

DIFFERENTIAL EQUATIONS

2nd order or higher

2ND ORDER WITH CONSTANT COEFFICIENTS

Question 1 ()**

Find a general solution of the differential equation

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 12(x + e^x).$$

$$\boxed{y = Ae^{-3x} + Be^{-2x} + e^x + 2x - \frac{5}{3}}$$

Handwritten solution for Question 1:

$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 12(x + e^x)$
 • AUXILIARY EQUATION: $\lambda^2 + 5\lambda + 6 = 0$
 $(\lambda + 3)(\lambda + 2) = 0$
 $\lambda = -3, -2$
 • PARTICULAR INTEGRAL
 Try $y = Px + Q + Re^x$
 $\frac{dy}{dx} = P + Re^x$
 $\frac{d^2 y}{dx^2} = Re^x$
 $Re^x + 5(P + Re^x) + 6(Px + Q + Re^x) = 12x + 12e^x$
 $12Re^x + 6Px + (5P + 6Q + 6R) = 12x + 12e^x$
 $R = 1, P = 2, 5P + 6Q + 6R = 12$
 $10 + 6Q + 6 = 12 \Rightarrow 6Q = -4 \Rightarrow Q = -\frac{2}{3}$
 $\therefore y = Ae^{-3x} + Be^{-2x} + e^x + 2x - \frac{5}{3}$

Question 2 ()**

Find a general solution of the differential equation

$$\frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 13y = 13x^2 - x + 22.$$

$$\boxed{y = e^{-3x} (A \cos 2x + B \sin 2x) + x^2 - x + 2}$$

Handwritten solution for Question 2:

$\frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 13y = 13x^2 - x + 22$
 • AUXILIARY EQUATION: $\lambda^2 + 6\lambda + 13 = 0$
 $(\lambda + 3)^2 + 4 = 0$
 $(\lambda + 3)^2 = -4$
 $\lambda + 3 = \pm 2i$
 $\lambda = -3 \pm 2i$
 C.F.: $y = e^{-3x} (A \cos 2x + B \sin 2x)$
 • PARTICULAR INTEGRAL
 $y = Px^2 + Qx + R$
 $\frac{dy}{dx} = 2Px + Q$
 $\frac{d^2 y}{dx^2} = 2P$
 Try by substituting into the O.D.E.
 $2P + 6(2Px + Q) + 13(Px^2 + Qx + R) = 13x^2 - x + 22$
 $13Px^2 + (12P + 13Q)x + (2P + 6Q + 13R) = 13x^2 - x + 22$
 $13P = 13 \Rightarrow P = 1$
 $12P + 13Q = -1 \Rightarrow 12 + 13Q = -1 \Rightarrow 13Q = -13 \Rightarrow Q = -1$
 $2P + 6Q + 13R = 22 \Rightarrow 2 - 6 + 13R = 22 \Rightarrow 13R = 26 \Rightarrow R = 2$
 $\therefore y = e^{-3x} (A \cos 2x + B \sin 2x) + x^2 - x + 2$

Question 3 (**)

Find a solution of the differential equation

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = 10 \sin x,$$

subject to the boundary conditions $y = 6$ and $\frac{dy}{dx} = 5$ at $x = 0$.

$$\boxed{}, \quad y = 2e^x + e^{2x} + 3\cos x + \sin x$$

Handwritten Solution:

Auxiliary Equation
 $\lambda^2 - 3\lambda + 2 = 0$
 $(\lambda - 2)(\lambda - 1) = 0$
 $\lambda = 1, 2$
Complementary Function
 $y = Ae^x + Be^{2x}$

Particular Solution by Inspection
 $y = P\cos x + Q\sin x$
 $y' = -P\sin x + Q\cos x$
 $y'' = -P\cos x - Q\sin x$

Substitute into the O.D.E
 $\Rightarrow (-P\cos x - Q\sin x) - 3(-P\sin x + Q\cos x) + 2(P\cos x + Q\sin x) = 10\sin x$
 $\Rightarrow \begin{cases} -P\cos x - Q\sin x \\ -3P\sin x + 3Q\cos x \\ +2P\cos x + 2Q\sin x \end{cases} = 10\sin x$
 $\Rightarrow (P - 3Q)\cos x + (3P + Q)\sin x = 10\sin x$

$\begin{cases} P - 3Q = 0 \\ 3P + Q = 10 \end{cases}$
 $P = 3Q$
 $3(3Q) + Q = 10$
 $10Q = 10$
 $Q = 1$
 $P = 3$

Particular Solution
 $y = 3\cos x + \sin x$

General Solution
 $y = Ae^x + Be^{2x} + 3\cos x + \sin x$

Differentiate to get 2 d Apply conditions
 $\frac{dy}{dx} = Ae^x + 2Be^{2x} - 3\sin x + \cos x$

$x=0, y=6 \Rightarrow \begin{cases} 6 = A + B + 3 \\ A + B = 3 \end{cases}$
 $x=0, \frac{dy}{dx}=5 \Rightarrow \begin{cases} 5 = A + 2B + 1 \\ A + 2B = 4 \end{cases}$
 $\therefore B = 1, A = 2$

Finalised full solution
 $y = 2e^x + e^{2x} + 3\cos x + \sin x$

Question 4 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^x.$$

$$\boxed{}, \quad y = (A + 2x)e^x + Be^{-2x}$$

START WITH AN AUXILIARY EQUATION
 $\lambda^2 + \lambda - 2 = 0$
 $(\lambda - 1)(\lambda + 2) = 0$
 $\lambda = -2$

COMPLEMENTARY SOLUTION
 $y = Ae^x + Be^{-2x}$

NOW FOR PARTICULAR INTEGRAL WE TRY $y = Pe^x$ AS e^x IS ALREADY PART OF THE COMPLEMENTARY FUNCTION
 $y = Pe^x$
 $\frac{dy}{dx} = P e^x + B e^{-2x} = P(1+x)e^x$
 $\frac{d^2y}{dx^2} = P e^x + B(-2)e^{-2x} = P e^x - 2B e^{-2x}$
 $P e^x - 2B e^{-2x} - 2(P e^x + B e^{-2x}) = 6e^x$
 $P e^x - 2B e^{-2x} - 2P e^x - 2B e^{-2x} = 6e^x$
 $-P e^x - 4B e^{-2x} = 6e^x$
 $-P = 6$
 $P = -6$

SUBSTITUTE INTO THE D.E.
 $P e^x + P(1+x)e^x - 2(P e^x + B e^{-2x}) = 6e^x$
 $P[2 + 1 + x - 2] = 6$
 $P x = 6$
 $P = 6$

GENERAL SOLUTION IS
 $y = Ae^x + Be^{-2x} + 6e^x$

Question 5 (**)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x,$$

subject to the boundary conditions $y = 1$ and $\frac{dy}{dx} = -5$ at $x = 0$.

$$y = 3\cos 2x - \sin 2x - e^{2x} - e^x$$

Handwritten solution for Question 5:

Given: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x$, $y = 1$, $\frac{dy}{dx} = -5$ at $x = 0$.

Homogeneous Solution:

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 2, 1$$

C.F. $y = Ae^{2x} + Be^x$

Particular Solution:

Try $y = P\cos 2x + Q\sin 2x$

$$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$$

$$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$$

Sub into the O.D.E.

$$-4P\cos 2x - 4Q\sin 2x - 3(-2P\sin 2x + 2Q\cos 2x) + 2(P\cos 2x + Q\sin 2x) = 20\sin 2x$$

$$(-4P - 6Q)\cos 2x + (6P - 4Q)\sin 2x = 20\sin 2x$$

$$\begin{cases} -4P - 6Q = 0 \\ 6P - 4Q = 20 \end{cases} \Rightarrow \begin{cases} P = -3Q \\ 6(-3Q) - 4Q = 20 \end{cases}$$

$$\Rightarrow -20Q = 20 \Rightarrow Q = -1, P = 3$$

$\therefore y = Ae^{2x} + Be^x + 3\cos 2x - \sin 2x$

Initial conditions:

$$x=0, y=1 \Rightarrow 1 = A + B + 3$$

$$x=0, \frac{dy}{dx} = -5 \Rightarrow -5 = 2A + B - 2$$

Subtract:

$$-6 = A - 5 \Rightarrow A = -1$$

$$B = -1$$

$\therefore y = 3\cos 2x - \sin 2x - e^{2x} - e^x$

Question 6 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12(e^{2x} - e^{-2x}).$$

$$y = (A + 4x)e^{2x} + Be^{-x} - 3e^{-2x}$$

Handwritten solution for Question 6:

Given: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12(e^{2x} - e^{-2x})$

Homogeneous Solution:

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2, -1$$

C.F. $y = Ae^{2x} + Be^{-x}$

Particular Solution:

Try $y = P e^{2x} + Q e^{-x} + R e^{-2x}$

$$\frac{dy}{dx} = 2P e^{2x} - Q e^{-x} - 2R e^{-2x}$$

$$\frac{d^2y}{dx^2} = 4P e^{2x} + Q e^{-x} - 4R e^{-2x}$$

Sub into the O.D.E.

$$4P e^{2x} + Q e^{-x} - 4R e^{-2x} - (2P e^{2x} - Q e^{-x} - 2R e^{-2x}) - 2(P e^{2x} + Q e^{-x} + R e^{-2x}) = 12(e^{2x} - e^{-2x})$$

$$(4P - 2P - 2P)e^{2x} + (Q + Q - 2Q)e^{-x} + (-4R + 2R - 2R)e^{-2x} = 12e^{2x} - 12e^{-2x}$$

$$0e^{2x} + 0e^{-x} - 4R e^{-2x} = 12e^{2x} - 12e^{-2x}$$

$$\begin{cases} 0 = 12 \\ -4R = -12 \end{cases} \Rightarrow \begin{cases} P = 3 \\ R = 3 \end{cases}$$

$\therefore y = 3e^{2x} + Be^{-x} - 3e^{-2x}$

Initial conditions:

$$y = (A + 4x)e^{2x} + Be^{-x} - 3e^{-2x}$$

$$y' = (A + 4x)2e^{2x} + Be^{-x} - 3e^{-2x}$$

Question 7 (**)

$$\frac{d^2y}{dx^2} + y = \sin 2x, \quad \text{with } y=0, \quad \frac{dy}{dx}=0 \quad \text{at } x=\frac{\pi}{2}.$$

Show that a solution of the above differential equation is

$$y = \frac{2}{3} \cos x (1 - \sin x).$$

proof

Handwritten solution for Question 7:

The differential equation is $\frac{d^2y}{dx^2} + y = \sin 2x$.

Homogeneous equation:

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

\therefore C.F. $y = A \cos x + B \sin x$

Particular integral:

Try $y = P \sin 2x + Q \cos 2x$ (since $\frac{dy}{dx}$ is missing)

$$\frac{dy}{dx} = 2P \cos 2x - 2Q \sin 2x$$

$$\frac{d^2y}{dx^2} = -4P \sin 2x - 4Q \cos 2x$$

Sub into the O.D.E.

$$-4P \sin 2x - 4Q \cos 2x + P \sin 2x + Q \cos 2x = \sin 2x$$

$$-3P \sin 2x - 3Q \cos 2x = \sin 2x$$

$$P = -\frac{1}{3}$$

\therefore particular solution is

$$y = A \cos x + B \sin x - \frac{1}{3} \sin 2x$$

Apply conditions:

At $x = \frac{\pi}{2}$, $y = 0$ and $\frac{dy}{dx} = 0$

1. $y = 0$ at $x = \frac{\pi}{2}$

$$0 = A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2} - \frac{1}{3} \sin \pi$$

$$0 = 0 + B - 0$$

$$B = 0$$

2. $\frac{dy}{dx} = 0$ at $x = \frac{\pi}{2}$

$$0 = -A \sin \frac{\pi}{2} - \frac{2}{3} \cos \pi$$

$$0 = -A - \frac{2}{3}$$

$$A = -\frac{2}{3}$$

\therefore General solution is

$$y = -\frac{2}{3} \cos x + 0 \sin x - \frac{1}{3} \sin 2x$$

$$y = -\frac{2}{3} \cos x - \frac{1}{3} \sin 2x$$

$$y = -\frac{2}{3} \cos x (1 - \sin x)$$

Referring to the question, the solution is

$$y = \frac{2}{3} \cos x (1 - \sin x)$$

Question 8 (**+)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^{-2x},$$

with $y = 3$ and $\frac{dy}{dx} = -2$ at $x = 0$.

Show that the solution of the above differential equation is

$$y = 2e^x + (1 - 2x)e^{-2x}.$$

, proof

Assume solution for the O.D.E is

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 1, -2$$

∴ complementary function

$$y = Ae^x + Be^{-2x}$$

As the R.H.S contains $6e^{-2x}$ which is part of the solution

For the particular integral we try

$$y = Pe^{-2x}$$

$$\frac{dy}{dx} = -2Pe^{-2x}$$

$$\frac{d^2y}{dx^2} = 4Pe^{-2x}$$

$$4Pe^{-2x} - 2Pe^{-2x} - 4Pe^{-2x} = 6e^{-2x}$$

$$-2Pe^{-2x} = 6e^{-2x}$$

$$P = -2$$

∴ particular integral is

$$y = -2e^{-2x}$$

∴ general solution is

$$y = Ae^x + Be^{-2x} - 2e^{-2x}$$

Differentiate and apply conditions

$$y = Ae^x + Be^{-2x} - 2e^{-2x}$$

$$\frac{dy}{dx} = Ae^x - 2Be^{-2x} + 4e^{-2x}$$

- $x=0, y=3 \Rightarrow 3 = A + B - 2$
- $x=0, \frac{dy}{dx} = -2 \Rightarrow -2 = A - 2B + 4$

$$A + B = 5$$

$$A - 2B = -6$$

$$3B = 11 \Rightarrow B = \frac{11}{3}$$

$$A = 5 - \frac{11}{3} = \frac{4}{3}$$

Finally we have

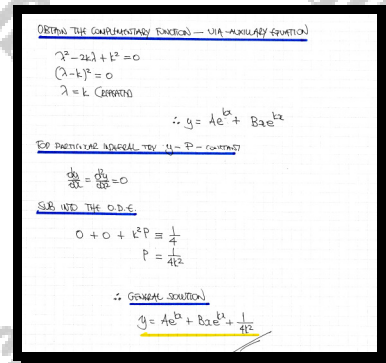
$$y = \frac{4}{3}e^x + \frac{11}{3}e^{-2x} - 2e^{-2x}$$

Question 9 (**+)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = \frac{1}{4}, \quad k > 0.$$

$$\boxed{}, \quad y = Ae^{kx} + Bxe^{kx} + \frac{1}{4k^2}$$



(OBTAIN THE COMPLEMENTARY FUNCTION) — VIA-AUXILIARY EQUATION
 $\lambda^2 - 2k\lambda + k^2 = 0$
 $(\lambda - k)^2 = 0$
 $\lambda = k$ (REPEATED)
 $\therefore y = Ae^{kx} + Bxe^{kx}$
 IF PARTICULAR INTEGRAL TRY $y = P$ — CONSTANT
 $\frac{dy}{dx} = \frac{dP}{dx} = 0$
 SUB INTO THE O.D.E.
 $0 + 0 + k^2 P = \frac{1}{4}$
 $P = \frac{1}{4k^2}$
 \therefore GENERAL SOLUTION
 $y = Ae^{kx} + Bxe^{kx} + \frac{1}{4k^2}$

Question 10 (**+)

Find the solution of the differential equation

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} = 2x + 3,$$

subject to the conditions $y = 2$, $\frac{dy}{dx} = -5$ at $x = 0$.

$$y = x^2 + x - 4 + 6e^{-x}$$

Handwritten solution for the differential equation problem:

Given: $\frac{d^2 y}{dx^2} + \frac{dy}{dx} = 2x + 3$

Homogeneous equation:

$$\lambda^2 + \lambda = 0$$

$$\lambda(\lambda + 1) = 0$$

$$\lambda = 0, -1$$

\therefore C.F. : $y = A + Be^{-x}$

Particular solution:

Try $y = Bx^2 + Cx$ (Suggested by the form of the R.H.S.)

$$\frac{dy}{dx} = 2Bx + C$$

$$\frac{d^2 y}{dx^2} = 2B$$

Sub into the O.D.E.

$$2B + (2Bx + C) = 2x + 3$$

$$2Bx + (2B + C) = 2x + 3$$

$2B = 2 \Rightarrow B = 1$
 $2B + C = 3 \Rightarrow 2 + C = 3 \Rightarrow C = 1$

\therefore G.P. solution: $y = A + Be^{-x} + x^2 + x$

Apply conditions:

At $x = 0$, $y = 2 \Rightarrow 2 = A + B$

At $x = 0$, $\frac{dy}{dx} = -5 \Rightarrow -5 = -B + 1$

$B = 6$
 $A = -4$

$\therefore y = 6e^{-x} + x^2 + x - 4$

Question 11 (**+)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34\cos 2x,$$

subject to the boundary conditions $y = 18$ and $\frac{dy}{dx} = 0$ at $x = 0$.

$$y = 2(8e^{-x} + 1)\cos 2x + 8\sin 2x$$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34\cos 2x$
 $\lambda^2 + 2\lambda + 5 = 0$
 $(\lambda + 1)^2 - 1 + 5 = 0$
 $(\lambda + 1)^2 = -4$
 $\lambda + 1 = \pm 2i$
 $\lambda = -1 \pm 2i$
 CF: $y = e^{-x}(A\cos 2x + B\sin 2x)$

$y = P\cos 2x + Q\sin 2x$
 $\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$
 $\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$
 $-4P\cos 2x - 4Q\sin 2x + 2(-2P\sin 2x + 2Q\cos 2x) + 5(P\cos 2x + Q\sin 2x) = 34\cos 2x$
 $-4P\cos 2x - 4Q\sin 2x - 4P\sin 2x + 4Q\cos 2x + 5P\cos 2x + 5Q\sin 2x = 34\cos 2x$
 $(-4P + 4Q + 5P)\cos 2x + (-4Q - 4P + 5Q)\sin 2x = 34\cos 2x$
 $(P + 4Q)\cos 2x + (-4P - Q)\sin 2x = 34\cos 2x$
 $P + 4Q = 34$
 $-4P - Q = 0$
 $4P + Q = 0$
 $5P = 34$
 $P = \frac{34}{5} = 6.8$
 $Q = -4P = -27.2$

$\therefore y = e^{-x}(A\cos 2x + B\sin 2x) + 6.8\cos 2x - 27.2\sin 2x$
 $\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) - 13.6\sin 2x + 54.4\cos 2x$
 $2=0 \quad y=18, \quad A+2=18 \quad \Rightarrow \quad A=16$
 $0=0 \quad \frac{dy}{dx}=0, \quad -A+2B=0 \quad \Rightarrow \quad -16+2B=0 \quad \Rightarrow \quad 2B=16 \quad \Rightarrow \quad B=8$
 $\therefore y = e^{-x}(16\cos 2x + 8\sin 2x) + 6.8\cos 2x - 27.2\sin 2x$
 $y = 2(8e^{-x} + 1)\cos 2x + 8\sin 2x$

Question 12 (**+)

The curve C has a local minimum at the origin and satisfies the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2.$$

Find an equation for C .

$$y = e^x (\sin 2x + \cos 2x) + (2x-1)^2$$

Handwritten solution for Question 12:

Given differential equation: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2$

Part 1: Homogeneous Solution

Assume $y = e^{\lambda x}$

$$\lambda^2 + 4\lambda + 8 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 32}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i$$

Therefore, the homogeneous solution is:

$$y_h = e^{-2x} (A \cos 2x + B \sin 2x)$$

Part 2: Particular Solution

Assume $y = Px^2 + Qx + R$

$$\frac{dy}{dx} = 2Px + Q$$

$$\frac{d^2y}{dx^2} = 2P$$

Substitute into the differential equation:

$$2P + 4(2Px + Q) + 8(Px^2 + Qx + R) = 32x^2$$

$$2P + 8Px + 4Q + 8Px^2 + 8Qx + 8R = 32x^2$$

$$8Px^2 + (8Q + 8P)x + (2P + 4Q + 8R) = 32x^2$$

Equating coefficients:

$$8P = 32 \Rightarrow P = 4$$

$$8Q + 8P = 0 \Rightarrow 8Q + 32 = 0 \Rightarrow Q = -4$$

$$2P + 4Q + 8R = 0 \Rightarrow 8 - 16 + 8R = 0 \Rightarrow 8R = 8 \Rightarrow R = 1$$

Therefore, the particular solution is:

$$y_p = 4x^2 - 4x + 1$$

General Solution

$$y = e^{-2x} (A \cos 2x + B \sin 2x) + 4x^2 - 4x + 1$$

Initial Conditions

The curve has a local minimum at the origin $(0, 0)$.

