$

• SUBSTITUTING EVERY INTO THE REQUIRED SURFACE INTEGRAL

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^3 \int_{u=0}^2 [(u+v)^2 (u-v)^2, u^2] \cdot [1, 1, -2] du dv$$

$$= \int_{v=0}^3 \int_{u=0}^2 [(u+v)^2 (u-v)^2 + u^2 - 2u^2] du dv$$

$$= \int_{v=0}^3 \int_{u=0}^2 [u^2 + 2uv + v^2 + u^2 - 2u^2] du dv$$

$$= \int_{v=0}^3 \int_{u=0}^2 [2uv + v^2] du dv$$

$$= \int_{v=0}^3 \left[u^2 v + u^2 v \right]_{u=0}^2 dv$$

$$= \int_{v=0}^3 4v^2 dv$$

$$= \left[\frac{4}{3} v^3 \right]_{v=0}^3$$

$$= \frac{4}{3} \times 27$$

$$= 36$$

Question 22

Evaluate the surface integral

$$\int_S \mathbf{z} \cdot \mathbf{k} \cdot d\mathbf{S},$$

where S is the surface represented parametrically by

$$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad 0 \leq \theta \leq \frac{1}{2}\pi, \quad 0 \leq \varphi \leq \frac{1}{2}\pi.$$

All integrations must be carried out in parametric.

$$\boxed{\frac{1}{6}\pi}$$

$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} \quad \begin{matrix} 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{matrix}$

$\int_S \mathbf{z} \cdot \mathbf{k} \cdot d\mathbf{S} = \int_S \mathbf{z} \cdot \hat{n} \cdot dS$

FIND THE UNIT NORMAL TO THE PARAMETRIZED SURFACE & CHECK THE ORIENTATION IN PARAMETRIC

$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} \quad \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi \\ 0 \end{bmatrix}$

$\therefore \hat{n} = \left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right| = \begin{vmatrix} 1 & 0 & 0 \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ -\sin \theta \sin \varphi & \sin \theta \cos \varphi & 0 \end{vmatrix}$

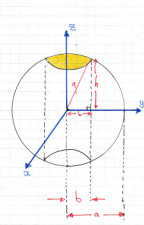
$= \begin{bmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \sin \theta \cos \varphi + \cos \theta \sin \theta \sin \varphi \end{bmatrix}$

$= \begin{bmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \sin \theta (\cos \varphi + \sin \varphi) \end{bmatrix}$

$= \begin{bmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \sin \theta \end{bmatrix}$

STARTING WITH A DIAPHRAM

SPHERE: $x^2 + y^2 + z^2 = a^2$
 CYLINDER: $x^2 + y^2 = b^2$
 $(a > b)$



AREA OF THE INNER CYLINDRICAL FACE IS GIVEN BY
 $2\pi r h = 2\pi b(2h)$
 $= 4\pi b h$
 $= 4\pi b(a^2 - b^2)^{\frac{1}{2}}$

NEXT WE FIND THE AREA OF ONE OF THE CAPS (SHOWN IN YELLOW) - PROJECT THE "TOP" CAP (Z > 0) ONTO THE XY PLANE

$z = \pm(a^2 - x^2 - y^2)^{\frac{1}{2}}$
 $\frac{\partial z}{\partial x} = -x(a^2 - x^2 - y^2)^{-\frac{1}{2}} \quad \frac{\partial z}{\partial y} = -y(a^2 - x^2 - y^2)^{-\frac{1}{2}}$
 $\Rightarrow dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dx \, dy$
 $\Rightarrow dS = \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} \, dx \, dy$

Question 23

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (u \sin v)\mathbf{j} + u\mathbf{k},$$

such that $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

$$\frac{2}{3}\pi$$

$\mathbf{F}(u, v) = [u \cos v, u \sin v, u]$ $0 \leq u \leq 1$
 $0 \leq v \leq 2\pi$

$\mathbf{F}(u, v) = (x, y, z)$

• FIRSTLY FIND THE JACOBIAN AND THE NORMAL

• $\frac{\partial \mathbf{r}}{\partial u} = [\cos v, \sin v, 1]$
 $\frac{\partial \mathbf{r}}{\partial v} = [-u \sin v, u \cos v, 0]$

• $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix}$
 $= [0 - u \cos v, -u \sin v - 0, u \cos^2 v + u \sin^2 v]$
 $= [-u \cos v, -u \sin v, u] \leftarrow \text{NORMAL}$

• $\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{(-u \cos v)^2 + (-u \sin v)^2 + u^2} = u \sqrt{\cos^2 v + \sin^2 v + 1} = u\sqrt{2} \leftarrow \text{JACOBIAN}$

• $\hat{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}$ $d\mathbf{S} = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$

• NOW THE FLUX INTEGRAL CAN BE COMPUTED

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} d\mathbf{S}$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 [u \cos v, u \sin v, u] \cdot \frac{1}{\sqrt{2}} [-u \cos v, -u \sin v, u] du dv$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 (-u^2 \cos^2 v - u^2 \sin^2 v + u^2) du dv$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 -u^2 (\cos^2 v + \sin^2 v) + u^2 du dv$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 u^2 du dv$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \left[\frac{1}{3} u^3 \right]_0^1 dv$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \frac{1}{3} dv$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \times 2\pi$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{3}$

Question 24

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (1 + \sin u \cos v)\mathbf{i} + (\sin u \sin v)\mathbf{j} + (\cos u)\mathbf{k},$$

such that $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

4π

The handwritten solution is divided into two parts. The left part defines the vector function $\mathbf{F}(u, v) = [1 + \sin u \cos v, \sin u \sin v, \cos u]$ and calculates the partial derivatives $\frac{\partial \mathbf{r}}{\partial u} = [\sin u \cos v, \sin u \sin v, -\cos u]$ and $\frac{\partial \mathbf{r}}{\partial v} = [-\sin u \sin v, \sin u \cos v, 0]$. It then finds the cross product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = [\sin^2 u \cos v, \sin^2 u \sin v, \sin u]$ and its magnitude $|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}| = \sin u$. The right part shows the integration $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \mathbf{F}(u, v) \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv = \int_0^{2\pi} \int_0^\pi (1 + \sin u \cos v) \sin u du dv$. It simplifies the integrand to $\sin u + \sin^2 u \cos v$ and integrates with respect to u to get $[-\cos u + \frac{1}{3} \cos 3u]_0^\pi$, which evaluates to 2 . Finally, it integrates with respect to v from 0 to 2π to get $2 \times 2\pi = 4\pi$.

Question 25

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (1 + u \sin v)\mathbf{j} + (u - 1)\mathbf{k},$$

such that $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

$$\boxed{\frac{1}{3}\pi}$$

Handwritten solution for Question 25:

$\mathbf{F}(u, v) = [u \cos v, 1 + u \sin v, u - 1]$ $0 \leq u \leq 1$
 $0 \leq v \leq 2\pi$
 $\mathbf{F}(u, v) = (x, y, z)$
 $\frac{\partial \mathbf{r}}{\partial u} = [\cos v, \sin v, 1]$
 $\frac{\partial \mathbf{r}}{\partial v} = [-u \sin v, u \cos v, 0]$
 $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix}$
 $= [0 - u \cos v, -u \sin v - u \cos v, -u \sin^2 v - u \cos^2 v]$
 $= [-u \cos v, -u \sin v, -u(\sin^2 v + \cos^2 v)]$
 $= [-u \cos v, -u \sin v, -u]$ $\leftarrow \text{MAGNITUDE}$
 $\hat{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}$ $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ $\leftarrow \text{Parametric, parametric!}$
 $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} dS$
 $= \int_0^{2\pi} \int_0^1 \mathbf{F}(u, v) \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv$
 $= \int_0^{2\pi} \int_0^1 [-u^2 \cos^2 v - u^2 \sin^2 v - u^2 + u] du dv$
 $= \int_0^{2\pi} \int_0^1 [-u^2(\cos^2 v + \sin^2 v) - u^2 + u] du dv$
 $= \int_0^{2\pi} \int_0^1 [-2u^2 + u] du dv$
 $= \int_0^{2\pi} \left[-\frac{2}{3}u^3 + \frac{1}{2}u^2 \right]_0^1 dv$
 $= \int_0^{2\pi} \left(-\frac{2}{3} + \frac{1}{2} \right) dv$
 $= \int_0^{2\pi} \left(-\frac{1}{6} \right) dv$
 $= -\frac{1}{6} \times 2\pi = -\frac{\pi}{3}$
 $\therefore \left| \int_S \mathbf{F} \cdot d\mathbf{S} \right| = \frac{\pi}{3}$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

$$\mathbf{r}(\theta, \varphi) = [(4 + \cos \theta) \cos \varphi] \mathbf{i} + [(4 + \cos \theta) \sin \varphi] \mathbf{j} + (\sin \theta) \mathbf{k},$$

All integrations must be carried out in parametric.

$$24\pi^2$$

$I(\varphi, \psi) = \begin{bmatrix} 1 + \cos(\varphi) \cos(\psi) & 4 + \cos(\varphi) \sin(\psi) \sin(\psi) \end{bmatrix} \quad \begin{matrix} 0 \leq \varphi \leq 2\pi \\ 0 \leq \psi \leq 2\pi \end{matrix}$
 $\Rightarrow F(\varphi, \psi) = \begin{pmatrix} x & y & z \end{pmatrix}$