At $x = 0$, $y = 0$:

$$0 = e^0 (A \cos 0 + B \sin 0) + 4(0)^2 - 4(0) + 1$$

$$0 = A + 1 \Rightarrow A = -1$$

At $x = 0$, $\frac{dy}{dx} = 0$ (local minimum):

$$0 = -2e^0 (A \sin 0 + B \cos 0) + 8(0) - 4$$

$$0 = -2B - 4 \Rightarrow -2B = 4 \Rightarrow B = -2$$

Final Equation for C

$$y = e^{-2x} (-\cos 2x - 2 \sin 2x) + 4x^2 - 4x + 1$$

Question 13 (**+)

$$\frac{d^2x}{dt^2} + 9x + 12 \sin 3t = 0, \quad t \geq 0,$$

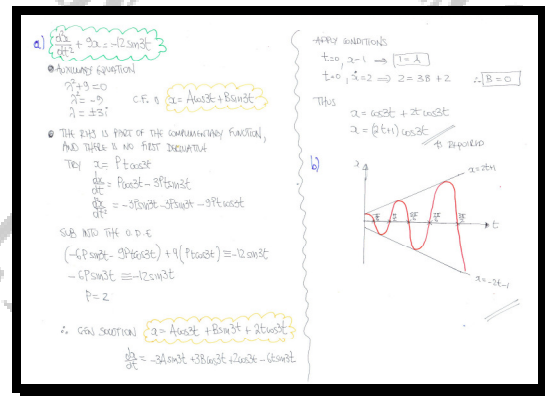
with $x = 1, \frac{dx}{dt} = 2$ at $t = 0$.

a) Show that a solution of the differential equation is

$$x = (2t + 1) \cos 3t.$$

b) Sketch the graph of x .

proof



Question 14 (**+)

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = 16 + 32e^{2x},$$

with $y = 8$ and $\frac{dy}{dx} = 0$ at $x = 0$.

Show that the solution of the above differential equation is

$$y = 8 \cosh^2 x.$$

proof

Handwritten solution for Question 14:

Given equation: $\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = 16 + 32e^{2x}$

Step 1: Auxiliary equation
 $\lambda^2 + 4\lambda + 4 = 0$
 $(\lambda + 2)^2 = 0$
 $\lambda = -2$ (Repeated)

Step 2: Particular integral
 Try $y = P + Qe^{2x}$
 $\frac{dy}{dx} = 2Qe^{2x}$
 $\frac{d^2 y}{dx^2} = 4Qe^{2x}$
 $4Qe^{2x} + 4(2Qe^{2x}) + 4(P + Qe^{2x}) = 16 + 32e^{2x}$
 $4P + 16Qe^{2x} = 16 + 32e^{2x}$
 $4P = 16$
 $P = 4$
 $16Q = 32$
 $Q = 2$

Step 3: General solution
 $y = 4 + 2e^{2x} + 2e^{-2x}$

Step 4: Apply boundary conditions
 At $x = 0$, $y = 8$
 $8 = 4 + 2 + 2$
 $8 = 8$ (Satisfied)

Step 5: Derivative condition
 At $x = 0$, $\frac{dy}{dx} = 0$
 $0 = 4 - 4$
 $0 = 0$ (Satisfied)

Step 6: Final solution
 $y = 4 + 2e^{2x} + 2e^{-2x}$
 $y = 4(1 + e^{2x} + e^{-2x})$
 $y = 4(2 \cosh^2 x)$
 $y = 8 \cosh^2 x$

Question 15 (**+)

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = 12x e^{kx}, \quad k > 0$$

a) Find a general solution of the differential equation given that $y = Px^3 e^{kx}$, where P is a constant, is part of the solution.

b) Given further that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$ show that

$$y = e^{kx} (2x^3 - kx + 1).$$

$$y = e^{kx} (2x^3 + Ax + B)$$

(a) $\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = 0$
 Homogeneous equation
 $\lambda^2 - 2k\lambda + k^2 = 0$
 $(\lambda - k)^2 = 0$
 $\lambda = k$ (repeated)
 For particular integral try
 $y = P_1 x^3 e^{kx}$
 $\frac{dy}{dx} = 3P_1 x^2 e^{kx} + P_1 k x^3 e^{kx}$
 $\frac{d^2y}{dx^2} = 6P_1 x e^{kx} + 3P_1 k x^2 e^{kx} + P_1 k^2 x^3 e^{kx}$
 Sub into the ODE
 $\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = 12x e^{kx}$
 $6P_1 x e^{kx} + 3P_1 k x^2 e^{kx} + P_1 k^2 x^3 e^{kx} - 2k(3P_1 x^2 e^{kx} + P_1 k x^3 e^{kx}) + k^2 P_1 x^3 e^{kx} = 12x e^{kx}$
 $6P_1 x e^{kx} + 3P_1 k x^2 e^{kx} + P_1 k^2 x^3 e^{kx} - 6P_1 k x^2 e^{kx} - 2P_1 k^2 x^3 e^{kx} + P_1 k^2 x^3 e^{kx} = 12x e^{kx}$
 $6P_1 x e^{kx} = 12x e^{kx}$
 $6P_1 = 12$
 $P_1 = 2$
 \therefore Gen solution $y = A e^{kx} + B e^{kx} + 2x^3 e^{kx}$
 $y = e^{kx} (A + B + 2x^3)$

(b) $\frac{dy}{dx} = k e^{kx} (A + B + 2x^3) + e^{kx} (B + 6x^2)$
 $x=0 \quad y=1 \Rightarrow \boxed{A+B=1}$
 $x=0 \quad \frac{dy}{dx}=0 \Rightarrow 0 = k(A+B) + B \quad \therefore \boxed{B=-k}$
 $\therefore y = e^{kx} (1 - k + 2x^3)$

Question 16 (**+)

Show that the solution of the differential equation

$$\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 16y = 24e^{4x},$$

subject to the boundary conditions $y = -1$, $\frac{dy}{dx} = -4$ at $x = 0$, can be written as

$$y = (12x^2 - 1)e^{4x}.$$

proof

Handwritten solution for Question 16:

The differential equation is $\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 16y = 24e^{4x}$.

Step 1: Homogeneous equation
 $\lambda^2 - 8\lambda + 16 = 0$
 $(\lambda - 4)^2 = 0$
 $\lambda = 4$ (repeated root)

Step 2: Particular solution
 Assume $y = Ae^{4x}$.
 $\frac{dy}{dx} = 4Ae^{4x}$
 $\frac{d^2 y}{dx^2} = 16Ae^{4x}$
 Substitute into the equation:
 $16Ae^{4x} - 8(4Ae^{4x}) + 16(Ae^{4x}) = 24e^{4x}$
 $16A - 32A + 16A = 24$
 $0 = 24$ (This is incorrect, so assume $y = (Ax^2 + Bx + C)e^{4x}$)

Step 3: Particular solution (corrected)
 Assume $y = (Ax^2 + Bx + C)e^{4x}$.
 $\frac{dy}{dx} = (2Ax + B + 4(Ax^2 + Bx + C))e^{4x}$
 $\frac{d^2 y}{dx^2} = (2A + 4(2Ax + B + 4(Ax^2 + Bx + C)))e^{4x}$
 $\frac{d^2 y}{dx^2} = (8Ax^2 + (12A + 8B)x + (2A + 4B + 16C))e^{4x}$
 Substitute into the equation:
 $(8Ax^2 + (12A + 8B)x + (2A + 4B + 16C))e^{4x} - 8(2Ax + B + 4(Ax^2 + Bx + C))e^{4x} + 16(Ax^2 + Bx + C)e^{4x} = 24e^{4x}$
 $(8Ax^2 + (12A + 8B)x + (2A + 4B + 16C))e^{4x} - (16Ax^2 + (8A + 8B)x + (8A + 8B + 32C))e^{4x} + (16Ax^2 + 16Bx + 16C)e^{4x} = 24e^{4x}$
 $(8A - 16A + 16A)x^2 + ((12A + 8B) - (8A + 8B) + 16B)x + ((2A + 4B + 16C) - (8A + 8B + 32C) + 16C) = 24$
 $8Ax^2 + (4A + 16B)x + (-6A + 8B + 0C) = 24$
 Equate coefficients:
 $8A = 0 \Rightarrow A = 0$
 $4A + 16B = 0 \Rightarrow 16B = 0 \Rightarrow B = 0$
 $-6A + 8B + 0C = 24 \Rightarrow 0C = 24 \Rightarrow C = 3$
 So the particular solution is $y = 3e^{4x}$.

Step 4: General solution
 $y = (Ax^2 + Bx + C)e^{4x} + 3e^{4x}$
 $y = (Ax^2 + Bx + C + 3)e^{4x}$

Step 5: Apply boundary conditions
 At $x = 0$, $y = -1$:
 $(A(0)^2 + B(0) + C + 3)e^{0} = -1$
 $C + 3 = -1 \Rightarrow C = -4$
 At $x = 0$, $\frac{dy}{dx} = -4$:
 $\frac{dy}{dx} = (2Ax + B + 4(Ax^2 + Bx + C + 3))e^{4x}$
 At $x = 0$, $\frac{dy}{dx} = (B + 12)e^{0} = -4$
 $B + 12 = -4 \Rightarrow B = -16$

Step 6: Final solution
 $y = (Ax^2 - 16x - 4 + 3)e^{4x}$
 $y = (Ax^2 - 16x - 1)e^{4x}$
 To find A , use the differential equation:
 $\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 16y = 24e^{4x}$
 $\frac{d^2 y}{dx^2} = (2A - 16 + 16(Ax^2 - 16x - 1))e^{4x}$
 $\frac{d^2 y}{dx^2} = (16Ax^2 - 32Ax + (-16 + 16A - 16))e^{4x}$
 $\frac{d^2 y}{dx^2} = (16Ax^2 - 32Ax + (16A - 32))e^{4x}$
 $\frac{dy}{dx} = (2Ax - 16 + 4(Ax^2 - 16x - 1))e^{4x}$
 $\frac{dy}{dx} = (4Ax^2 + (2A - 64)x + (-16 - 4))e^{4x}$
 $\frac{dy}{dx} = (4Ax^2 + (2A - 64)x - 20)e^{4x}$
 At $x = 0$, $\frac{dy}{dx} = -20 = -4$ (This is incorrect, so assume $y = (Ax^2 + Bx + C)e^{4x}$)

Step 7: Final solution (corrected)
 $y = (12x^2 - 1)e^{4x}$

Question 17 (***)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}.$$

- a) Find a solution of the differential equation given that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$.
- b) Sketch the graph of y .

The sketch must include ...

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any stationary points of the curve.
- clear indications of how the graph looks for large positive or negative values of x .

$$y = e^{3x}(2x^2 - 3x + 1)$$

Q17 $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$

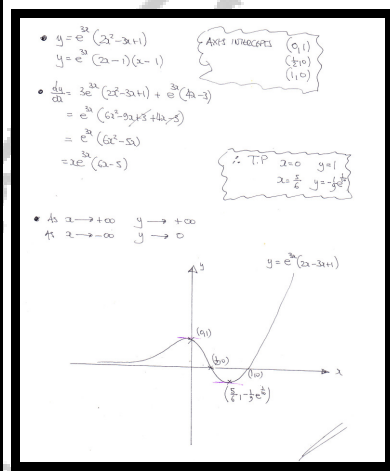
Assume $y = e^{\lambda x}$
 $\lambda^2 - 6\lambda + 9 = 0$
 $(\lambda - 3)^2 = 0$
 $\lambda = 3$ (repeated)

For particular integral try $y = Pxe^{3x}$
 $\frac{dy}{dx} = 2Pe^{3x} + 3Pe^{3x}$
 $\frac{d^2y}{dx^2} = 2Pe^{3x} + 6Pe^{3x} + 9Pe^{3x} = 17Pe^{3x}$

Sub into diff eqn
 $17Pe^{3x} - 6(2Pe^{3x} + 3Pe^{3x}) + 9Pe^{3x} = 4e^{3x}$
 $17P - 12P - 18P + 9P = 4$
 $-12P = 4$
 $P = -\frac{1}{3}$

$\therefore y = -\frac{1}{3}e^{3x} + 3xe^{3x} + 2x^2e^{3x}$
 $y = e^{3x}(2x^2 - 3x + 1)$
 $\frac{dy}{dx} = 3e^{3x}(2x^2 - 3x + 1) + e^{3x}(4x - 3)$
 $\frac{dy}{dx} = e^{3x}(6x^2 - 9x + 3 + 4x - 3)$
 $\frac{dy}{dx} = e^{3x}(6x^2 - 5x)$

• $x=0, y=1 \Rightarrow 1 = 1$
 • $x=0, \frac{dy}{dx}=0 \Rightarrow 0 = 0$
 $y = e^{3x}(2x^2 - 3x + 1)$



Question 18 (***)

The curve with equation $y = f(x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8\sin 2x.$$

The first two non zero terms in Maclaurin series expansion of $f(x)$ are $x + kx^2$, where k is a constant.

Determine in any order the value of k and the exact value of $f\left(\frac{1}{4}\pi\right)$.

$$\boxed{}, \quad k = 2, \quad f\left(\frac{1}{4}\pi\right) = \frac{1}{2}(3\pi - 4)e^{\frac{1}{2}\pi}$$

START USING THE DIFFERENTIAL EQUATION

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2$$

$\therefore C.F. = (A + Bx)e^{2x}$

PARTICULAR INTEGRAL BY INSPECTION OR BY OPERATOR

TRY $y = P\cos 2x + Q\sin 2x$

$$y' = 2P(-\sin 2x) + 2Q\cos 2x$$

$$y'' = -4P\cos 2x - 4Q\sin 2x$$

By inspection

$$y = \frac{1}{2} \int \frac{8\cos 2x}{2^2 - 4} = \frac{1}{2} \int \frac{8\cos 2x}{-4} = -\frac{2}{1} \int \cos 2x = -\frac{1}{2} \sin 2x$$

Now we have

$$y' = 2Be^{2x} + 2(A+Bx)e^{2x} - 2\sin 2x$$

$$y'' = 4Be^{2x} + 4(A+Bx)e^{2x} - 4\cos 2x$$

And substituting at $x=0$

$$y_0 = A + 1, \quad y'_0 = 2A + 8, \quad y''_0 = 4B + 4A - 4$$

NOW THE MACLAURIN SERIES EXPANSION OF THE O.D.E IS

$$y = y_0 + 2y'_0x + \frac{1}{2}y''_0x^2 + \dots$$

$$y = (A+1) + (2A+8)x + \frac{1}{2}(4A+4B-4)x^2$$

$$y = 0 + x + 2x^2$$

COMPARING COEFFICIENTS

- $A+1=0 \Rightarrow A=-1$
- $2A+8=1 \Rightarrow -2+8=1 \Rightarrow 6=1 \Rightarrow \text{No solution}$
- $4A+4B-4=2 \Rightarrow -4+4B-4=2 \Rightarrow 4B=10 \Rightarrow B=2.5$

FINALLY WE HAVE

$$y = f(x) = (3x-1)e^{2x} + \cos 2x$$

$$f\left(\frac{1}{4}\pi\right) = \left(\frac{3}{4}\pi - 1\right)e^{\frac{1}{2}\pi} + \cos \frac{\pi}{2}$$

$$f\left(\frac{1}{4}\pi\right) = \left(\frac{3}{4}\pi - 1\right)e^{\frac{1}{2}\pi}$$

ALTERNATIVE METHOD: STANDARD EXPANSIONS

$$y = (A+Bx)^2 + \cos 2x = (A+Bx)(1+2x+2^2x^2+\dots) + (1-2^2x^2+\dots)$$

$$= A + 2Ax + 2Ax^2 + Bx + 2Bx^2 + \dots$$

$$= (A+1) + (2A+B)x + (2A+2B-2)x^2 + \dots$$

- $A+1=0 \Rightarrow A=-1$
- $2A+B=1 \Rightarrow -2+B=1 \Rightarrow B=3$
- $2A+2B-2=2 \Rightarrow -2+6-2=2 \Rightarrow 2=2$

etc etc etc

Question 19 (*)**

The function $y = f(x)$ satisfies the following differential equation

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 5y = 2e^{-x}(\sin 2x - 2\cos 2x),$$

subject to the boundary conditions $y = 0$, $\frac{dy}{dx} = 2$ at $x = 0$.

Solve the differential equation to show that

$$y = \cosh x \sinh 2x.$$

No credit will be given for verification methods.

, proof

FIND THE COMPLEMENTARY FUNCTION FIRST

$$\begin{aligned} \lambda^2 - 2\lambda + 5 &= 0 \\ (\lambda - 1)^2 - 1 + 5 &= 0 \\ (\lambda - 1)^2 &= -4 \\ \lambda - 1 &= \pm 2i \\ \lambda &= 1 \pm 2i \end{aligned}$$

∴ COMPLEMENTARY SOLUTION

$$y = e^x (A \cos 2x + B \sin 2x)$$

FOR PARTICULAR INTEGRAL

- $y = e^x (P \cos 2x + Q \sin 2x)$
- $\frac{dy}{dx} = e^x (P \cos 2x + Q \sin 2x) + e^x (-2P \sin 2x + 2Q \cos 2x)$
 $= e^x (-P \sin 2x + Q \cos 2x + 2P \cos 2x - 2Q \sin 2x)$
 $= e^x [(P + 2Q) \cos 2x - (P - 2Q) \sin 2x]$
- $\frac{d^2 y}{dx^2} = e^x [(P + 2Q) \cos 2x - (P - 2Q) \sin 2x] + e^x [2(P + 2Q) \sin 2x + 2(P - 2Q) \cos 2x]$
 $= e^x [(P + 2Q + 2P + 4Q) \sin 2x + (-P - 2Q + 2P - 4Q) \cos 2x]$
 $= e^x [(3P + 6Q) \sin 2x - (P + 6Q) \cos 2x]$

SUB INTO THE O.D.E

$$\begin{aligned} \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 5y &= 2e^{-x}(\sin 2x - 2\cos 2x) \\ (3P + 6Q) e^x \sin 2x - (P + 6Q) e^x \cos 2x - 2e^x [(P + 2Q) \cos 2x - (P - 2Q) \sin 2x] + 5e^x (P \cos 2x + Q \sin 2x) &= 2e^{-x}(\sin 2x - 2\cos 2x) \end{aligned}$$

EQUATING COEFFICIENTS W/ COEFF

$$\begin{aligned} 4P + 6Q &= 2 & 3P + 6Q &= 2 \\ -P + 2Q &= -4 & -P + 2Q &= -4 \end{aligned} \Rightarrow Q = 0 \quad P = \frac{1}{2}$$

∴ GENERAL SOLUTION IS

$$y = e^x (A \cos 2x + B \sin 2x) + \frac{1}{2} e^x \sin 2x$$

APPLY CONDITIONS - FIRSTLY $x=0$, $y=0$

$$\begin{aligned} 0 &= A \\ y &= B e^x \sin 2x + \frac{1}{2} e^x \sin 2x \\ y &= (B + \frac{1}{2}) e^x \sin 2x \end{aligned}$$

DIFFERENTIATE & APPLY THE SECOND CONDITION $x=0$, $\frac{dy}{dx}=2$

$$\begin{aligned} \frac{dy}{dx} &= (B + \frac{1}{2}) e^x \sin 2x + (B + \frac{1}{2}) e^x (2 \cos 2x) \\ \therefore 2 &= (B + \frac{1}{2}) \times 2 \\ B &= \frac{1}{2} \end{aligned}$$

∴ $y = (\frac{1}{2} e^x + \frac{1}{2} e^x) \sin 2x$

$$y = \cosh x \sinh 2x$$

As required

2ND ORDER or HIGHER CAUCHY EULER TYPE

Question 1 (**)

Find the general solution of the following differential equation.

$$4t^2 \frac{d^2x}{dt^2} + 4t \frac{dx}{dt} - x = 0.$$

$$x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}$$

$4t^2 \frac{d^2x}{dt^2} + 4t \frac{dx}{dt} - x = 0$
 TRY A SOLUTION OF THE FORM $x = t^n$, WHERE n IS A CONSTANT TO BE FOUND
 $\frac{dx}{dt} = nt^{n-1}$
 $\frac{d^2x}{dt^2} = n(n-1)t^{n-2}$
 SUB INTO THE O.D.E
 $\Rightarrow 4t^2 [n(n-1)t^{n-2}] + 4t [nt^{n-1}] - t^n = 0$
 $\Rightarrow 4n(n-1)t^n + 4nt^n - t^n = 0$
 $\Rightarrow [4n(n-1) + 4n - 1] t^n = 0$
 $\Rightarrow 4n^2 - 4n + 4n - 1 = 0$
 $\Rightarrow (4n^2 - 1) = 0$
 $n = \pm \frac{1}{2}$
 $\therefore x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}$

Question 2 (**+)

Find the general solution of the following differential equation.

$$4t^2 \frac{d^2y}{dt^2} + 4t \frac{dy}{dt} + y = 0.$$

$$y = P \cos[\ln \sqrt{t}] + P \sin[\ln \sqrt{t}]$$

$4t^2 \frac{d^2y}{dt^2} + 4t \frac{dy}{dt} + y = 0$
 USE A TRIAL SOLUTION OF THE FORM $y = t^n$
 $\frac{dy}{dt} = nt^{n-1}$
 $\frac{d^2y}{dt^2} = n(n-1)t^{n-2}$
 SUB INTO THE O.D.E
 $\Rightarrow 4t^2 [n(n-1)t^{n-2}] + 4t [nt^{n-1}] + t^n = 0$
 $\Rightarrow [4n(n-1) + 4n + 1] t^n = 0$
 $\Rightarrow [4n^2 - 4n + 4n + 1] t^n = 0$
 $\Rightarrow 4n^2 + 1 = 0$
 $\Rightarrow n = \pm \frac{1}{2}i$
 $\therefore y = At^{\frac{1}{2}i} + Bt^{-\frac{1}{2}i}$
 $y = A e^{i \ln(t^{\frac{1}{2}})} + B e^{-i \ln(t^{\frac{1}{2}})}$
 $y = A e^{i \ln t^{\frac{1}{2}}} + B e^{-i \ln t^{\frac{1}{2}}}$
 $y = A \cos(\ln t^{\frac{1}{2}}) + A \sin(\ln t^{\frac{1}{2}})$
 $y = B \cos(\ln t^{\frac{1}{2}}) + B \sin(-\ln t^{\frac{1}{2}})$
 $y = (A+B) \cos(\ln t^{\frac{1}{2}}) + [A-B] \sin(\ln t^{\frac{1}{2}})$
 $y = P \cos(\ln \sqrt{t}) + Q \sin(\ln \sqrt{t})$

Question 3 (***)

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = 9x^8.$$

Determine the solution of the above differential equation subject to the boundary conditions

$$y = \frac{3}{2}, \quad \frac{dy}{dx} = 2 \quad \text{at } x = 1.$$

$$\boxed{\frac{1}{4}}, \quad y = \frac{1}{4} x^4 (x^4 + 1) + \frac{1}{x}$$

$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = 9x^8$ with $x=1, y=\frac{3}{2}, \frac{dy}{dx}=2$

● ASSUME A SOLUTION OF THE FORM $y = x^2$
 $y' = 2x^{2-1} = 2x$
 $y'' = 2(2-1)x^{2-2} = 2$

● SUBSTITUTE INTO THE L.H.S OF THE O.D.E (GIVEN R.H.S.)
 $\Rightarrow x^2 [2(2-1)x^{2-2}] - 2x [2x^{2-1}] - 4[x^2] = 0$
 $\Rightarrow 2(2-1)x^2 - 2(2)x^2 - 4x^2 = 0$
 $\Rightarrow [2(2-1) - 2(2) - 4] x^2 = 0$
 $\Rightarrow 2^2 - 2(2) - 4 = 0$
 $\Rightarrow (2-4)(2+1) = 0$
 $\Rightarrow 2 = -1$

∴ COMPLEMENTARY FUNCTION $y = Ax^{-1} + Bx^4$

● PARTIAL DIFFERENTIAL BY INSPECTION
 $y = Px^8$
 $y' = 8Px^7$
 $y'' = 56Px^6$
 $\Rightarrow x^2 [56Px^6] - 2x [8Px^7] - 4Px^8 = 9x^8$
 $\Rightarrow 56Px^8 - 16Px^8 - 4Px^8 = 9x^8$
 $\Rightarrow 36P = 9$
 $\Rightarrow P = \frac{1}{4}$

∴ GENERAL SOLUTION IS
 $y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$

● APPLYING CONDITIONS $x=1, y=\frac{3}{2}, \frac{dy}{dx}=2$
 $y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$
 $\frac{dy}{dx} = -\frac{A}{x^2} + 4Bx^3 + 2x^7$
 $\left. \begin{aligned} \frac{3}{2} &= A + B + \frac{1}{4} \\ 2 &= -A + 4B + 2 \end{aligned} \right\} \Rightarrow \begin{aligned} 2 &= -A + 4B + 2 \\ \frac{3}{2} &= A + B + \frac{1}{4} \end{aligned}$ ADDING
 $\Rightarrow 0 = 5B + \frac{1}{2}$
 $\Rightarrow 5B = -\frac{1}{2}$
 $\Rightarrow B = -\frac{1}{10}$
 $\therefore A = 4B \Rightarrow A = -\frac{2}{5}$

● FINAL WE HAVE A SOLUTION
 $y = \frac{1}{x} + \frac{1}{4}x^8 + \frac{1}{4}x^8$
 $y = \frac{1}{x} + \frac{1}{2}x^8$

Question 4 (***)

Find the general solution of the following differential equation

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - n(n+1)y = 0.$$

$$y = Ax^n + \frac{B}{x^{n+1}}$$

$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - n(n+1)y = 0$
 This is a standard "Euler type" equation
 Let $y = x^\lambda$
 $\frac{dy}{dx} = \lambda x^{\lambda-1}$
 $\frac{d^2 y}{dx^2} = \lambda(\lambda-1)x^{\lambda-2}$
 Sub into the O.D.E.
 $\Rightarrow x^2 \lambda(\lambda-1)x^{\lambda-2} + 2x [\lambda x^{\lambda-1}] - n(n+1)x^\lambda = 0$
 $\Rightarrow \lambda(\lambda-1)x^\lambda + 2\lambda x^\lambda - n(n+1)x^\lambda = 0$
 $\Rightarrow [\lambda(\lambda-1) + 2\lambda - n(n+1)]x^\lambda = 0$
 $\Rightarrow \lambda^2 - \lambda - n^2 - n = 0$
 $\Rightarrow \lambda^2 + \lambda = n^2 + n$
 $\Rightarrow (\lambda + \frac{1}{2})^2 - \frac{1}{4} = n^2 + n$
 $\Rightarrow (\lambda + \frac{1}{2})^2 = n^2 + n + \frac{1}{4}$
 $\Rightarrow (\lambda + \frac{1}{2})^2 = (n + \frac{1}{2})^2$
 $\Rightarrow \lambda + \frac{1}{2} = \pm (n + \frac{1}{2})$
 $\Rightarrow \lambda + \frac{1}{2} = n + \frac{1}{2}$
 $\Rightarrow \lambda = n$
 $\Rightarrow \lambda = -n-1$
 \therefore Gen solution $y = Ax^n + Bx^{-n-1}$
 $y = Ax^n + \frac{B}{x^{n+1}}$

Question 5 (***)

Given that if $x = e^t$ and $y = f(x)$, show clearly that ...

a) ... $x \frac{dy}{dx} = \frac{dy}{dt}$.

b) ... $x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$.

The following differential equation is to be solved

$$x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2 \ln x$$

subject to the boundary conditions $y = \frac{1}{2}, \frac{dy}{dx} = \frac{3}{2}$ at $x = 1$.

c) Use the substitution $x = e^t$ to solve the above differential equation.

$$y = \frac{1}{2} + \frac{1}{2}(2x^2 + 1)\ln x$$

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t = 2\sqrt{x}$$

a) DEPENDENT WITH RESPECT TO y

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \alpha \cdot \frac{dt}{dx}$$

$$2 \frac{dy}{dx} = \frac{dy}{dt} \quad // \text{ mit } t = \sqrt{x}$$

b) DEPENDENT ON α WRT α

$$\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$$

$$1 \cdot \frac{dy}{dx} + 2 \cdot \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} + 2 \cdot \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} + 2 \cdot \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{2\sqrt{x}}$$

$$3 \cdot \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{6\sqrt{x}}$$

4. BEWERTUNG

c) $y = 2x^2 - 3x \frac{dy}{dx} + 4y = 2 \ln(x^2)$

$$\left(\frac{dy}{dx} - \frac{3}{2} \frac{dy}{dx} \right) \cdot \frac{1}{2} \frac{dy}{dx} + 4y = 2 \ln(x^2)$$

$$\frac{dy}{dx} - \frac{1}{2} \frac{dy}{dx} + 4y = 2 \ln(x^2)$$

• AUXILIARY EQUATION
 $\lambda^2 - 4\lambda + 1 = 0$
 $(\lambda - 2)^2 = 0$
 $\lambda = 2$
 $y = A e^{2x} + B e^{2x}$

• PARTICULAR INTEGRAL TRY $y = P e^t + Q$
 $y = P$
 $y = Q$
 SUBSTITUTION IN ODE:
 $0 = 2P + 4(0 - P) = 2P - 4P = -2P$
 $4P = 4(0 - P) = -4P$
 $P = 0 \quad Q = \frac{1}{2}$

• GEN. SOLUTION $y = A e^{2x} + B e^{2x} + \frac{1}{2} + \frac{1}{2}$

$$\frac{dy}{dx} = 2A e^{2x} + 2B e^{2x} + \frac{1}{2} + \frac{1}{2}$$

$$y = A e^{2x} + B e^{2x} + \frac{1}{2} + \frac{1}{2}$$

• WRT $x = 1, y = \frac{1}{2} \Rightarrow \frac{1}{2} = A + \frac{1}{2} \Rightarrow A = 0$

$$\frac{dy}{dx} = 2A x + 2B(2x) + 2A + \frac{1}{2}$$

$$y = 1, \frac{dy}{dx} = \frac{1}{2} \Rightarrow \frac{1}{2} = 2A + \frac{1}{2} + \frac{1}{2}$$

$$\frac{1}{2} = 2A + \frac{1}{2} + \frac{1}{2} \Rightarrow \frac{1}{2} = 2A + 1$$

$$2A = \frac{1}{2} - 1 = -\frac{1}{2} \Rightarrow A = -\frac{1}{4}$$

$$y = -\frac{1}{4} e^{2x} + \frac{1}{4} e^{2x} + \frac{1}{2} + \frac{1}{2}$$

$$y = -\frac{1}{4} (e^{2x} - e^{2x}) + \frac{1}{2} + \frac{1}{2}$$

Question 6 (***)

$$x^3 \frac{d^2 y}{dx^2} - 2x^2 \frac{dy}{dx} - 4xy = 5.$$

Find the solution of the above differential equation subject to the boundary conditions

$$y = 4, \quad \frac{dy}{dx} = 20 \quad \text{at } x = 0.$$

$$y = 5x^4 - \frac{1}{x}(1 + \ln x)$$

$x^3 \frac{d^2 y}{dx^2} - 2x^2 \frac{dy}{dx} - 4xy = 5, \quad x=0, y=4, \frac{dy}{dx}=20$
 ASSUME A SOLUTION OF THE FORM $y = x^n$
 $y' = nx^{n-1}$
 $y'' = n(n-1)x^{n-2}$
 DETERMINE n THROUGH, THEN SUB INTO THE HOMOGENEOUS O.D.E
 $x^3 \frac{d^2 y}{dx^2} - 2x^2 \frac{dy}{dx} - 4xy = 0$
 $\Rightarrow x^3 [n(n-1)x^{n-2}] - 2x^2 [nx^{n-1}] - 4x^n = 0$
 $\Rightarrow [n(n-1) - 2n - 4] x^n = 0$
 $\Rightarrow n^2 - 3n - 4 = 0$
 $\Rightarrow (n-4)(n+1) = 0$
 $n = 4, -1$
 \therefore C.F. : $y = Ax^4 + Bx^{-1}$
 FOR PARTICULAR INTEGRAL TRY $y = \frac{P}{x^2} \ln x$ (SINCE $\frac{1}{x^2} = x^{-2}$ IS PART OF C.F.)
 $y' = -\frac{P}{x^2} \ln x + \frac{P}{x^2} = \frac{P}{x^2} [1 - \ln x]$
 $y'' = -\frac{2P}{x^3} [1 - \ln x] - \frac{P}{x^3} = -\frac{P}{x^3} [2 - 2\ln x + 1]$
 $= -\frac{P}{x^3} [3 - 2\ln x]$
 SUB INTO THE O.D.E $x^3 \left[-\frac{P}{x^3} (3 - 2\ln x) \right] - 2x^2 \left[\frac{P}{x^2} (1 - \ln x) \right] - 4 \left[\frac{P}{x^2} \ln x \right] = 5$
 $-\frac{P}{x^3} [3 - 2\ln x] - 2P [1 - \ln x] - 4P \ln x = 5$
 $-5P = 5$
 $P = -1$
 $\therefore y = Ax^4 + \frac{B}{x} - \frac{1}{x^2} \ln x$
 $\frac{dy}{dx} = 4Ax^3 - \frac{B}{x^2} - \frac{1}{x^2} \ln x - \frac{1}{x}$
 $\left. \begin{aligned} 4 &= 4A + B \\ 20 &= 4A - B - 1 \end{aligned} \right\} \Rightarrow \begin{aligned} 5A &= 25 \\ A &= 5 \\ B &=-1 \end{aligned}$
 $\therefore y = 5x^4 - \frac{1}{x} - \frac{1}{x^2} \ln x$
 $y = 5x^4 - \frac{1}{x} (1 + \ln x)$

Question 7 (***)

Find the general solution of the following differential equation.

$$\frac{d^4 y}{dx^4} + \frac{2}{x} \frac{d^3 y}{dx^3} - \frac{1}{x^2} \frac{d^2 y}{dx^2} + \frac{1}{x^3} \frac{dy}{dx} = 0, \quad x > 0.$$



$$y = A \ln x + Bx^2 + Cx^2 \ln x + D$$

$\frac{d^4 y}{dx^4} + \frac{2}{x} \frac{d^3 y}{dx^3} - \frac{1}{x^2} \frac{d^2 y}{dx^2} + \frac{1}{x^3} \frac{dy}{dx} = 0$
 $\Rightarrow x^4 \frac{d^4 y}{dx^4} + 2x^3 \frac{d^3 y}{dx^3} - x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0$
 • Let $z = \frac{dy}{dx}$, then $\frac{dz}{dx} = \frac{d^2 y}{dx^2}$, $\frac{d^2 z}{dx^2} = \frac{d^3 y}{dx^3}$, $\frac{d^3 z}{dx^3} = \frac{d^4 y}{dx^4}$
 $\Rightarrow x^4 \frac{d^3 z}{dx^3} + 2x^3 \frac{d^2 z}{dx^2} - x^2 \frac{dz}{dx} + z = 0$
 Try solution of the form $z = x^m$
 $\frac{dz}{dx} = mx^{m-1}$, $\frac{d^2 z}{dx^2} = m(m-1)x^{m-2}$, $\frac{d^3 z}{dx^3} = m(m-1)(m-2)x^{m-3}$
 $\Rightarrow m(m-1)(m-2)x^m + 2m(m-1)x^m - m^2 x^m + x^m = 0$
 $\Rightarrow m^3 - 3m^2 + 2m + 2m^2 - 2m - m + 1 = 0$
 $\Rightarrow m^3 - m^2 - m + 1 = 0$
 $\Rightarrow m^2(m-1) - (m-1) = 0$
 $\Rightarrow (m-1)(m^2 - 1) = 0$
 $\Rightarrow (m-1)(m+1)(m-1) = 0$
 $\Rightarrow m = 1$ (repeated)
 Thus $z = Ax^1 + Bx^2 + Cx^2 \ln x$
 $\frac{dy}{dx} = Ax + Bx^2 + Cx^2 \ln x$ (by parts)
 $y = \frac{A}{2}x^2 + \frac{B}{3}x^3 + C \left[\frac{x^3}{3} \ln x - \int \frac{1}{3} x^2 dx \right]$
 $y = \frac{A}{2}x^2 + \frac{B}{3}x^3 + \frac{C}{3}x^3 \ln x - \frac{C}{9}x^3 + D$
 $y = A \ln x + Bx^2 + Cx^2 \ln x + D$

Question 8 (***)

The curve with equation $y = f(x)$ satisfies

$$x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0, \quad x > 0.$$

By using the substitution $x = e^t$, or otherwise, determine an equation for $y = f(x)$,

given further that $y = 1$ and $\frac{dy}{dx} = -2$ at $x = 1$.

$$y = \frac{\cos(3 \ln x)}{x^2}$$

Working:

Let $x = e^t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\frac{dt}{dx} = \frac{1}{e^t} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right)$

$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dt} \right)$

$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d}{dt} \left(\frac{dy}{dt} \right) \cdot \frac{dt}{dx}$

$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \cdot \frac{1}{x}$

$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}$

Substitute into the original equation:

$x^2 \left(-\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2} \right) + 5x \left(\frac{1}{x} \frac{dy}{dt} \right) + 13y = 0$

$-\frac{dy}{dt} + \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 13y = 0$

$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 0$

Characteristic equation:

$m^2 + 4m + 13 = 0$

$m = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$

General solution:

$y = e^{-2t} (A \cos(3t) + B \sin(3t))$

Substitute back $t = \ln x$:

$y = e^{-2 \ln x} (A \cos(3 \ln x) + B \sin(3 \ln x))$

$y = \frac{1}{x^2} (A \cos(3 \ln x) + B \sin(3 \ln x))$

Initial conditions:

At $x = 1$, $y = 1$ and $\frac{dy}{dx} = -2$

When $x = 1$, $t = \ln 1 = 0$

$y = \frac{1}{1^2} (A \cos(0) + B \sin(0)) = A = 1$

$\frac{dy}{dx} = -2$ at $x = 1$

$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

$\frac{dy}{dt} = -2x$ at $x = 1$

$\frac{dy}{dt} = -2$ at $t = 0$

$\frac{d}{dt} \left(\frac{1}{x^2} (A \cos(3 \ln x) + B \sin(3 \ln x)) \right) = -2$ at $t = 0$

$\frac{d}{dt} \left(\frac{1}{e^{2t}} (A \cos(3t) + B \sin(3t)) \right) = -2$ at $t = 0$

$\frac{d}{dt} \left(e^{-2t} (A \cos(3t) + B \sin(3t)) \right) = -2$ at $t = 0$

$-2e^{-2t} (A \cos(3t) + B \sin(3t)) + e^{-2t} (-3A \sin(3t) + 3B \cos(3t)) = -2$ at $t = 0$

$-2(1)(1) + 1(3B) = -2$

$-2 + 3B = -2$

$3B = 0$

$B = 0$

Final solution:

$y = \frac{\cos(3 \ln x)}{x^2}$

Question 9 (***)

$$x^2 \frac{d^2 y}{dx^2} - 8x \frac{dy}{dx} + 9y = 0, \quad x > 0.$$

Use the fact that $y = Ax^{\frac{3}{2}}$ satisfies the above differential equation, to find the full solution subject to $y = 4$ and $\frac{dy}{dx} = 10$ at $x = 1$.

$$\boxed{}, \quad y = 4x^{\frac{3}{2}}(1 + \ln x)$$

The image shows two handwritten solutions for the differential equation $x^2 \frac{d^2 y}{dx^2} - 8x \frac{dy}{dx} + 9y = 0$ with initial conditions $y(1) = 4$ and $y'(1) = 10$.