$\frac{\partial F}{\partial \varphi} = \begin{bmatrix} -\sin(\varphi) \cos(\psi) & -\sin(\varphi) \sin(\psi) \cos(\psi) \end{bmatrix} \quad \& \quad \frac{\partial F}{\partial \psi} = \begin{bmatrix} -4 \cos(\varphi) \sin(\psi) \sin(\psi) & 4 \cos(\varphi) + \cos(\varphi) \sin^2(\psi) \end{bmatrix}$

$\frac{\partial F}{\partial \varphi} \cdot \frac{\partial F}{\partial \psi} = \begin{vmatrix} -\sin(\varphi) \cos(\psi) & -\sin(\varphi) \sin(\psi) \cos(\psi) \\ -4 \cos(\varphi) \sin(\psi) \sin(\psi) & 4 \cos(\varphi) + \cos(\varphi) \sin^2(\psi) \end{vmatrix}$

$= \begin{bmatrix} 0 - 4 \sin^2(\varphi) \sin^2(\psi) \cos(\psi) & -4 \cos^2(\varphi) \sin(\psi) \cos(\psi) - 0 & -4 \sin(\varphi) \cos(\psi) - \sin(\varphi) \cos(\psi) \sin^2(\psi) + \cos(\varphi) \sin^3(\psi) \end{bmatrix}$

$= \begin{bmatrix} -4 \sin^2(\varphi) \sin^2(\psi) \cos(\psi) & -4 \sin^2(\varphi) \cos(\psi) \sin(\psi) & -4 \sin(\varphi) (\cos(\psi) + \cos(\psi) \sin^2(\psi)) - \sin(\varphi) \cos(\psi) (\sin^2(\psi) + \sin^2(\psi)) \end{bmatrix}$

$= \begin{bmatrix} -4 \sin^2(\varphi) \sin^2(\psi) \cos(\psi) & -4 \cos^2(\varphi) \sin(\psi) \cos(\psi) & -8 \sin(\varphi) (\cos(\psi) + \cos(\psi) \sin^2(\psi)) \end{bmatrix}$

$= (4 + \cos(\varphi)) \begin{bmatrix} -\cos(\psi) \sin(\psi) & -\cos(\psi) \sin(\psi) & -\sin(\psi) (1 + \cos(\psi)) \end{bmatrix}$

\Rightarrow NORMAL VECTOR IS

$\frac{1}{\|v\|} = \frac{1}{\sqrt{\left(\frac{\partial F}{\partial \varphi}\right)^2 + \left(\frac{\partial F}{\partial \psi}\right)^2}}$
 $\Rightarrow d\vec{\sigma} = \left| \frac{\partial F}{\partial \varphi} \frac{\partial F}{\partial \psi} \right| d\varphi d\psi$

"PARAMETRIC AREA"

HERE THE FORMULAS OF $\left(\frac{\partial F}{\partial \varphi}\right)^2 + \left(\frac{\partial F}{\partial \psi}\right)^2$ IS NOT MENTIONED IN THIS TYPE OF QUESTION, AS IT WILL COME

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} \, dA = \int F(\theta, \phi) \left(\frac{2\pi - 2\pi}{2\pi} \right) \left(\frac{2\pi - 2\pi}{2\pi} \right) \left(\frac{2\pi - 2\pi}{2\pi} \right) dA \\ &= \int_{\mathcal{S}} \left[(4 + \cos\theta) \cos\theta (4 + \cos\theta) \sin\theta \sin\phi \right] \cdot \left[\cos\theta \cos\phi \cos\theta \sin\theta \sin\phi \right] (4 + \cos\theta) \, d\theta \, d\phi \\ &= \int_{\mathcal{S}} (4 + \cos\theta) \left[(4 + \cos\theta) \cos\theta \cos^2\theta \sin\theta \sin\phi + \cos^2\theta \right] \, d\theta \, d\phi \\ &= \int_{\mathcal{S}} -(4 + \cos\theta) \left[(4 + \cos\theta) \cos\theta (\cos^2\theta \sin\theta) + \sin\theta \right] \, d\theta \, d\phi \\ &= \int_{\mathcal{S}} -(4 + \cos\theta) \left[4\cos\theta + \cos^3\theta + \sin\theta \right] \, d\theta \, d\phi \\ &= \int_{\mathcal{S}} -(4 + \cos\theta) (4 \cos\theta + 1) \, d\theta \, d\phi = - \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} (16\cos\theta + 4\cos\theta + \cos\theta) \, d\theta \, d\phi \\ &= - \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} (4 + \frac{1}{2} \cos\theta) \, d\theta \, d\phi = - \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} 4 \, d\theta \, d\phi \\ &= -6 \times 2\pi \times 2\pi = -24\pi^2 \end{aligned}$$