Left Solution (Variation of Parameters):

- Assumes a solution of the form $y = Ax^{\frac{3}{2}}$.
- Substitutes into the ODE and simplifies to $\frac{d^2 A}{dx^2} = 0$.
- Integrates to find $A = 2x + C$.
- Substitutes back to get $y = (2x + C)x^{\frac{3}{2}}$.
- Applies initial conditions to find $C = 4$.
- Final solution: $y = 4x^{\frac{3}{2}}(1 + \ln x)$.

Right Solution (Reduction of Order):

- Assumes a solution of the form $y = v(x)x^{\frac{3}{2}}$.
- Substitutes into the ODE and simplifies to $x \frac{d^2 v}{dx^2} - 5 \frac{dv}{dx} = 0$.
- Let $w = \frac{dv}{dx}$, then $x \frac{dw}{dx} - 5w = 0$.
- Solves for $w = \frac{B}{x^4}$.
- Integrates to find $v = -\frac{B}{3x^3} + C$.
- Substitutes back to get $y = (-\frac{B}{3x^3} + C)x^{\frac{3}{2}}$.
- Applies initial conditions to find $B = 4$ and $C = 4$.
- Final solution: $y = 4x^{\frac{3}{2}}(1 + \ln x)$.

Question 10 (****)

Find the general solution of the following differential equation.

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 2x, \quad x > 0.$$

$$\boxed{}, \quad y = Ax + B \cos(\ln x) + C \sin(\ln x) + x \ln x$$

• LOOKING AT THE LHS OF THE O.D.E WE TRY A SOLUTION OF THE FORM

$$y = x^2$$

$$\frac{dy}{dx} = 2x^{2-1}$$

$$\frac{d^2 y}{dx^2} = 2(2-1)x^{2-2}$$

$$\frac{d^3 y}{dx^3} = 2(2-1)(2-2)x^{2-3}$$

• SUBSTITUTING INTO THE O.D.E (R.H.S = 0)

$$\Rightarrow 2(2-1)(2-2)x^2 + 2(2-1)x^2 + 2x^2 - x^2 = 0$$

$$\Rightarrow x^2 [2(2-1)(2-2) + 2(2-1) + 2 - 1] = 0$$

$$\Rightarrow (2-1)[2(2-2) + 2 + 1] = 0$$

$$\Rightarrow (2-1)(2+1) = 0$$

$$\lambda = \begin{matrix} 1 \\ 1 \\ -1 \end{matrix} \quad y = Ax^1 + Bx^1 + Cx^{-1}$$

• NOW NOTE THAT

$$Bx^1 + Cx^{-1} = B e^{i \ln x} + C e^{-i \ln x}$$

$$= B [\cos(\ln x) + i \sin(\ln x)] + C [\cos(\ln x) - i \sin(\ln x)]$$

$$= (B+C) \cos(\ln x) + i(B-C) \sin(\ln x)$$

$$= D \cos(\ln x) + E \sin(\ln x)$$

• FOR PARTICULAR INTEGRAL, BY INSPECTION, WE TRY $y = P \ln x$

$$y = P \ln x$$

$$\frac{dy}{dx} = P + P \ln x$$

$$\frac{d^2 y}{dx^2} = \frac{P}{x}$$

$$\frac{d^3 y}{dx^3} = -\frac{P}{x^2}$$

• SUB INTO THE O.D.E GIVES

$$-\frac{P}{x^2} + 2\frac{P}{x} + P \ln x - P \ln x = 2x$$

$$\therefore P = 1$$

• HENCE THE GENERAL SOLUTION IS

$$y = x \ln x + D \cos(\ln x) + E \sin(\ln x) + Ax$$

Question 11 (****)

Use variation of parameters to determine the specific solution of the following differential equation

$$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 16 \ln x,$$

given further that $y = \frac{1}{2}$, $\frac{dy}{dx} = 2$ at $x = 1$.

$$y = \frac{1}{2} + (1 + x^4) \ln x$$

$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 16 \ln x$ SUBJECT TO $x=1$
 $y = \frac{1}{2}$
 $\frac{dy}{dx} = 2$

● ASSUME SOLUTION OF THE FORM
 $y = x^2 \Rightarrow y' = 2x^{2-1} \Rightarrow y' = 2x$
 $y'' = 2$

SUB INTO THE O.D.E WITH R.H.S. ZERO
 $2(1-1)x^2 - 7(2)x + 16x^2 = 0$
 $2(1-7)x + 16x = 0$
 $(2-14+16)x = 0$
 $(4)x = 0$
 $x = 4$ C.E.R.R.E.N.T.O

● PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS
 $y_p = A x^2 + B x \ln x$

WE CAN ALSO
 $y_p = A x^2 + B x \ln x$

● THIS THE PARTICULAR INTEGRAL IS GIVEN BY
 $y_p = -C \int \frac{Q(x)}{A(x)} dx + C_2 \int \frac{Q(x)}{A(x)} dx$
 $y_p = -C \int \frac{16 \ln x}{x^2} dx + C_2 \int \frac{16 \ln x}{x^2} dx$
 $y_p = -C \int \frac{16 \ln x}{x^2} dx + C_2 \int \frac{16 \ln x}{x^2} dx$

● EACH BY PARTS
 $\int \frac{16 \ln x}{x^2} dx$
 $= -\frac{16}{x} \ln x + \int \frac{16}{x} dx$
 $= -\frac{16}{x} \ln x + 16 \ln x + C$
 $= 16 \ln x + C$

● WE CAN ALSO
 $y_p = A x^2 + B x \ln x$

SO $y_p = \ln x + \frac{1}{2}$

● FIND SOLUTION
 $y = A x^2 + B x \ln x + \ln x + \frac{1}{2}$

● APPLY CONDITIONS $x=1$ $y = \frac{1}{2}$
 $\frac{1}{2} = A + \frac{1}{2} \Rightarrow A = 0$

● APPLY CONDITIONS $x=1$ $\frac{dy}{dx} = 2$
 $2 = B + 1$
 $B = 1$

● $y = \frac{1}{2} + (1 + x^4) \ln x$

2ND ORDER ODEs WITH MISSING INDEPENDENT VARIABLE

Question 1 (****+)

The curve C , has gradient $\frac{2}{9}$ at the point with coordinates $(\ln 2, \frac{2}{3})$, and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = (1-2y) \frac{dy}{dx}, \quad y < \frac{1}{2}.$$

Find an equation for C , giving the answer in the form $y = f(x)$.

$$y = \frac{e^x}{1+e^x} = \frac{1}{e^x + e^{-x}} = \frac{1}{2} \operatorname{sech} x$$

$\frac{d^2y}{dx^2} = (1-2y) \frac{dy}{dx}$

At the independent variable is missing we try $p = \frac{dy}{dx}$

Then differentiate both sides w.r.t. x

$$\frac{dp}{dx} = \frac{dy}{dx} \frac{dp}{dy} = \frac{dy}{dx} \cdot \frac{1}{p} \Rightarrow \frac{dp}{dx} = \frac{1}{p} \frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2} = p \frac{dp}{dy}$$

By partial fractions

$$\Rightarrow \frac{dp}{dy} = \frac{1}{1-2y} \Rightarrow \int \frac{1}{1-2y} dy = \int 1 dx$$

$$\Rightarrow \int \frac{1}{1-2y} dy = -\frac{1}{2} \ln|1-2y| = x + C$$

$$\Rightarrow \ln|1-2y| = -2x + C$$

$$\Rightarrow \frac{1-2y}{1} = e^{-2x+C} = e^{-2x} \cdot e^C$$

$$\Rightarrow \frac{1-2y}{1} = B e^{-2x} \quad [B = e^C]$$

Apply condition $x = \ln 2, y = \frac{2}{3}$

$$\frac{1-2(\frac{2}{3})}{1} = B e^{-2 \ln 2} = B e^{-\ln 4} = \frac{B}{4}$$

$$\frac{1-\frac{4}{3}}{1} = \frac{B}{4} \Rightarrow -\frac{1}{3} = \frac{B}{4} \Rightarrow B = -\frac{4}{3}$$

$$\Rightarrow \frac{1-2y}{1} = -\frac{4}{3} e^{-2x}$$

$$\Rightarrow 1-2y = -\frac{4}{3} e^{-2x}$$

$$\Rightarrow 2y = 1 + \frac{4}{3} e^{-2x}$$

$$\Rightarrow y = \frac{1}{2} + \frac{2}{3} e^{-2x}$$

ALTERNATIVE 4/6/6

$\frac{d^2y}{dx^2} = (1-2y) \frac{dy}{dx}$

Integrate both sides with respect to x , subject to $y = \frac{2}{3}, \frac{dy}{dx} = \frac{2}{9}$

$$\Rightarrow \int \frac{d^2y}{dx^2} dx = \int (1-2y) \frac{dy}{dx} dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{9}$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = \left(\frac{2}{9} \right)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{9} \quad (\text{SQUARE BOTH SIDES})$$

$$\Rightarrow \frac{1}{y-1/2} dy = 1 dx$$

$$\Rightarrow \int \frac{1}{y-1/2} dy = \int 1 dx$$

$$\Rightarrow \ln|y-1/2| = x + C$$

$$\Rightarrow \ln\left|\frac{y}{1+y}\right| = x + C$$

$$\Rightarrow \frac{y}{1+y} = e^{x+C} = e^x \cdot e^C$$

$$\Rightarrow \frac{y}{1+y} = B e^x \quad [B = e^C]$$

Question 2 (****)

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2 y}{dx^2} = -\frac{144}{y^3}, \quad y(0) = 6, \quad \left. \frac{dy}{dx} \right|_{x=0} = 0.$$

$$\boxed{}, \quad \frac{x^2}{9} + \frac{y^2}{36} = 1$$

The image shows two handwritten solutions for the differential equation $\frac{d^2 y}{dx^2} = -\frac{144}{y^3}$ with initial conditions $y(0) = 6$ and $\left. \frac{dy}{dx} \right|_{x=0} = 0$.

Left Solution (Separation of Variables):

- Let $p = \frac{dy}{dx}$ so we have $\frac{dp}{dy} = -\frac{144}{y^3}$ with $x=0, y=6, \frac{dp}{dy}=0$.
- Integrate: $\int \frac{dp}{dy} = \int -\frac{144}{y^3} dy$
- $\frac{1}{2} p^2 = \frac{72}{y^2} + A$
- $p^2 = \frac{144}{y^2} + B$
- Apply $x=0, y=6, p=0 \Rightarrow 0 = \frac{144}{36} + B \Rightarrow B = -4$
- $\therefore p^2 = \frac{144}{y^2} - 4$

Right Solution (Energy Method):

- $p^2 = \frac{144 - 4y^2}{y^2}$
- $p = \pm \frac{\sqrt{144 - 4y^2}}{y}$
- $\frac{dy}{dx} = \pm \frac{\sqrt{144 - 4y^2}}{y}$
- Manipulate to form $\frac{dy}{dx} = \pm \frac{\sqrt{144 - 4y^2}}{y}$
- Separate variables and integrate by inspection: $\int \frac{1}{y} dy = \int \pm \frac{\sqrt{144 - 4y^2}}{y^2} dy$
- $x = \pm \frac{1}{4} (144 - 4y^2)^{\frac{1}{2}} + C$
- Apply $x=0, y=6$
 $0 = \pm 0 + C$
 $C = 0$
- Finalize the answer: $x^2 = \left[\pm \frac{1}{4} (144 - 4y^2)^{\frac{1}{2}} \right]^2$
- $x^2 = \frac{1}{16} (144 - 4y^2)$
- $16x^2 = 144 - 4y^2$
- $4x^2 + y^2 = 36$ or $\frac{x^2}{9} + \frac{y^2}{36} = 1$

Question 3 (****+)

The curve C , has a stationary point at $(0,2)$ and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = \frac{4}{y^3}, \quad y \neq 0.$$

a) Given further that $\frac{dy}{dx} \geq 0$ along C , determine a simplified expression for the Cartesian equation of C .

b) Verify by differentiation the answer to part (a).

$$y^2 - x^2 = 4$$

a) $\frac{d^2y}{dx^2} = \frac{4}{y^3}$ stationary point at $(0,2)$, $\frac{dy}{dx} \geq 0$
 let $p = \frac{dy}{dx}$ (since the independent variable is missing)
 $\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} = \frac{4}{y^3} \cdot p$
 $\frac{dp}{dy} = p \cdot \frac{4}{y^3}$
 $\Rightarrow p \cdot \frac{dp}{dy} = \frac{4}{y^3}$
 $\Rightarrow p \cdot dp = \int \frac{4}{y^3} dy$
 $\Rightarrow \frac{1}{2} p^2 = -\frac{2}{y^2} + C$
 $\Rightarrow p^2 = C - \frac{4}{y^2}$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = C - \frac{4}{y^2}$
 Apply condition, $y=2$, $\frac{dy}{dx}=0$
 $0 = C - \frac{4}{2^2}$
 $C = 1$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = 1 - \frac{4}{y^2}$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{y^2 - 4}{y^2}$
 $\Rightarrow \frac{dy}{dx} = \pm \frac{\sqrt{y^2 - 4}}{y}$ ($\frac{dy}{dx} \geq 0$)
 $\Rightarrow \frac{dy}{\sqrt{y^2 - 4}} = \pm \frac{1}{y} dx$

$\Rightarrow \int \frac{dy}{\sqrt{y^2 - 4}} = \pm \int \frac{1}{y} dx$
 $\Rightarrow \ln|y^2 - 4| = \pm x + 8$
 Apply condition, $x=0$, $y=2$
 $0 = 0 + 8$
 $8 = 0$
 $x = (y^2 - 4)^{\frac{1}{2}}$
 $x^2 = y^2 - 4$
 $y^2 - x^2 = 4$

b) $y^2 - x^2 = 4$ Differentiate w.r.t x
 $\Rightarrow 2y \frac{dy}{dx} = 2x$
 $\Rightarrow y \frac{dy}{dx} = x$
 Differentiate w.r.t y again
 $\Rightarrow \frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2} = 1$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 + y \frac{d^2y}{dx^2} = 1$
 But $\frac{dy}{dx} = \frac{x}{y}$
 $\Rightarrow \frac{x^2}{y^2} + y \frac{d^2y}{dx^2} = 1$
 $\Rightarrow \frac{x^2}{y^2} + y \frac{d^2y}{dx^2} = 1$
 $\Rightarrow y \frac{d^2y}{dx^2} = 1 - \frac{x^2}{y^2}$
 $\Rightarrow \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$
 $\Rightarrow \frac{d^2y}{dx^2} = \frac{4}{y^3}$
 As required

Question 4 (****+)

The curve C , has a stationary point at $(0,4)$ and satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{y^2}, \quad y \neq 0.$$

- a)** Given further that $\frac{dy}{dx} \geq 0$ along C , determine a simplified expression for the Cartesian equation of C , giving the answer in the form $x = f(y)$.
- b)** Verify by differentiation the answer to part (a).

$$x = 4 \operatorname{arcosh} \left(\frac{1}{2} \sqrt{y} \right) + \sqrt{y^2 - 4y}$$

[illegible]

$2x=0, y=4$
 $4 \cos(0) + 0 = 0 + 4$
 $B=0$

$\therefore \alpha = 4 \cos(\frac{\pi}{2}) + \sqrt{3^2 - 4^2}$

$b) \alpha = 4 \cos(\frac{\pi}{2}) + (y^2 - 4)^{\frac{1}{2}}$

$\frac{d\alpha}{dy} = A' \times \frac{1}{\sqrt{y^2 - 4}} \times \frac{1}{2} y^{\frac{1}{2}} y^{\frac{1}{2}} + \frac{1}{2} (y^2 - 4)^{-\frac{1}{2}} (2y)$

$\frac{dy}{dy} = \frac{1}{\sqrt{y^2 - 4}} + \frac{y-2}{(y^2 - 4)^{\frac{1}{2}}}$

$\frac{d\alpha}{dy} = \frac{1}{\sqrt{y^2 - 4}} + \frac{y-2}{\sqrt{y^2 - 4}} = \frac{y}{\sqrt{y^2 - 4}}$

$\frac{d\alpha}{dy} = \frac{\sqrt{y^2 - 4}}{y} = y^{\frac{1}{2}} (y^2 - 4)^{-\frac{1}{2}} = y^{\frac{1}{2}} \times y^{\frac{1}{2}} (1 - 4y^{-2})^{\frac{1}{2}} = (1 - \frac{4}{y^2})^{\frac{1}{2}}$

$\frac{d\alpha}{dy} = \frac{d}{dy} (1 - \frac{4}{y^2})^{\frac{1}{2}} = \frac{1}{2} (1 - \frac{4}{y^2})^{-\frac{1}{2}} \times \frac{8}{y^3} = \frac{4}{y^3} (1 - \frac{4}{y^2})^{-\frac{1}{2}} \frac{dy}{dy}$

$\text{Hence } \frac{d\alpha}{dy} (1 - \frac{4}{y^2})^{\frac{1}{2}}$

$\frac{d^2\alpha}{dy^2} = \frac{2}{y^4} (1 - \frac{4}{y^2})^{-\frac{1}{2}} (1 - \frac{4}{y^2})^{\frac{1}{2}}$

$\frac{d^2\alpha}{dy^2} = \frac{2}{y^4}$

is simplified

Question 5 (****+)

The curve C with Cartesian equation $f(x, y) = 0$, satisfies the differential equation

$$(1-y)y'' = (2-y)(y')^2.$$

It is further given that $y(0) = 0$ and $y'(0) = 1$

- Determine a simplified expression for the Cartesian equation of C .
- Verify by differentiation the answer to part (a).

$$x = ye^{-y}$$

Handwritten Solution (Left Page):

a) $(1-y) \frac{d^2y}{dx^2} = (2-y) \left(\frac{dy}{dx}\right)^2$ $x=0, y=0, \frac{dy}{dx}=1$

• Since the dependent variable x is missing, we use the standard substitution $\frac{dy}{dx} = p$.

Diff w.r.t y :

$$\frac{d}{dy} \left(\frac{dy}{dx} \right) = \frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2} \frac{dy}{dx} = \frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2} \cdot p = \frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{p} \frac{dp}{dy}$$

• $(1-y) p \frac{dp}{dy} = (2-y) p^2$

$$\Rightarrow (1-y) \frac{dp}{dy} = (2-y) p$$

$$\Rightarrow \frac{1}{p} dp = \frac{(2-y)}{(1-y)} dy$$

$$\Rightarrow \frac{1}{p} dp = \frac{1+(1-y)}{1-y} dy$$

$$\Rightarrow \frac{1}{p} dp = \frac{1}{1-y} + 1 dy$$

$$\Rightarrow \ln p = -\ln(1-y) + y + C$$

$$\Rightarrow p = e^{-\ln(1-y)} (A \cdot e^y)$$

$$\Rightarrow p = \frac{Ae^y}{1-y}$$

... Apply condition $y=0, p=\frac{dy}{dx}=1$

• This gives $1 = \frac{Ae^0}{1-0} \Rightarrow A=1$

• Thus $p = \frac{e^y}{1-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^y}{1-y}$$

By partials:

$$\frac{1-y}{e^y} = \frac{1}{e^y} - \frac{y}{e^y}$$

$$\Rightarrow -\frac{(1-y)e^{-y}}{e^{-2y}} = \int \frac{1}{e^y} dy = x + D$$

$$\Rightarrow -e^y + ye^y = x + D$$

$$\Rightarrow ye^y = x + D$$

Handwritten Solution (Right Page):

Apply condition $x=0, y=0 \Rightarrow D=0$

$\therefore x = ye^y$ or $xe^{-y} = y$

b) Diff $x = ye^{-y}$ w.r.t y

$$\frac{dx}{dy} = 1e^{-y} + y(-e^{-y})$$

$$\frac{dx}{dy} = e^{-y} - ye^{-y}$$

$$\frac{dx}{dy} = e^{-y}(1-y)$$

$$\frac{dy}{dx} = \frac{e^y}{1-y}$$

Diff w.r.t x

$$-\frac{dy}{dx} \frac{dx}{dx} + (1-y) \frac{d^2y}{dx^2} = \frac{e^y}{1-y} \frac{dy}{dx}$$

$$-\left(\frac{dy}{dx}\right)^2 + (1-y) \frac{d^2y}{dx^2} = (1-y) \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2$$

$$(1-y) \frac{d^2y}{dx^2} = (1-y) \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2$$

$$(1-y) \frac{d^2y}{dx^2} = (2-y) \left(\frac{dy}{dx}\right)^2$$

As Required

Question 6 (****)

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2 y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 = 2y \ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2 \ln 3.$$

Solve the above differential equation to show that $y = 3^{x^2+2x}$.

, proof

USING THE SUBSTITUTION $p = \frac{dy}{dx}$ AS THE INDEPENDENT VARIABLE IS USEFUL

$$p = \frac{dy}{dx} \Rightarrow \frac{dp}{dx} = \frac{dp}{dy} \left(\frac{dy}{dx} \right) = \frac{dp}{dy} \times \frac{dy}{dx}$$

$$\Rightarrow \frac{dp}{dy} = \frac{dp}{dx} \times \frac{dx}{dy}$$

$$\Rightarrow \frac{dp}{dy} = p \frac{dp}{dy}$$

TRANSFORMING THE O.D.E

$$\Rightarrow \frac{dp}{dy} - \frac{1}{y} p^2 = 2y \ln 3$$

$$\Rightarrow p \frac{dp}{dy} - \frac{1}{y} p^2 = 2y \ln 3$$

$$\Rightarrow \frac{dp}{dy} - \frac{1}{y} p^2 = 2y \ln 3$$

USING ANOTHER SUBSTITUTION $v = \frac{1}{y}$

$$p = \frac{dy}{dx} \Rightarrow \frac{dp}{dy} = \frac{dp}{dv} \left(\frac{dv}{dy} \right) = \frac{dp}{dv} \times \frac{dv}{dy}$$

$$\Rightarrow \frac{dp}{dv} \times \frac{dv}{dy} - \frac{1}{y} p^2 = 2y \ln 3$$

TRANSFORMING THE O.D.E FURTHER

$$\Rightarrow \left(\frac{dp}{dv} + v \right) - v = \frac{2}{y} \ln 3$$

$$\Rightarrow \frac{dp}{dv} = \frac{2 \ln 3}{y}$$

$$\Rightarrow \int v dv = (2 \ln 3) \int \frac{1}{y} dy$$

$$\Rightarrow \frac{1}{2} v^2 = (2 \ln 3) (\ln y) + C$$

$$\Rightarrow v^2 = (4 \ln 3) (\ln y) + C$$

APPLY CONDITIONS ONCE THE O.D.E TRANSFORMATION IS REVERSED

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = (4 \ln 3) (\ln y) + C$$

$$\Rightarrow p^2 = (4 \ln 3) (y^2 \ln y) + C y^2$$

$y=0, \quad y=1, \quad \frac{dy}{dx} = p = 2 \ln 3$

$$\Rightarrow (2 \ln 3)^2 = (4 \ln 3) (1^2 \times \ln 1) + C \times 1^2$$

$$\Rightarrow C = 4 (\ln 3)^2$$

$$\Rightarrow p^2 = (4 \ln 3) y^2 \ln y + 4 y^2 (\ln 3)^2$$

$$\Rightarrow p^2 = 4 y^2 \ln 3 [\ln y + \ln 3]$$

$$\Rightarrow p = \frac{dy}{dx} = \sqrt{4 y^2 \ln 3 \ln 3 y}$$

$$\Rightarrow \frac{dy}{dx} = 2 y \sqrt{\ln 3 \ln 3 y}$$

$$\Rightarrow \frac{dy}{dx} = 2 y (\ln 3)^{\frac{1}{2}} (\ln 3 y)^{\frac{1}{2}}$$

REPRESENT UNDEVELOPED

$$\Rightarrow \int \frac{1}{2 (\ln 3 y)^{\frac{1}{2}}} dy = \int 2 (\ln 3)^{\frac{1}{2}} dy$$

BY SUBSTITUTION OF $z = \ln 3 y$ INTO THE INTEGRATION

$$\Rightarrow \int \frac{1}{2 (\ln 3)^{\frac{1}{2}}} dz = 2 (\ln 3)^{\frac{1}{2}} z + 4$$

$$\Rightarrow 2 (\ln 3 y)^{\frac{1}{2}} = 2 (\ln 3)^{\frac{1}{2}} z + 4$$

$$\Rightarrow \sqrt{\ln 3 y} = 2 \sqrt{\ln 3} + 2$$

APPLY CONDITIONS $z=0, \quad y=1$

$$\Rightarrow \sqrt{\ln 3} = 0 \sqrt{\ln 3} + 2$$

$$\Rightarrow 2 = \ln 3$$

FINALLY USE CRAMER

$$\Rightarrow \sqrt{\ln 3 y} = 2 \sqrt{\ln 3} + 2$$

$$\Rightarrow \sqrt{\ln 3 y} = (2+1) \sqrt{\ln 3}$$

$$\Rightarrow \ln 3 y = (2+1)^2 \ln 3$$

$$\Rightarrow 3 y = e^{(2+1)^2 \ln 3}$$

$$\Rightarrow 3 y = 3^{(2+1)^2}$$

$$\Rightarrow 3 y = 3^{2^2+2+1}$$

$$\Rightarrow y = 3^{2^2+2+1}$$

$$\Rightarrow y = 3^{x^2+2x}$$

As expected

Question 7 (*****)

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2 y}{dx^2} = 6y^2 + 4y, \quad \frac{dy}{dx} \geq 0.$$

If $y = 3$, $\frac{dy}{dx} = 12$ at $x = -\frac{1}{2} \ln 3$, solve the differential equation to show that

$$y = \operatorname{cosech}^2 x.$$

proof

Step 1: Substitution and Separation of Variables

Given $\frac{d^2 y}{dx^2} = 6y^2 + 4y$. Let $u = \frac{dy}{dx}$. Then $\frac{du}{dx} = 6y^2 + 4y$.
 Since the independent variable is missing, we use $p = \frac{dy}{dx}$.
 So $\frac{dp}{dy} = \frac{1}{u} \frac{du}{dy} = \frac{1}{u} (6y^2 + 4y) = \frac{6y^2 + 4y}{u}$.
 Thus $\frac{dp}{dy} = p \frac{dy}{dy}$.
 The ODE transforms to:
 $\Rightarrow p \frac{dp}{dy} = 6y^2 + 4y$
 $\Rightarrow p dp = (6y^2 + 4y) dy$
 $\Rightarrow \int p dp = \int (6y^2 + 4y) dy$
 $\Rightarrow \frac{1}{2} p^2 = 2y^3 + 2y^2 + C$
 $\Rightarrow p^2 = 4y^3 + 4y^2 + C$
 Apply condition $y=3$, $p = \frac{dy}{dx} = 12$:
 $144 = 4(27) + 4(9) + C$
 $144 = 108 + 36 + C$
 $C = 0$
 $\Rightarrow p^2 = 4y^3 + 4y^2$
 $\Rightarrow p^2 = 4y^2(y+1)$
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y^2(y+1)$
 $\Rightarrow \frac{dy}{dx} = 2y(y+1)^{\frac{1}{2}}$ (since $\frac{dy}{dx} \geq 0$)

Step 2: Separation of Variables

$\Rightarrow \int \frac{1}{y(y+1)^{\frac{1}{2}}} dy = \int 2 dx$
 $\Rightarrow \int \frac{1}{(y^2+1)^{\frac{1}{2}}} (2y dy) = \int 2 dx$
 $\Rightarrow \int \frac{2}{u^2-1} du = \int 2 dx$ (where $u = y^2+1$)
 $\Rightarrow \int \frac{2}{(u-1)(u+1)} du = \int 2 dx$
 By partial fractions:
 $\Rightarrow \int \frac{1}{u-1} - \frac{1}{u+1} du = \int 2 dx$
 $\Rightarrow \ln|u-1| - \ln|u+1| = 2x + k$
 $\Rightarrow \ln \left| \frac{u-1}{u+1} \right| = 2x + k$
 $\Rightarrow \frac{u-1}{u+1} = e^{2x+k}$
 $\Rightarrow \frac{u-1}{u+1} = A e^{2x}$ ($A = e^k$)
 $\Rightarrow u-1 = A u e^{2x} + A e^{2x}$
 $\Rightarrow u - A u e^{2x} = 1 + A e^{2x}$
 $\Rightarrow u(1 - A e^{2x}) = 1 + A e^{2x}$
 $\Rightarrow u = \frac{1 + A e^{2x}}{1 - A e^{2x}}$
 $\Rightarrow (y^2+1)^{\frac{1}{2}} = \frac{1 + A e^{2x}}{1 - A e^{2x}}$

Step 3: Apply the Initial Condition

$y=3$, $x = -\frac{1}{2} \ln 3$ ($e^{2x} = \frac{1}{3}$)
 $\sqrt{3+1} = \frac{1 + \frac{1}{3}A}{1 - \frac{1}{3}A}$
 $2 = \frac{3 + A}{3 - A}$
 $6 - 2A = 3 + A$
 $3 = 3A$
 $A = 1$
 Hence
 $\Rightarrow \sqrt{y+1} = \frac{1 + e^{2x}}{1 - e^{2x}}$
 $\Rightarrow \sqrt{y+1} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
 $\Rightarrow \sqrt{y+1} = -\frac{e^x + e^{-x}}{e^x - e^{-x}}$
 $\Rightarrow \sqrt{y+1} = -\coth x$
 $\Rightarrow y+1 = \coth^2 x$
 $\Rightarrow y = \coth^2 x - 1$
 $\Rightarrow y = \operatorname{cosech}^2 x$
 (Note: $\coth^2 x - 1 = \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} = \operatorname{cosech}^2 x$)

Question 8 (*****)

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y.$$

Given further that the curve has a stationary point at $\left(\frac{1}{2}, \frac{1}{4}\right)$, solve the differential equation to show that

$$y = x^2 + x + \frac{1}{2}.$$

proof

Left Page Working:

Given $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y$. Substituting $y = \frac{1}{p}$, $\frac{dy}{dx} = -\frac{1}{p^2} \frac{dp}{dx}$, At $x = \frac{1}{2}$, $y = \frac{1}{4}$.

• Since the independent variable is missing (a), let $p = \frac{dy}{dx}$.

Then $\frac{dp}{dy} = \frac{dy/dx}{dy/dx} = \frac{dy/dx}{dy/dx} = \frac{dy/dx}{dy/dx} = \frac{dy/dx}{dy/dx}$.

• $p = \frac{dy}{dx}$ and $\frac{dp}{dy} = \frac{dy/dx}{dy/dx} = \frac{dy/dx}{dy/dx}$.

• $\frac{dp}{dy} + 2p^2 = 8y$

$\Rightarrow \frac{dp}{dy} + 2p^2 = \frac{8}{p}$

$\Rightarrow \frac{dp}{dy} + 2p^2 = 8p^{-1}$

Standard Bernoulli type

Substitution $z = \frac{1}{p+1}$

Then $z = \frac{1}{p+1} \Rightarrow p = \frac{1}{z} - 1$

Let $z = p^2$

$\frac{dz}{dy} = 2p \frac{dp}{dy}$

• Now

Integrating factor is $e^{\int 4 dy} = e^{4y}$

$\Rightarrow \frac{d}{dy}(ze^{4y}) = 16ye^{4y}$

$\Rightarrow ze^{4y} = \int 16ye^{4y} dy$ (by parts)

Right Page Working:

$\Rightarrow ze^{4y} = \int 16ye^{4y} dy$

$\Rightarrow ze^{4y} = 4ye^{4y} - \int 4e^{4y} dy$

$\Rightarrow ze^{4y} = 4ye^{4y} - e^{4y} + A$

$\Rightarrow z = (4y-1) + Ae^{-4y}$

$\Rightarrow p^2 = 4y-1 + Ae^{-4y}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y-1 + Ae^{-4y}$

Now stationary at $\left(\frac{1}{2}, \frac{1}{4}\right) \Rightarrow 0 = 4\left(\frac{1}{2}\right) - 1 + Ae^{-4\left(\frac{1}{2}\right)}$

$0 = 1 - 1 + Ae^{-2}$

$Ae^{-2} = 0$

$A = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y-1$

$\Rightarrow \frac{dy}{dx} = \pm(4y-1)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{\pm(4y-1)^{\frac{1}{2}}} dy = 1 dx$

$\Rightarrow \int \frac{1}{\pm(4y-1)^{\frac{1}{2}}} dy = \int 1 dx$

$\Rightarrow \pm \frac{1}{2}(4y-1)^{\frac{1}{2}} = x + C$

Now $\left(\frac{1}{2}, \frac{1}{4}\right) \Rightarrow \pm \frac{1}{2} \times 0 = \frac{1}{2} + C$

$\Rightarrow 0 = \frac{1}{2} + C$

$\Rightarrow C = -\frac{1}{2}$

$\Rightarrow \pm \frac{1}{2}(4y-1)^{\frac{1}{2}} = x - \frac{1}{2}$

$\Rightarrow \pm(4y-1)^{\frac{1}{2}} = 2x - 1$ (square)

$\Rightarrow 4y-1 = 4x^2 - 4x + 1$

$\Rightarrow 4y = 4x^2 - 4x + 2$

$\Rightarrow y = x^2 - x + \frac{1}{2}$

Question 9 (****)

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2 y}{dx^2} + e^{-y} = 0, \quad \frac{dy}{dx} \geq 0.$$

If $y = 0$, $\frac{dy}{dx} = -1$ at $x = \frac{1}{2}\pi$, solve the differential equation to show that

$$y = \ln(1 - \cos x).$$

, proof

As the independent variable x is missing, use the standard substitution $p = \frac{dy}{dx}$.

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{dp}{dx} \left(\frac{dx}{dy} \right) = \frac{dp}{dx} \frac{dy}{dy} = \frac{dp}{dy} \cdot \frac{1}{p}$$

$$\Rightarrow \frac{dp}{dy} = p \frac{dy}{dy}$$

Compare with the standard non-linear differential equation: $\frac{dp}{dy} = p \frac{dy}{dy}$

Transforming the O.D.E

$$\Rightarrow \frac{dp}{dy} + e^{-y} = 0 \quad \left[x = \frac{\pi}{2}, y = 0, \frac{dy}{dx} = p = -1 \right]$$

$$\Rightarrow p \frac{dp}{dy} = -e^{-y}$$

$$\Rightarrow \int p \, dp = \int -e^{-y} \, dy$$

$$\Rightarrow \left[\frac{1}{2} p^2 \right]_{-1}^p = \left[-e^{-y} \right]_0^{-y}$$

$$\Rightarrow \frac{1}{2} p^2 - \frac{1}{2} = -e^{-y} - (-1)$$

$$\Rightarrow p^2 - 1 = 2e^{-y} - 2$$

$$\Rightarrow p^2 = 2e^{-y} - 1$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = 2e^{-y} - 1$$

$$\Rightarrow \frac{dy}{dx} = + \frac{\sqrt{2 - e^y}}{e^{y/2}}$$

Separating variables: $\frac{dy}{dx} = \frac{\sqrt{2 - e^y}}{e^{y/2}}$

$$\Rightarrow \int \frac{e^{y/2}}{\sqrt{2 - e^y}} \, dy = \int \frac{1}{dx}$$

$$\Rightarrow \int \frac{e^{y/2}}{\sqrt{2 - e^y}} \, dy = \int \frac{1}{dx}$$

Using a trigonometric substitution on the integral on the R.H.S

$$e^{y/2} = 2 \sin \theta \quad \left[e^{y/2} = \sqrt{2} \sin \theta \quad \text{so} \quad \theta = \arcsin \left(\frac{e^{y/2}}{\sqrt{2}} \right) \right]$$

$$\Rightarrow e^{y/2} dy = 4 \sin \theta \cos \theta \, d\theta$$

$$\Rightarrow dy = \frac{4 \sin \theta \cos \theta}{e^{y/2}} \, d\theta = \frac{4 \sin \theta \cos \theta}{2 \sin \theta} \, d\theta = \frac{2 \cos \theta}{\sin \theta} \, d\theta$$

Units transform to

$$y = 0 \quad \mapsto \quad \theta = \frac{\pi}{4}$$

$$y \quad \mapsto \quad \theta = \arcsin \left(\frac{e^{y/2}}{\sqrt{2}} \right)$$

Returning to the O.D.E

$$\Rightarrow \int \frac{2 \cos \theta}{\sin \theta} \, d\theta = \int \frac{1}{dx}$$

$$\Rightarrow 2 \ln \left(\frac{e^{y/2}}{\sqrt{2}} \right) = \ln \left(\frac{e^{y/2}}{\sqrt{2}} \right) + \frac{2 \cos \theta}{\sin \theta} \, d\theta$$

$$\Rightarrow 2 \ln \left(\frac{e^{y/2}}{\sqrt{2}} \right) = \ln \left(\frac{e^{y/2}}{\sqrt{2}} \right) + \frac{2 \cos \theta}{\sin \theta} \, d\theta$$

$$\Rightarrow 2 - \frac{\pi}{2} = 2 \left[\theta \right]_{\frac{\pi}{4}}^{\arcsin \left(\frac{e^{y/2}}{\sqrt{2}} \right)}$$

$$\Rightarrow 2 - \frac{\pi}{2} = 2 \left[\arcsin \left(\frac{e^{y/2}}{\sqrt{2}} \right) - \frac{\pi}{4} \right]$$

$$\Rightarrow 2 = 2 \arcsin \left(\frac{e^{y/2}}{\sqrt{2}} \right)$$

$$\Rightarrow \arcsin \left(\frac{e^{y/2}}{\sqrt{2}} \right) = \frac{\pi}{4}$$

$$\Rightarrow \sin \left(\frac{\pi}{4} \right) = \frac{e^{y/2}}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{e^{y/2}}{\sqrt{2}}$$

$$\Rightarrow 2 \sin^2 \frac{\pi}{4} = e^y$$

$$\Rightarrow \cos 2 \cdot \frac{\pi}{4} = 1 - e^y$$

$$\Rightarrow \cos 2 = 1 - e^y$$

$$\Rightarrow e^y = 1 - \cos 2$$

$$\Rightarrow y = \ln(1 - \cos 2)$$

At $x = \frac{\pi}{2}$

Question 10 (****)

The curve C , has gradient 1 at the origin and satisfies the differential relationship

$$\frac{d^2y}{dx^2}\sqrt{1-2y} = \frac{dy}{dx}(3y-2), \quad y < \frac{1}{2}.$$

Find an equation for C , giving the answer in the form $y = f(x)$.

$$y = \frac{\sin x}{1 + \sin x} = (\sec x - \tan x) \tan x$$

$\frac{\partial^2}{\partial x^2} = \frac{dy}{dx} \left(\frac{\partial}{\partial y} \right) \left(\frac{3y-2}{\sqrt{1-2y}} \right) \quad 2=0, 3=0, \frac{dy}{dx}=1$

LET $p = \frac{dy}{dx}$ DIFF W.R.T y

$\frac{dp}{dy} = \frac{d^2y}{dx^2} = \frac{d^2y}{dx^2} \cdot \frac{dx}{dy} = \frac{d^2y}{dx^2} \cdot \frac{1}{\frac{dy}{dx}}$

$\frac{dp}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\frac{d^2y}{dx^2} = p \frac{dp}{dy}$