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with Cartesian equation

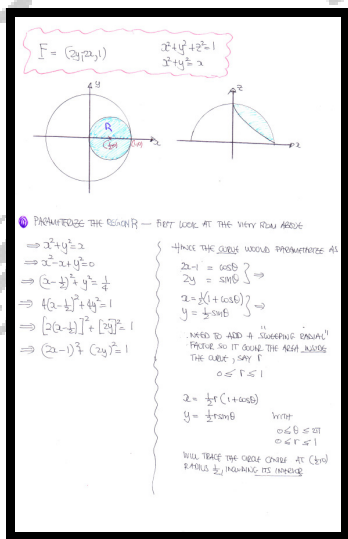
$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

cut off by the cylinder with cartesian equation

$$x^2 + y^2 = x.$$

You **must** find a suitable parameterization for S , and carry out the **integration in parametric**, without using any integral theorems.

$$\frac{\pi}{4}$$



④ BC: $\frac{x^2+y^2+z^2=1}{x^2+y^2=1-z^2}$
 $\frac{z^2}{1-z^2} = 1 - \frac{1}{2}r(1+\cos\theta)$
 $z = (1 - \frac{1}{2}r(1+\cos\theta))^{\frac{1}{2}}$
 Hence
 $dz = \left[\frac{1}{2}(1+\cos\theta) \cdot \frac{1}{2}r \cos\theta, \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} \right]$
 $0 \leq r \leq 1$
 $0 \leq \theta \leq 2\pi$

⑤ $\frac{\partial z}{\partial x} = \left[\frac{1}{2}(1+\cos\theta) \cdot \frac{1}{2}r \cos\theta, -\frac{1}{4}(1+\cos\theta) \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} \right]$
 $\frac{\partial z}{\partial y} = \left[-\frac{1}{2}r \sin\theta, \frac{1}{2}r \sin\theta, \frac{1}{2}r \cos\theta \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} \right]$
 $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = \left[\frac{1}{8}r^2 \cos^2\theta \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} + \frac{1}{8}r^2 \cos\theta(1+\cos\theta) \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} \right]$
 $+ \left[\frac{1}{8}r^2 \sin^2\theta(1+\cos\theta) \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} + \frac{1}{8}r^2 \sin\theta(1+\cos\theta) \left[1 - \frac{1}{2}r(1+\cos\theta) \right]^{\frac{1}{2}} \right]$
 $+ \left[\frac{1}{4}r \cos\theta(1+\cos\theta) + \frac{1}{4}r \sin^2\theta \right] \cdot \frac{1}{2}$

⑥ Now $\int_A \vec{F} \cdot \vec{n} \, dA = \int_A \vec{F}(\vec{r}) \cdot \left(\frac{\partial \vec{r}}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial \vec{r}}{\partial y} \frac{\partial z}{\partial x} \right) \, dr \, d\theta$
 $= \int_A \vec{F}(\vec{r}) \cdot \left(\frac{\partial \vec{r}}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial \vec{r}}{\partial y} \frac{\partial z}{\partial x} \right) \, dr \, d\theta$

$$\begin{aligned} \text{Now } F(\Omega_1, \Omega_2) &= (\Omega_1, \Omega_2, 1) \\ F(r, \phi) &= (\cos \phi, -\sin \phi, \cos \phi) \\ \text{So } F(r, \phi) \cdot \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial \phi} \right) &= \frac{1}{r^2} \sin^2 \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} \\ &\quad + \frac{1}{2} r^2 \sin \phi \cos \phi (1 + \cos \phi) \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} \\ &\quad + \frac{1}{2} r^2 \cos^2 \phi (1 + \cos \phi) + \frac{1}{2} r^2 \cos \phi \\ \text{Similarly} \\ &= \frac{1}{r^2} \sin^2 \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} \left[\sin \phi + \cos \phi (1 + \cos \phi) \right] \\ &\quad + \frac{1}{2} r^2 \sin \phi \cos \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} \left[\sin \phi + \cos \phi (1 + \cos \phi) \right] \\ &\quad + \frac{1}{2} r^2 \cos^2 \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} \left[\sin \phi + \cos \phi (1 + \cos \phi) \right] + \frac{1}{2} r C(1 + \cos \phi) \\ &= \frac{1}{r^2} \sin^2 \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} C(1 + \cos \phi) + \frac{1}{2} r C(1 + \cos \phi) \\ \text{Similarly} \\ &= \frac{1}{r^2} \sin^2 \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} + \frac{1}{2} r \sin \phi \cos \phi \left[1 - \frac{1}{2} r C(1 + \cos \phi) \right]^{\frac{1}{2}} + \frac{1}{2} r^2 \cos^2 \phi \end{aligned}$$