Thus

$x \frac{dp}{dy} = x \frac{3y-2}{\sqrt{1-2y}}$

$\Rightarrow \int x \, dp = \int \frac{3y-2}{\sqrt{1-2y}} \, dy$

$\Rightarrow p = \int \frac{3y-2}{\sqrt{1-2y}} \, dy$

BY SUBSTITUTION

$u = \sqrt{1-2y}$
 $u^2 = 1-2y$
 $2udu = -2dy$
 $dy = -u \, du$

$\Rightarrow \frac{dy}{dx} = \left(\frac{3y-2}{u} \right) (-u \, du)$

$\Rightarrow \frac{dy}{dx} = \int 2-3y \, dy$

$\Rightarrow \frac{dy}{dx} = \int 2 - \frac{3}{2}(u^2) \, du$

$\Rightarrow \frac{dy}{dx} = \int 2 - \frac{3}{2}(u^2) \, du$

$\Rightarrow \frac{dy}{dx} = \int 2 - \frac{3}{2} + \frac{3}{2}u^2 \, du$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}u + \frac{1}{2}u^3 + C$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}u(1+u^2) + C$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{1-2y}(1+1-2y) + C$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{1-2y}(2-2y) + C$

$\Rightarrow \frac{dy}{dx} = (1-y)\sqrt{1-2y} + C$

① ARBEIT UNTERSUCHEN
 $y=0 \quad \frac{dy}{dx}=1$
 $1=1+c$
 $C=0$
 $\frac{dy}{dx} = (1-y)(1-2y)^2$

② SEPARIERE VARIABLEN
 $\int \frac{1}{(1-y)(1-2y)^2} dy = \int 1 dx$

③ DIE A. BESTIMMEN
 $2y = \sin \theta$
 $2dy = 2 \cos \theta d\theta$
 $dy = \cos \theta d\theta$

$\Rightarrow z = \int \frac{\sin \theta \cos \theta}{(1 - \frac{1}{2} \sin \theta)(\cos \theta)^2} d\theta$
 $\Rightarrow z = \int \frac{2 \sin \theta}{2 \cos \theta - \sin \theta} d\theta$
 $\Rightarrow z = \int \frac{2 \sin \theta}{2 \cos \theta - \sin \theta} d\theta$
 $\Rightarrow z = \int \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta - \sin^2 \theta} d\theta$
 $\Rightarrow z = \int \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta - (1 - \cos^2 \theta)} d\theta$

$\Rightarrow z = \int \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + 1} d\theta$

$\Rightarrow z = \int \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + 1} d\theta$

④ PARTIELLE FRAKTIONEN
 $\frac{dy}{dx} = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + 1}$
 $\Rightarrow z = \int \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + 1} d\theta$

⑤ PARTIELLE FRAKTIONEN
 $\frac{dy}{dx} = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + 1}$
 $\Rightarrow z = \int \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + 1} d\theta$

⑥ NEU
 $\sin \theta = 2y$
 $\sqrt{1-y^2}$
 $\sqrt{1-2y^2}$
 $\therefore \sin \theta = \frac{1-y^2}{1+y^2}$

$\Rightarrow z = 2 \arctan \left(\frac{1-y^2}{1+y^2} \right) + C$

$\Rightarrow z = 2 \arctan \left(\frac{1-y^2}{1+y^2} \right) + C$

⑦ ARBEIT UNTERSUCHEN
 $z=0, y=0 \Rightarrow 0 = -2x \frac{z}{x} + \frac{1}{x}$
 $k = \frac{1}{x}$
 $z = -2 \arctan \sqrt{1-2y^2} + \frac{1}{x}$

⑧ TIPPS FÜR DIE PRÜFUNG

$$\begin{aligned} \Rightarrow 2 \arctan \sqrt{1-2y} &= \frac{\pi}{2} - x \\ \Rightarrow \arctan \sqrt{1-2y} &= \frac{\pi}{4} - \frac{x}{2} \\ \Rightarrow \sqrt{1-2y} &= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \end{aligned}$$

↑
trig identity

$$\begin{aligned} \frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}} &= \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} \quad \left(\begin{array}{l} \text{MULTIPLY NUM \& DENOM BY } \cos \frac{x}{2} \end{array} \right) \\ &= \frac{(\cos \frac{x}{2} - \sin \frac{x}{2})(\cos \frac{x}{2} - \sin \frac{x}{2})}{(\cos \frac{x}{2} + \sin \frac{x}{2})(\cos \frac{x}{2} - \sin \frac{x}{2})} \\ &= \frac{\cos^2 \frac{x}{2} - 2\cos \frac{x}{2} \sin \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \\ &= \frac{1 - \sin x}{\cos x} \end{aligned}$$

$$\Rightarrow \sqrt{1-2y} = \frac{1 - \sin x}{\cos x}$$

$$\Rightarrow \sqrt{1-2y} = \sec x - \tan x$$

$$\Rightarrow 1-2y = \sec^2 x + \tan^2 x - 2\sec x \tan x$$

$$\Rightarrow 1 - \sec^2 x - \tan^2 x + 2\sec x \tan x = 2y$$

$$\Rightarrow -2\tan^2 x + 2\sec x \tan x = 2y$$

$$\Rightarrow y = \sec x \tan x - \tan^2 x$$

$$\Rightarrow y = \tan x (\sec x - \tan x)$$

ALTERNATIVE APPROACH

$$\Rightarrow \sqrt{1-2y} = \frac{1-\sin x}{\cos x}$$
$$\Rightarrow 1-2y = \frac{(1-\sin x)^2}{\cos^2 x}$$
$$\Rightarrow 1-2y = \frac{(1-\sin x)^2}{1-\sin^2 x}$$
$$\Rightarrow 1-2y = \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$$
$$\Rightarrow 1-2y = \frac{1-\sin x}{1+\sin x}$$
$$\Rightarrow 1 - \frac{1-\sin x}{1+\sin x} = 2y$$
$$\Rightarrow 2y = \frac{1+\sin x - 1 + \sin x}{1+\sin x}$$
$$\Rightarrow 2y = \frac{2\sin x}{1+\sin x}$$
$$\Rightarrow y = \frac{\sin x}{1+\sin x}$$

Question 11 (****)

The curve C , has gradient $\frac{1}{8}$ at the point with coordinates $(1, \frac{1}{2})$ and further satisfies the differential relationship

$$2y^2 \frac{d^2 y}{dx^2} + (2y+1)(y-1)^2 \frac{dy}{dx} = 0, \quad y \neq 0.$$

Find an equation for C , giving the answer in the form $y = f(x)$.

$$y = \frac{\sqrt{x}}{1 + \sqrt{x}}$$

$$\begin{aligned}
 & 2y^2 \frac{d^2 y}{dx^2} + (2y+1)(y-1) \frac{dy}{dx} = 0 \quad \Rightarrow 1, y=1, \frac{dy}{dx} = \frac{1}{8} \\
 & \Rightarrow 2y^2 \frac{d^2 y}{dx^2} = -(2y+1)(y-1) \frac{dy}{dx} \\
 & \Rightarrow \frac{d^2 y}{dx^2} = -\frac{(2y+1)(y-1)}{2y^2} \frac{dy}{dx} \\
 & \Rightarrow \frac{d^2 y}{dx^2} = -\frac{2y^2 - y^2 + 2y + 1}{2y^2} \frac{dy}{dx} \\
 & \Rightarrow \frac{d^2 y}{dx^2} = -\frac{y^2 + 3y + 1}{2y^2} \frac{dy}{dx} \\
 & \Rightarrow \frac{d^2 y}{dx^2} = \left(-\frac{1}{2} + \frac{3}{2y} + \frac{1}{2y^2}\right) \frac{dy}{dx} \\
 & \bullet \text{ INTEGRATE BOTH SIDES WITH RESPECT TO } x \\
 & \text{SOLUTION TO } y=1 \Rightarrow \frac{dy}{dx} = \frac{1}{8} \\
 & \Rightarrow \int \frac{d^2 y}{dx^2} dx = \int \left(-\frac{1}{2} + \frac{3}{2y} + \frac{1}{2y^2}\right) \frac{dy}{dx} dx \\
 & \Rightarrow \left[\frac{dy}{dx}\right]_8^y = \left[-\frac{1}{2}y + \frac{3}{2}\ln|y| + \frac{1}{2}y^{-1}\right]_8^y \\
 & \Rightarrow \frac{dy}{dx} \cdot \frac{1}{8} = \left(-\frac{1}{2}y + \frac{3}{2}\ln|y| + \frac{1}{2y}\right) - \left(-\frac{1}{2} \cdot \frac{1}{8} + \frac{3}{2}\ln|8| + \frac{1}{2 \cdot 8}\right) \\
 & \Rightarrow \frac{dy}{dx} = -\frac{1}{2}y + \frac{3}{2}\ln|y| - \frac{3}{2} + \frac{1}{2y} \\
 & \text{TRY AGAIN} \\
 & \Rightarrow \frac{dy}{dx} = \frac{1}{2y} \left[-\frac{1}{2}y^2 + 3\ln|y| + \frac{1}{y}\right]
 \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_{\frac{1}{2}}^{1-\frac{1}{2}} \frac{2-2u}{u^3} (-du) &= \int_1^{\frac{1}{2}} \frac{1}{u} du \\ \Rightarrow 2 \int_{\frac{1}{2}}^{1-\frac{1}{2}} -u^{-2} + u^{-2} du &= \int_1^{\frac{1}{2}} \frac{1}{u} du \\ \Rightarrow 2 \left[\frac{1}{2} u^{-2} - u^{-1} \right]_{\frac{1}{2}}^{1-\frac{1}{2}} &= \left[\frac{1}{u} \right]_1^{\frac{1}{2}} \\ \Rightarrow \left[\frac{1}{u^2} - \frac{1}{u} \right]_{\frac{1}{2}}^{1-\frac{1}{2}} &= 2 - 1 \\ \Rightarrow \left[\frac{1-2u}{u^2} \right]_{\frac{1}{2}}^{1-\frac{1}{2}} &= 2 - 1 \\ \Rightarrow \frac{1-2(1-\frac{1}{2})}{(1-\frac{1}{2})^2} - 0 &= 2 - 1 \\ \Rightarrow \frac{2y-1}{(y-1)^2} &= 2 - 1 \\ \Rightarrow 2y-1 &= (x-1)(y^2-2y+1) \\ \Rightarrow 2y-1 &= (x-1)y^2 - 2(x-1)y + (x-1) \\ \Rightarrow 2y-1 &= (x-1)y^2 - 2xy + 2y + x - 1 \\ \Rightarrow (x-1)y^2 - 2xy + x &= 0 \quad \leftarrow \left[y(x-1)-x \right] \left[y(x-1)-x \right] = 0 \\ \text{Factorize or complete the square} \\ \Rightarrow y^2 - \frac{2x}{x-1}y + \frac{x}{x-1} &= 0 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \left[y - \frac{x}{x-1} \right]^2 - \frac{x^2}{(x-1)^2} + \frac{x}{x-1} = 0 \\ &\Rightarrow \left[y - \frac{x}{x-1} \right]^2 + \frac{-x^2 + x(x-1)}{(x-1)^2} = 0 \\ &\Rightarrow \left[y - \frac{x}{x-1} \right]^2 + \frac{-x}{(x-1)^2} = 0 \\ &\Rightarrow \left[y - \frac{x}{x-1} \right]^2 = \frac{-x}{x-1} \\ &\Rightarrow y - \frac{x}{x-1} = \pm \sqrt{\frac{-x}{x-1}} \\ &\Rightarrow y = \frac{x \pm \sqrt{x}}{x-1} \\ &\Rightarrow y = \frac{\frac{x^2}{(\sqrt{x})^2} \pm 1}{(\sqrt{x})^2 - 1} \\ &\Rightarrow y = \frac{\sqrt{x} \left[\sqrt{x} \pm 1 \right]}{(\sqrt{x}-1)(\sqrt{x}+1)} \\ &\Rightarrow y = \frac{\sqrt{x}}{\sqrt{x}+1} \quad \begin{matrix} \text{NOT } \frac{\sqrt{x}}{\sqrt{x}-1} \\ \text{AT } x=1 \end{matrix} \\ &\Rightarrow y = \frac{\sqrt{x}}{\sqrt{x}+1} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{2y-1}{(y-1)^2} &= x-1 \\ \Rightarrow x &= \frac{2y-1}{(y-1)^2} + 1 \\ \Rightarrow x &= \frac{(2y-1) + (y-1)^2}{(y-1)^2} \\ \Rightarrow x &= \frac{2y-1+y^2-2y+1}{(y-1)^2} \\ \Rightarrow x &= \frac{y^2}{(y-1)^2} \\ \Rightarrow \frac{y}{y-1} &= \begin{cases} \sqrt{x} \\ -\sqrt{x} \end{cases} \end{aligned}$$

DEGREE 4 IS SATISFIED BY $x=1, y=\frac{1}{2}$

ANS:

$$\frac{y}{y-1} = -\sqrt{x}$$

$$y = -y^2\sqrt{x} + \sqrt{x}$$

$$y + y^2\sqrt{x} = \sqrt{x}$$

$$y(1+y^2\sqrt{x}) = \sqrt{x}$$

$$y = \frac{\sqrt{x}}{1+y^2\sqrt{x}}$$

~~to BODI~~

2ND ORDER BY SUBSTITUTIONS

Question 1 (***)

$$2y \frac{d^2 y}{dx^2} - 8y \frac{dy}{dx} + 16y^2 = \left(\frac{dy}{dx} \right)^2, \quad y \neq 0,$$

Find the general solution of the above differential equation by using the transformation equation $t = \sqrt{y}$.

Give the answer in the form $y = f(x)$.

$$y = \left(Ae^{2x} + Bxe^{2x} \right)^2$$

Handwritten solution for the differential equation using the transformation $t = \sqrt{y}$.

Given: $2y \frac{d^2 y}{dx^2} - 8y \frac{dy}{dx} + 16y^2 = \left(\frac{dy}{dx} \right)^2$

Let $t = \sqrt{y}$

$\frac{dy}{dx} = 2t \frac{dt}{dx}$

$\frac{d^2 y}{dx^2} = 2 \left(\frac{dt}{dx} \right)^2 + 2t \frac{d^2 t}{dx^2}$

Substitute into the equation:

$$2(2t) \left(2 \left(\frac{dt}{dx} \right)^2 + 2t \frac{d^2 t}{dx^2} \right) - 8t \left(2t \frac{dt}{dx} \right) + 16t^4 = \left(2t \frac{dt}{dx} \right)^2$$

$$4t \left(2 \left(\frac{dt}{dx} \right)^2 + 2t \frac{d^2 t}{dx^2} \right) - 16t^2 \frac{dt}{dx} + 16t^4 = 4t^2 \left(\frac{dt}{dx} \right)^2$$

$$8t \left(\frac{dt}{dx} \right)^2 + 8t^2 \frac{d^2 t}{dx^2} - 16t^2 \frac{dt}{dx} + 16t^4 = 4t^2 \left(\frac{dt}{dx} \right)^2$$

$$4t^2 \frac{d^2 t}{dx^2} - 8t^2 \frac{dt}{dx} + 16t^4 = 0$$

Divide by $4t^2$ (since $t \neq 0$):

$$\frac{d^2 t}{dx^2} - 2 \frac{dt}{dx} + 4t^2 = 0$$

Let $u = \frac{dt}{dx}$

$$\frac{du}{dx} - 2u + 4t^2 = 0$$

Let $v = u - 2t^2$

$$\frac{dv}{dx} + 4t^2 = 0$$

Let $w = 4t^2$

$$\frac{dw}{dx} + 2w = 0$$

Separate variables:

$$\frac{dw}{w} = -2 dx$$

Integrate:

$$\ln w = -2x + C$$

$$w = Ae^{-2x}$$

Since $w = 4t^2$:

$$4t^2 = Ae^{-2x}$$

$$t^2 = \frac{A}{4} e^{-2x}$$

Let $A/4 = B^2$:

$$t = B e^{-x}$$

Since $t = \sqrt{y}$:

$$\sqrt{y} = B e^{-x}$$

$$y = B^2 e^{-2x}$$

General solution:

$$y = \left(Ae^{2x} + Bxe^{2x} \right)^2$$

Question 2 (*)**

The differential equation

$$x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 3x, \quad x \neq 0,$$

is to be solved subject to the boundary conditions $y = \frac{3}{2}, \frac{dy}{dx} = \frac{1}{2}$ at $x=1$.

- a) Show that the substitution $v = \frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx} + \frac{2v}{x} = 3.$$

- b) Hence find the solution of the original differential equation, giving the answer in the form $y = f(x)$.

$$y = \frac{1}{2} \left(x^2 + \frac{1}{x} + 1 \right)$$

(a) $x \frac{dv}{dx} + 2v = 3x$
 $\frac{dv}{dx} + \frac{2v}{x} = 3$

(b) $\frac{dv}{dx} + \frac{2v}{x} = 3$
 $\int \frac{1}{x^2} dv = \int -\frac{3}{x} dx$
 $\ln|v| = -2\ln|x| + C$
 $\ln|v| = \ln\left|\frac{A}{x^2}\right|$
 $v = \frac{A}{x^2}$
 $\therefore v = \frac{A}{x^2} + 3x$ (or do it by integrating factor)
 $\Rightarrow \frac{dv}{dx} = -\frac{2A}{x^3} + 3$
 $\Rightarrow y = -\frac{A}{x^2} + \frac{3}{2}x^2 + B$
 • Apply condition $x=1, \frac{dy}{dx} = \frac{1}{2} \Rightarrow \frac{1}{2} = -\frac{A}{1^2} + \frac{3}{2}(1)^2$
 $\Rightarrow \frac{1}{2} = -A + \frac{3}{2}$
 $\Rightarrow A = 1$
 • Apply condition $x=1, y = \frac{3}{2} \Rightarrow \frac{3}{2} = -\frac{1}{1^2} + \frac{3}{2}(1)^2 + B$
 $\Rightarrow \frac{3}{2} = -1 + \frac{3}{2} + B$
 $\Rightarrow B = \frac{1}{2}$
 $\therefore y = -\frac{1}{x^2} + \frac{3}{2}x^2 + \frac{1}{2}$
 $y = \frac{1}{2} \left(x^2 + \frac{1}{x^2} + 1 \right)$

Question 3 (*)**

The curve C has equation $y = f(x)$ and satisfies the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3 e^x, \quad x \neq 0$$

is to be solved subject to the boundary conditions $y = \frac{3}{2}, \frac{dy}{dx} = \frac{1}{2}$ at $x = 1$.

- a) Show that the substitution $y = xv$, where v is a function of x transforms the above differential equation into

$$\frac{d^2 v}{dx^2} - 4v = 3e^x.$$

It is further given that C meets the x axis at $x = \ln 2$ and has a finite value for y as x gets infinitely negatively large.

- b) Express the equation of C in the form $y = f(x)$.

$$y = \frac{1}{2} x e^{2x} - x e^x$$

Handwritten solution for Question 3b:

a) $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3 e^x$
 SUBSTITUTION: $y = xv$
 $\frac{dy}{dx} = v + x \frac{dv}{dx}$
 $\frac{d^2 y}{dx^2} = \frac{dv}{dx} + 2v$
 $x^2 (\frac{dv}{dx} + 2v) - 2x(v + x \frac{dv}{dx}) - 2(xv)(2x^2 - 1) = 3x^3 e^x$
 $x^2 \frac{dv}{dx} + 2x^2 v - 2xv - 2x^2 \frac{dv}{dx} - 4x^3 v + 2x^2 v = 3x^3 e^x$
 $-x^2 \frac{dv}{dx} - 4x^3 v = 3x^3 e^x$
 $\frac{dv}{dx} - 4v = 3e^x$ (As required)

b) Homogeneous Solution
 $\lambda^2 - 4 = 0$
 $\lambda = \pm 2$
 $y_h = A e^{2x} + B e^{-2x}$

Particular Solution
 Try $y = P e^x$
 $\frac{dy}{dx} = P e^x$
 Sub into the ODE:
 $P e^x - 4P e^x = 3e^x$
 $-3P = 3$
 $P = -1$

General Solution
 $y = A e^{2x} + B e^{-2x} - e^x$

Boundary Conditions
 Solution is finite as $x \rightarrow -\infty \Rightarrow B = 0$
 $y = A e^{2x} - e^x$
 Curve crosses the x-axis at $x = \ln 2$
 $0 = A \ln 2 e^{2 \ln 2} - e^{\ln 2}$
 $0 = 4A \ln 2 - 2$
 $0 = 2A - 1$
 $A = \frac{1}{2}$
 $\therefore y = \frac{1}{2} x e^{2x} - x e^x$

Question 4 (***)

The differential equation

$$(x^3 + 1) \frac{d^2 y}{dx^2} - 3x^2 \frac{dy}{dx} = 2 - 4x^3,$$

is to be solved subject to the boundary conditions $y = 0, \frac{dy}{dx} = 4$ at $x = 0$.

Use the substitution $u = \frac{dy}{dx} - 2x$, where u is a function of x , to show that the solution of the above differential equation is

$$y = x^4 + x^2 + 4x.$$

$\frac{dy}{dx} = 4$

,

proof

USING THE SUBSTITUTION GIVEN

$$\Rightarrow u = \frac{dy}{dx} - 2x$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx} \left(\frac{dy}{dx} - 2x \right) = \frac{d^2 y}{dx^2} - 2$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{du}{dx} + 2$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow (x^3 + 1) \left(\frac{du}{dx} + 2 \right) - 3x^2 \frac{du}{dx} = 2 - 4x^3$$

$$\Rightarrow (x^3 + 1) \frac{du}{dx} + 2x^3 + 2 - 3x^2 \frac{du}{dx} = 2 - 4x^3$$

$$\Rightarrow (x^3 + 1) \frac{du}{dx} - 3x^2 \frac{du}{dx} = 2 - 4x^3 - 2x^3 - 2$$

$$\Rightarrow (x^3 + 1) \frac{du}{dx} - 3x^2 \frac{du}{dx} = -4x^3$$

$$\Rightarrow (x^3 + 1) \frac{du}{dx} = 3x^2 \frac{du}{dx} - 4x^3$$

SEPARATE VARIABLES

$$\Rightarrow \frac{1}{u} du = \frac{3x^2}{x^3 + 1} dx$$

$$\Rightarrow \int \frac{1}{u} du = \int \frac{3x^2}{x^3 + 1} dx$$

$$\Rightarrow \ln|u| = \ln|x^3 + 1| + \ln A$$

$$\Rightarrow |u| = A|x^3 + 1|$$

$$\Rightarrow u = A(x^3 + 1)$$

REVERSING THE TRANSFORMATION

$$\Rightarrow \frac{du}{dx} - 2x = A(x^3 + 1)$$

$$\Rightarrow \frac{du}{dx} = A(x^3 + 1) + 2x$$

INTEGRATING W.R.T. x

$$\Rightarrow u = A \left(\frac{x^4}{4} + 2x \right) + x^2 + B$$

USING THE CONDITION GIVEN

$$x=0, u=0 \Rightarrow 0 = B$$

$$x=0, \frac{du}{dx} = 4 \Rightarrow 4 = A$$

$$\therefore u = 4 \left(\frac{x^4}{4} + 2x \right) + x^2$$

$$u = x^4 + 8x + x^2$$

$$u = \frac{dy}{dx} - 2x$$

$$\frac{dy}{dx} = x^4 + 8x + x^2 + 2x$$

$$\frac{dy}{dx} = x^4 + x^2 + 4x$$

Question 5 (***)

$$\frac{d^2 y}{dx^2} - (1 - 6e^x) \frac{dy}{dx} + 10ye^{2x} = 5e^{2x} \sin(2e^x).$$

- a) By using the substitution $x = \ln t$ or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 10y = 5 \sin 2t.$$

- b) Hence find a general solution for the original differential equation.

$$\boxed{}, \quad y = e^{-3e^x} \left[A \cos(e^x) + B \sin(e^x) \right] + \frac{1}{6} \sin(2e^x) - \frac{1}{3} \cos(2e^x)$$

a) START BY SUBSTITUTION 'REPLACEMENTS' FOR $\frac{dy}{dx}$ & $\frac{d^2 y}{dx^2}$

- $x = \ln t$
DIFFERENTIATE W.R.T y
 $\Rightarrow \frac{dy}{dx} = \frac{1}{t} \frac{dy}{dt}$
 $\Rightarrow \frac{dy}{dx} = t \frac{dy}{dt}$
- ALSO NOTE THAT
 $x = \ln t$
 $\frac{dx}{dt} = \frac{1}{t}$
 $\frac{dy}{dx} = t$

DIFFERENTIATE W.R.T x
 $\Rightarrow \frac{d^2 y}{dx^2} = \frac{dy}{dt} \frac{dt}{dx} + t \frac{d}{dt} \left(\frac{dy}{dt} \right)$
 $\Rightarrow \frac{d^2 y}{dx^2} = \frac{dy}{dt} + t \frac{d^2 y}{dt^2}$
 $\Rightarrow \frac{d^2 y}{dx^2} = t \frac{dy}{dt} + \frac{d^2 y}{dt^2} \cdot t^2$

SUBSTITUTING INTO THE O.D.E. AND SIMPLIFY, NOTING
FURTHER THAT $e^x = t$

$\Rightarrow \frac{d^2 y}{dt^2} + (1 - 6t) \frac{dy}{dt} + 10y = 5t^2 \sin(2t)$
 $\Rightarrow \left[t \frac{dy}{dt} + t \frac{d^2 y}{dt^2} \right] - (1 - 6t) \left(t \frac{dy}{dt} \right) + 10y = 5t^2 \sin(2t)$
 $\Rightarrow t \frac{d^2 y}{dt^2} + t \frac{dy}{dt} - t \frac{dy}{dt} + 6t^2 \frac{dy}{dt} + 10ty = 5t^2 \sin(2t)$
 $\Rightarrow t \frac{d^2 y}{dt^2} + 6t^2 \frac{dy}{dt} + 10ty = 5t^2 \sin(2t)$
 $\Rightarrow \frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 10y = 5 \sin(2t)$

- AS REQUIRED

b) SOLVING THE TRANSFORMED EQUATION

- AUXILIARY EQUATION
 $\Rightarrow \lambda^2 + 6\lambda + 10 = 0$
 $\Rightarrow (2+3)^2 - 9 + 10 = 0$
 $\Rightarrow (2+3)^2 = -1$
 $\Rightarrow 2+3 = \pm i$
 $\Rightarrow \lambda = -3 \pm i$
COMPLEMENTARY FUNCTION
 $y = e^{-3t} (A \cos t + B \sin t)$
- PARTICULAR INTEGRAL
 $y = P \cos 2t + Q \sin 2t$
 $\ddot{y} = -2P \cos 2t + 2Q \sin 2t$
 $\dot{y} = -4P \sin 2t + 4Q \cos 2t$
SUB INTO THE O.D.E.
 $\ddot{y} = -4P \sin 2t + 4Q \cos 2t$
 $-16P \cos 2t + 16Q \sin 2t$
 $+10y = 10P \cos 2t + 10Q \sin 2t$
Equate and compare
 $(-4P + 10Q) \cos 2t = 5 \cos 2t$
 $(16Q - 10P) \sin 2t = 5 \sin 2t$
 $\begin{cases} -4P + 10Q = 5 \\ 16Q - 10P = 5 \end{cases} \Rightarrow P = -2P$
 $6Q + 21Q = 5 \Rightarrow 27Q = 5$
 $\Rightarrow Q = \frac{5}{27}$
 $\Rightarrow P = -\frac{5}{27}$

HENCE THE PARTICULAR SOLUTION CAN BE FOUND

$\Rightarrow y = e^{-3t} (A \cos t + B \sin t) - \frac{5}{27} \cos 2t + \frac{5}{27} \sin 2t$
 $\Rightarrow y = e^{-3e^x} [A \cos(e^x) + B \sin(e^x)] - \frac{5}{27} \cos(2e^x) + \frac{5}{27} \sin(2e^x)$

Question 6 (***)

Solve the differential equation

$$x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0,$$

subject to the boundary conditions $y = 2$, $\frac{dy}{dx} = -1$ at $x = 1$.

$$y = \frac{2e^{2x}}{e^{2x} + x}$$

$x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0$ subject to $y = 2$, $\frac{dy}{dx} = -1$ at $x = 1$

FIRST METHOD - REDUCE THE ORDER
 Let $p = \frac{dy}{dx}$
 $\rightarrow x \frac{d}{dx} \left(\frac{dy}{dx} \right) + 2 \left(\frac{dy}{dx} \right) = 0$
 $\rightarrow x \frac{dp}{dx} + 2p = 0$
 $\rightarrow x \frac{dp}{dx} = -2p$
 $\rightarrow \left[\frac{1}{p} dp \right] = \left[-\frac{2}{x} dx \right]$
 $\rightarrow \ln p = -2 \ln x + \ln C$
 $\rightarrow \ln p = \ln \left(\frac{C}{x^2} \right)$
 $\rightarrow p = \frac{C}{x^2}$
 $\rightarrow \frac{dy}{dx} = \frac{C}{x^2}$
 $y = \frac{A}{x} + B$
 At $x=1, y=2 \rightarrow 2 = A + B$
 $y' = -\frac{A}{x^2}$
 At $x=1, y'=-1 \rightarrow -1 = -\frac{A}{1^2}$
 $A = 1$
 $B = 1$
 $\therefore y = \frac{1}{x} + 1$

SECOND METHOD - BY INSPECTION
 TRY SOLUTION $y = x^2$
 $y' = 2x^{2-1}$
 $y' = 2(x-1)x^{2-2}$
 SEE INTO THE O.D.E.
 $2(x-1)x^{2-1} + 2x^{2-1} = 0$
 $[2(x-1) + 2]x^{2-1} = 0$
 $2x + 2 = 0$
 $2(x+1) = 0$
 $x = -1$
GEN. SOLUTION
 $y = p x^2 + q x^{-1}$
 $y = p + \frac{q}{x}$
 APPLY CONDITIONS AT BOUNDARY

Question 7 (***)

$$x \frac{d^2 y}{dx^2} + (6x + 2) \frac{dy}{dx} + 9xy = 27x - 6y.$$

Use the substitution $u = xy$, where u is a function of x , to find a general solution of the above differential equation.

$$\boxed{}, \quad y = \frac{A}{x} e^{-3x} + B e^{-3x} + 3 - \frac{2}{x}$$

(Using the substitution) Given: $u(x) = 2.96$

$$\frac{d}{dx}(u) = \frac{d}{dx}(xy)$$

$$\frac{du}{dx} = x \times \frac{dy}{dx} + 1 \times y$$

$$\frac{du}{dx} = x \frac{dy}{dx} + y$$

$$x \frac{dy}{dx} = \frac{du}{dx} - y$$

Differentiate the above equation with respect to x

$$\frac{d}{dx} \left[x \frac{dy}{dx} \right] = \frac{d}{dx} \left[\frac{du}{dx} - y \right]$$

$$1 \times \frac{dy}{dx} + x \frac{d^2 y}{dx^2} = \frac{d^2 u}{dx^2} - \frac{dy}{dx}$$

$$x \frac{d^2 y}{dx^2} = \frac{d^2 u}{dx^2} - 2 \frac{dy}{dx}$$

Transfer the r.h.s.

$$\Rightarrow x \frac{d^2 y}{dx^2} + (6x + 2) \frac{dy}{dx} + 9xy = 27x - 6y$$

$$\Rightarrow \frac{d^2 u}{dx^2} - 2 \frac{dy}{dx} + (6x + 2) \frac{dy}{dx} + 9xy = 27x - 6y$$

$$\Rightarrow \frac{d^2 u}{dx^2} + 6 \left(\frac{dy}{dx} - y \right) + 9y = 27x - 6y$$

$$\Rightarrow \frac{d^2 u}{dx^2} + 6 \frac{du}{dx} - 6y + 9y = 27x - 6y$$

$$\Rightarrow \frac{d^2 u}{dx^2} + 6 \frac{du}{dx} + 3u = 27x$$

The auxiliary equation for the LHS is

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3$$

Complementary function

$$u = A e^{-3x} + B x e^{-3x}$$

Particular integral by inspection

$$u = Px + Q$$

$$u' = P$$

$$u'' = 0$$

$$\therefore 0 + 6P + 9(Px + Q) = 27x$$

$$(6P + 9Q) + 9Px = 27x$$

$$P = 3 \quad \begin{cases} 6P + 9Q = 0 \\ 18 + 9Q = 0 \\ Q = -2 \end{cases}$$

Thus we have

$$u(x) = (A + Bx) e^{-3x} + 3x - 2$$

Reversing the transformation

$$xy = (A + Bx) e^{-3x} + 3x - 2$$

$$y = \left(\frac{A}{x} + B \right) e^{-3x} + 3 - \frac{2}{x}$$

Question 8 (***)

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} \tan x - y \sec^4 x = 0.$$

The above differential equation is to be solved by a substitution.

a) If $t = \tan x$ show that ...

i. ... $\frac{dy}{dx} = \frac{dy}{dt} \sec^2 x$

ii. ... $\frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \sec^4 x + 2 \frac{dy}{dt} \sec^2 x \tan x$

b) Use the results obtained in part (a) to find a general solution of the differential equation in the form $y = f(x)$.

$$y = Ae^{\tan x} + Be^{-\tan x}$$

97 DIFFERENTIATING WITH RESPECT TO y

$$t = \tan x \Rightarrow \frac{dt}{dx} = \frac{d}{dx}(\tan x)$$

$$\Rightarrow \frac{dt}{dx} = \sec^2 x = \frac{dy}{dy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 x} \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x \frac{dy}{dt}$$

100 DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\sec^2 x \frac{dy}{dt} \right)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 2 \sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2 y}{dt^2}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 2 \sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2 y}{dt^2}$$

BUT IF $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 2 \sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2 y}{dt^2}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \sec^2 x + 2 \frac{dy}{dt} \sec^2 x \tan x$$

101 TRANSFORMING THE GIVEN O.D.E.

$$\Rightarrow \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} \tan x - y \sec^4 x = 0$$

$$\Rightarrow \left(\frac{d^2 y}{dt^2} \sec^2 x + 2 \frac{dy}{dt} \sec^2 x \tan x \right) - 2 \left(\sec^2 x \frac{dy}{dt} \right) \tan x - y \sec^4 x = 0$$

102 SOLVING THE QUADRATIC

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

GENERAL SOLUTION IS

$$y = Ae^t + Be^{-t} \quad \text{or} \quad y = \text{Pseud} + \text{Quint}$$

$$y = Ae^{\tan x} + Be^{-\tan x} \quad \text{or} \quad y = \text{Pseud}(\tan x) + \text{Quint}(\tan x)$$

Question 9 (***)

Show clearly that the substitution $z = \sin x$, transforms the differential equation

$$\frac{d^2 y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2 \cos^5 x,$$

into the differential equation

$$\frac{d^2 y}{dz^2} - 2y = 2(1 - z^2)$$

proof

Handwritten proof showing the transformation of the differential equation from x to $z = \sin x$.

Given: $z = \sin x$

Step 1: Find $\frac{dy}{dz}$ and $\frac{d^2 y}{dz^2}$ in terms of x .

Using the chain rule:

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{dy}{dx} \cdot \frac{1}{\cos x}$$

$$\frac{d^2 y}{dz^2} = \frac{d}{dz} \left(\frac{dy}{dz} \right) = \frac{d}{dx} \left(\frac{dy}{dz} \right) \cdot \frac{dx}{dz} = \frac{d}{dx} \left(\frac{dy}{dx} \cdot \frac{1}{\cos x} \right) \cdot \frac{1}{\cos x}$$

Step 2: Substitute into the original equation.

Original equation:

$$\frac{d^2 y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2 \cos^5 x$$

Substitute $\frac{d^2 y}{dx^2} = \cos x \frac{d^2 y}{dz^2} - \sin x \frac{dy}{dz}$ and $\frac{dy}{dx} = \frac{dy}{dz} \cos x$:

$$(\cos x \frac{d^2 y}{dz^2} - \sin x \frac{dy}{dz}) \cos x + (\frac{dy}{dz} \cos x) \sin x - 2y \cos^3 x = 2 \cos^5 x$$

Simplify:

$$\cos^2 x \frac{d^2 y}{dz^2} - \sin^2 x \frac{dy}{dz} + \sin x \cos x \frac{dy}{dz} - 2y \cos^3 x = 2 \cos^5 x$$

$$\cos^2 x \frac{d^2 y}{dz^2} - \cos^3 x \frac{dy}{dz} - 2y \cos^3 x = 2 \cos^5 x$$

Divide through by $\cos^3 x$:

$$\frac{d^2 y}{dz^2} - 2y = 2(1 - z^2)$$

As required.

Question 10 (***)

By using the substitution $z = \frac{dy}{dx}$, or otherwise, solve the differential equation

$$(x^2 + 1) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 6x^2 + 2,$$

subject to the conditions $x = 0$, $y = 2$, $\frac{dy}{dx} = 1$

$$y = x^2 + 2 + \arctan x$$

Handwritten solution for Question 10:

Let $z = \frac{dy}{dx}$

Then $(x^2 + 1) \frac{dz}{dx} + 2xz = 6x^2 + 2$

$\frac{dz}{dx} + \frac{2z}{x^2 + 1} = \frac{6x^2 + 2}{x^2 + 1}$

Integrating factor $I.F. = e^{\int \frac{2}{x^2 + 1} dx} = e^{2 \arctan x} = x^2 + 1$

Multiplying through by $I.F.$:

$$\frac{d}{dx} (z(x^2 + 1)) = \frac{6x^2 + 2}{x^2 + 1} (x^2 + 1)$$

$$z(x^2 + 1) = \int (6x^2 + 2) dx$$

$$z(x^2 + 1) = 2x^3 + 2x + C$$

When $x = 0$, $\frac{dy}{dx} = 1$

$$1 = \frac{C}{1} \Rightarrow C = 1$$

$\therefore z(x^2 + 1) = 2x^3 + 2x + 1$

$\Rightarrow z = \frac{2x^3 + 2x + 1}{x^2 + 1}$

$\Rightarrow \frac{dy}{dx} = \frac{2x^3 + 2x + 1}{x^2 + 1}$

$\Rightarrow y = \int \frac{2x^3 + 2x + 1}{x^2 + 1} dx$

$\Rightarrow y = \int \frac{2x^3 + 2x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx$

$\Rightarrow y = \int 2x dx + \int \frac{1}{x^2 + 1} dx$

$\Rightarrow y = x^2 + \arctan x + D$

Apply condition $x = 0, y = 2$

$$2 = 0 + 0 + D$$

$\therefore D = 2$

$\therefore y = x^2 + \arctan x + 2$

Question 11 (****)

Use the substitution $z = \sqrt{y}$, where $y = f(x)$, to solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0,$$

subject to the boundary conditions $y = 4, \frac{dy}{dx} = 44$ at $x = 0$.

Give the answer in the form $y = f(x)$.

$$y = 9e^{6x} - 6e^x + e^{-4x}$$

$$2y \frac{\partial y}{\partial x_1} - \frac{\partial y}{\partial x_1} [2y + \frac{\partial y}{\partial x_1}] - 2y^2 = 0$$

$\bullet y = z^2 \text{ od } e = y^4$
 $\bullet \frac{\partial y}{\partial x_1} = 2z \frac{\partial z}{\partial x_1}$
 $\bullet \frac{\partial y}{\partial x_1} = 2 \frac{\partial z^2}{\partial x_1}, \text{ ze } \frac{\partial z}{\partial x_1}$

$\Rightarrow 2z^2 \left[2 \frac{\partial z}{\partial x_1} + 2z \frac{\partial z}{\partial x_1} \right] - 2z^4 \left[2z^2 + 2z \frac{\partial z}{\partial x_1} \right] - 2z^6 = 0$
 $= 4z^2 \frac{\partial z}{\partial x_1} + 4z^3 \frac{\partial z}{\partial x_1} - 4z^6 - 4z^5 \frac{\partial z}{\partial x_1} - 2z^6 = 0$
 $\Rightarrow \frac{4z^2}{\partial x_1} + \frac{4z^3}{\partial x_1} - 6z^6 = 0$

$\Rightarrow \frac{4z^2}{\partial x_1} + \frac{4z^3}{\partial x_1} - 6z^6 = 0$
 $\Rightarrow \frac{4z^2}{\partial x_1} + \frac{4z^3}{\partial x_1} - 6z^6 = 0$

AN(LINE) (LINEAR)

$2x - y - 6 = 0$

$(1-3)(1+2) = 0$

$1 = 0 \Rightarrow$

AN(E)

$= 4e^{2x} + 6e^{2x}$

$y^4 = 4e^x + 6e^{2x}$

$q^2 + 4e^x + 6e^{2x}$

$\frac{d}{dx} \text{ wrt } x$

$\frac{d}{dx} (q^2 + 4e^x + 6e^{2x})$

$2eq, y=4, \frac{dy}{dx} = 4$

$2 = A+B$

$\frac{1}{2} \cdot 4 \cdot 4 = 3A - 2B$

$\Rightarrow \begin{cases} A+B=2 \\ 3A-2B=11 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=-1 \end{cases}$

$\therefore y = 3e^{2x} - e^{2x}$

$y = (3e^{2x} - e^{2x})$

$y = 9e^{2x} - e^{2x}$

Question 12 (****)

$$2x \frac{d^2 y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0.$$

The above differential equation is to be solved by a substitution.

a) Given that $y = f(x)$ and $t = x^{\frac{1}{2}}$, show clearly that ...

i. ... $\frac{dy}{dx} = \frac{1}{2t} \frac{dy}{dt}.$

ii. ... $\frac{d^2 y}{dx^2} = \frac{1}{4t^2} \frac{d^2 y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt}.$

b) Hence show further that the differential equation

$$2x \frac{d^2 y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0,$$

can be transformed to the differential equation

$$\frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = 0.$$

c) Find a general solution of the **original** differential equation, giving the answer in the form $y = f(x)$.

$$y = Ae^{\sqrt{x}} + Be^{2\sqrt{x}}$$

Handwritten solution for Question 12c:

a) $t = x^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2t}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{2t} \frac{dy}{dt}$
 $\Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{4t^2} \frac{d^2 y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt}$

b) Substituting into the original equation:
 $2x \left(\frac{1}{4t^2} \frac{d^2 y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt} \right) + \left(1 - 3x^{\frac{1}{2}} \right) \frac{1}{2t} \frac{dy}{dt} + y = 0$
 $\Rightarrow \frac{1}{2t} \frac{d^2 y}{dt^2} - \frac{1}{2t^2} \frac{dy}{dt} + \left(1 - \frac{3}{2t} \right) \frac{dy}{dt} + y = 0$
 $\Rightarrow \frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = 0$

c) Aux equation:
 $\lambda^2 - 3\lambda + 2 = 0$
 $(\lambda - 2)(\lambda - 1) = 0$
 $\lambda = 1, 2$
 $\therefore y = Ae^t + Be^{2t}$
 $\text{But } t = x^{\frac{1}{2}} = \sqrt{x}$
 $y = Ae^{\sqrt{x}} + Be^{2\sqrt{x}}$

Question 13 (****)

Show clearly that the substitution $z = y^2$, where $y = f(x)$, transforms the differential equation

$$\frac{d^2 y}{dx^2} + \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2 z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0$$

proof

Handwritten proof showing the transformation of the differential equation using the substitution $z = y^2$.

Given: $z = y^2$

Diff wrt x :

$$\frac{dz}{dx} = 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y} \frac{dz}{dx}$$

Diff wrt x again:

$$\frac{d^2 z}{dx^2} = 2y \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} \frac{dz}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{2y} \frac{d^2 z}{dx^2} - \frac{1}{y^2} \left(\frac{dz}{dx} \right)^2$$

Substitute into the original equation:

$$\left(\frac{1}{2y} \frac{d^2 z}{dx^2} - \frac{1}{y^2} \left(\frac{dz}{dx} \right)^2 \right) + \frac{1}{y} \left(\frac{1}{2y} \frac{dz}{dx} \right)^2 - 5 \left(\frac{1}{2y} \frac{dz}{dx} \right) + 2y = 0$$

$$\Rightarrow \frac{1}{2y} \frac{d^2 z}{dx^2} - \frac{1}{y^2} \left(\frac{dz}{dx} \right)^2 + \frac{1}{4y^3} \left(\frac{dz}{dx} \right)^2 - \frac{5}{2y} \frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{1}{2y} \frac{d^2 z}{dx^2} - \frac{4}{4y^3} \left(\frac{dz}{dx} \right)^2 + \frac{1}{4y^3} \left(\frac{dz}{dx} \right)^2 - \frac{5}{2y} \frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{1}{2y} \frac{d^2 z}{dx^2} - \frac{3}{4y^3} \left(\frac{dz}{dx} \right)^2 - \frac{5}{2y} \frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{d^2 z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0$$

As required

Given that if $x = t^{\frac{1}{2}}$, where $y = f(x)$, show clearly that

b) $\frac{d^2 y}{dx^2} = 4t \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt}.$

$$x \frac{d^2 y}{dx^2} - (8x^2 + 1) \frac{dy}{dx} + 12x^3 y = 12x^5,$$

c) Show further that the substitution $x = t^{\frac{1}{2}}$, where $y = f(x)$, transforms the above differential equation into the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t.$$

$$y = e^{3x^2} + e^{x^2} + x^2 + \frac{4}{3}.$$

(a) $z = t^2$
 Diff w.r.t. y
 $\Rightarrow \frac{dz}{dy} = \frac{1}{2} t^{-\frac{1}{2}} \cdot \frac{dt}{dy}$
 $\Rightarrow \frac{dz}{dy} = \frac{1}{2t^{\frac{1}{2}}} \cdot \frac{dt}{dy}$
 $\Rightarrow \frac{dz}{dz} = 2t^{\frac{1}{2}} \frac{dz}{dt}$
 // as $\frac{dz}{dz} = 1$

(b) Diff about answer w.r.t z
 $\Rightarrow \frac{dz}{dz} = t^{-\frac{1}{2}} \frac{dz}{dt} + 2t^{\frac{1}{2}} \frac{dz}{dt} \frac{dt}{dz}$
 $z = t^2 \Rightarrow \frac{dz}{dz} = \frac{1}{2} t^{-\frac{1}{2}} \cdot \frac{dt}{dz}$
 $\Rightarrow \frac{dz}{dz} = t^{-\frac{1}{2}} \cdot 2t^{\frac{1}{2}} \frac{dz}{dt} + 2t^{\frac{1}{2}} \cdot \frac{1}{2t^{\frac{1}{2}}} \cdot \frac{dt}{dz} \cdot 2t^{\frac{1}{2}}$
 $\Rightarrow \frac{dz}{dz} = 2 \frac{dz}{dt} + 4t \frac{dz}{dz}$
 // as $\frac{dz}{dz} = 1$

(c) $2 \frac{dz}{dz} = (8t^2 + 1) \frac{dz}{dt} + 12t^{\frac{3}{2}}$
 $\Rightarrow t^{\frac{1}{2}} \left[4t \frac{dz}{dt} + 2 \frac{dz}{dt} \right] = (8t + 1) \times 2t^{\frac{3}{2}} \frac{dz}{dt} + 12t^{\frac{3}{2}}$
 // as $\frac{dz}{dz} = 1$
 $\Rightarrow 4t \frac{dz}{dt} + 2 \frac{dz}{dt} = (16t + 2) \frac{dz}{dt} + 12t^{\frac{3}{2}}$
 $\Rightarrow 4t \frac{dz}{dt} + 2 \frac{dz}{dt} - 16t \frac{dz}{dt} - 2 \frac{dz}{dt} = 12t^{\frac{3}{2}}$
 $\Rightarrow 4t \frac{dz}{dt} - 16t \frac{dz}{dt} + 12t^{\frac{3}{2}} = 12t^{\frac{3}{2}}$
 $\Rightarrow \frac{dz}{dt} - 4 \frac{dz}{dt} + 3y = 3t$ // as $\frac{dz}{dz} = 1$

(d) 1st Equation
 $y^2 - 4y + 3 = 0$
 $(y-3)(y-1) = 0$
 $y = 1, 3$
2nd Equation
 $y = Ae^{3t} + Be^{-3t}$

For $y = 1$
 $1 = Ae^{3t} + Be^{-3t}$
 $1 = A + B$
 $A + B = 1$

For $y = 3$
 $3 = Ae^{3t} + Be^{-3t}$
 $3 = A + B$
 $A + B = 3$

Solving these two equations
 $1 = A + B$
 $3 = A + B$
 $2 = 0$
 $A = 0, B = 1$
 $y = 1$

Solving these two equations
 $1 = A + B$
 $3 = A + B$
 $2 = 0$
 $A = 0, B = 1$
 $y = 1$

Question 16 (****)

$$x \frac{d^2 y}{dx^2} - \frac{dy}{dx} - x^3 y + x^5 = 0.$$

Use the substitution $x = z^{\frac{1}{2}}$, where $y = f(x)$, to find a general solution of the above differential equation.

$$\mathbf{V}, \quad \boxed{}, \quad y = Ae^{\frac{1}{2}x^2} + Be^{-\frac{1}{2}x^2} + x^2$$

[illegible]

$\Rightarrow \frac{1}{2} \frac{d^2}{dx^2} - y^2 + z^2 = 0$
 $\Rightarrow \frac{d^2 y}{dx^2} - y + z = 0$

ANALYTIC EQUATION FOR $y \frac{d^2 z}{dx^2} - z = 3$

$\lambda^2 - 1 = 0$
 $\lambda = \pm 1$
 $z = e^{\pm x}$

PERIODIC SOLUTION (BY INSPECTION)

$y = z$

GENERAL SOLUTION y, z

$y = A e^{ix} + B e^{-ix} + z$
 $y = A e^{ix} + B e^{-ix} + z^2$

$z^2 = z^2$
 $z^3 = z^3$

Use a suitable substitution to solve the differential equation

subject to the boundary conditions $y(1)=1, \frac{dy}{dx}(1)=3$

$$\boxed{}, \quad y = x^3 + (\ln x)^2$$

Created by T. Madas

Question 18 (****)

Use a suitable trigonometric substitution to solve the following differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0, \quad y(0)=1 \quad \frac{dy}{dx}(0)=4.$$

$$y = 3x - \cos(\arcsin x)$$

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad x=0, y=1, \frac{dy}{dx}=4$

LET $x = \sin \theta$ (or $\cos \theta$)

DIFFERENTIATE W.R.T x

$\frac{dy}{dx} = \cos \theta \frac{dy}{d\theta}$
 $\frac{dy}{dx} = \frac{1}{\cos \theta} \frac{dy}{d\theta}$
 $\frac{dy}{dx} = \sec \theta \frac{dy}{d\theta}$

$\frac{d^2y}{dx^2} = \sec \theta \frac{d}{d\theta} \left[\frac{dy}{d\theta} \cos \theta + \frac{dy}{d\theta} \sin \theta \right]$
 $\frac{d^2y}{dx^2} = \sec \theta \left[\frac{dy}{d\theta} \cos \theta + \frac{dy}{d\theta} \sin \theta \right]$
 $\frac{d^2y}{dx^2} = \sec \theta \left[\frac{dy}{d\theta} \cos \theta + \frac{dy}{d\theta} \sin \theta \right]$

SUB INTO THE O.D.E TO OBTAIN

$(1-\sin^2 \theta) \sec \theta \left[\frac{dy}{d\theta} \cos \theta + \frac{dy}{d\theta} \sin \theta \right] - \sin \theta \sec \theta \frac{dy}{d\theta} + y = 0$
 $\cos^2 \theta \sec \theta \left[\frac{dy}{d\theta} \cos \theta + \frac{dy}{d\theta} \sin \theta \right] - \sin \theta \sec \theta \frac{dy}{d\theta} + y = 0$
 $\cos^2 \theta \left[\frac{dy}{d\theta} \cos \theta + \frac{dy}{d\theta} \sin \theta \right] - \sin \theta \sec \theta \frac{dy}{d\theta} + y = 0$
 $\cos^2 \theta \frac{dy}{d\theta} + y = 0$

THIS IS A 'SIMPLE HARMONIC MOTION' O.D.E WITH GENERAL SOLUTION

$y = A \cos \theta + B \sin \theta$
 $y = A \cos(\arcsin x) + B \sin(\arcsin x)$
 $y = A \cos(\arcsin x) + Bx$

Question 19 (****)

$$4x \frac{d^2 y}{dx^2} + 4x \left(\frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} = 1.$$

By using the substitution $t = \sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y = A - \sqrt{x} + \ln \left[1 + B e^{2\sqrt{x}} \right],$$

where A and B are arbitrary constants.

, proof

Panel 1: Substitution and Derivation

Let $t = \sqrt{x}$ DIFFERENTIATE w.r.t. t

$\frac{dx}{dt} = 2t$ DIFFERENTIATE w.r.t. x

$\frac{dy}{dx} = \frac{1}{2t} \frac{dy}{dt}$ DIFFERENTIATE w.r.t. t

$\frac{d^2 y}{dx^2} = \frac{1}{2t} \frac{d}{dt} \left(\frac{1}{2t} \frac{dy}{dt} \right) = -\frac{1}{4t^2} \frac{dy}{dt} + \frac{1}{2t} \frac{d^2 y}{dt^2}$

Panel 2: Separation of Variables

$\Rightarrow \frac{d^2 y}{dt^2} + \left(\frac{dy}{dt} \right)^2 = 1$

$\Rightarrow \frac{d}{dt} \left(\frac{dy}{dt} \right) + \left(\frac{dy}{dt} \right)^2 = 1$

ANOTHER CRUCIAL SUBSTITUTION IS $z = \frac{dy}{dt}$

$\Rightarrow \frac{dz}{dt} + z^2 = 1$

$\Rightarrow \frac{dz}{1-z^2} = 1 dt$

$\Rightarrow \int \frac{1}{(1-z)(1+z)} dz = \int 1 dt$

$\Rightarrow \int \frac{\frac{1}{2}}{1-z} + \frac{\frac{1}{2}}{1+z} dz = \int 1 dt$

$\Rightarrow \frac{1}{2} \ln|1-z| + \frac{1}{2} \ln|1+z| = t + C$

$\Rightarrow \ln|1-z| + \ln|1+z| = 2t + C$

$\Rightarrow \ln|(1-z)(1+z)| = 2t + C$

$\Rightarrow (1-z)(1+z) = A e^{2t}$ (1 = e^0)

$\Rightarrow 1 - z^2 = A e^{2t}$

$\Rightarrow z^2 = 1 - A e^{2t}$

$\Rightarrow z = \pm \sqrt{1 - A e^{2t}}$ i.e. $\frac{dy}{dt} = \pm \sqrt{1 - A e^{2t}}$

Panel 3: Integration with Substitution

PROCEED BY INTEGRATION USING A SUBSTITUTION

$\Rightarrow y = \int \frac{1}{2e^{2t} + 1} dt$ Let $v = A e^{2t} + 1$

$\Rightarrow y = \int \frac{1}{v} \cdot \frac{dv}{2v} = \frac{1}{2} \int \frac{1}{v} dv$ $\frac{dv}{dt} = 2A e^{2t}$

$\Rightarrow y = \frac{1}{2} \ln|v| + B$ $\frac{dv}{dt} = \frac{dv}{2A e^{2t}}$

$\Rightarrow y = \frac{1}{2} \ln|A e^{2t} + 1| + B$

$\Rightarrow y = \frac{1}{2} \ln|A e^{2t} + 1| + B$

$\Rightarrow y = \frac{1}{2} \ln|A e^{2t} + 1| + B$

$\Rightarrow y = \frac{1}{2} \ln|A e^{2t} + 1| + B$

VARIOUS TYPES

Question 1 (***)

Find the general solution of the following differential equation.

$$\frac{d^4\psi}{dx^4} + 2\lambda \frac{d^2\psi}{dx^2} + \lambda^4\psi = 0.$$

$$\psi = A \cos \lambda x + B \sin \lambda x$$

$\frac{d^4\psi}{dx^4} + 2\lambda^2 \frac{d^2\psi}{dx^2} + \lambda^4\psi = 0$
 Auxiliary equation
 $w^4 + 2\lambda^2 w^2 + \lambda^4 = 0$
 $(w^2 + \lambda^2)^2 = 0$
 $w^2 + \lambda^2 = 0$
 $w^2 = -\lambda^2$
 $w = \pm \lambda i$
 $\therefore \psi = A \cos \lambda x + B \sin \lambda x$

Question 2 (***)

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 1,$$

given that $y = -\frac{1}{4}$ and $\frac{dy}{dx} = 1$ at $x = 0$, giving the answer in the form $y = f(x)$.

$$y = \frac{1}{2} \left[2x - e^{-2x} \right]$$

$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 1$
 Let $p = \frac{dy}{dx}$
 $\frac{dp}{dx} + 2p = 1$
 $\Rightarrow \frac{dp}{dx} = 1 - 2p$
 $\Rightarrow \frac{1}{1-2p} dp = 1 dx$
 $\Rightarrow \int \frac{-2}{1-2p} dp = \int 1 dx$
 $\Rightarrow \ln|1-2p| = -2x + C$
 $\Rightarrow 1-2p = e^{-2x+C}$
 $\Rightarrow 1-2p = A e^{-2x}$ ($A = e^C$)
 $\Rightarrow 1 + A e^{-2x} = 2p$
 $\Rightarrow p = \frac{1}{2} + B e^{-2x}$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{2} + B e^{-2x}$
 Integrate again
 $y = \frac{1}{2}x - \frac{1}{2}e^{-2x} + D$
 Apply condition
 $x=0, y = -\frac{1}{4}$
 $-\frac{1}{4} = -\frac{1}{2} + D$
 $D = 0$
 $\therefore y = \frac{1}{2}x - \frac{1}{2}e^{-2x}$
 $y = \frac{1}{2} [2x - e^{-2x}]$
 Apply condition
 $x=0, \frac{dy}{dx} = 1$
 $1 = \frac{1}{2} + B$
 $B = \frac{1}{2}$
 $\frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}e^{-2x}$

Question 3 (***)

Solve the differential equation

$$\frac{d^2 y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 = 1,$$

given that $y = 0$ and $\frac{dy}{dx} = \frac{1}{6}$ at $x = 0$, giving the answer in the form $y = f(x)$.

$$y = \frac{1}{4} \ln \left[\frac{1 + 2e^{4x}}{3} \right] - \frac{1}{2} x$$

Method 1: Substitution

Let $p = \frac{dy}{dx}$
 $\frac{dp}{dx} = \frac{d^2 y}{dx^2}$
 $\Rightarrow \frac{dp}{dx} + 4p^2 = 1$
 $\Rightarrow \frac{dp}{1 - 4p^2} = 1 \, dx$
 $\Rightarrow \frac{1}{(1-2p)(1+2p)} dp = 1 \, dx$
BY PARTIAL FRACTIONS
 $\Rightarrow \frac{1}{1-2p} + \frac{1}{1+2p} dp = 1 \, dx$
 $\Rightarrow \left(\frac{1}{1-2p} + \frac{1}{1+2p} \right) dp = 1 \, dx$
 $\Rightarrow \left(\ln \left| \frac{1+2p}{1-2p} \right| \right) = dx + C$
 $\Rightarrow \frac{1+2p}{1-2p} = Ae^x$ ($A = e^C$)

Method 2: Partial Fractions

THUS $y = \frac{1}{4} \int \frac{(u-1)-1}{u} \times \frac{1}{4(u-1)} du$
 $y = \frac{1}{8} \int \frac{u-2}{u(u-1)} du$
BY PARTIAL FRACTIONS
 $\Rightarrow y = \frac{1}{8} \int \left(\frac{2}{u} - \frac{1}{u-1} \right) du$
 $\Rightarrow y = \frac{1}{8} \left(2 \ln u - \ln |u-1| \right) + D$
 $\Rightarrow y = \frac{1}{8} \ln \left(\frac{u^2}{u-1} \right) + D$
 $\Rightarrow y = \frac{1}{8} \ln \left(\frac{(2e^{4x}+1)^2}{e^{4x}} \right) + D$
 $\Rightarrow y = \frac{1}{4} \ln \left(\frac{2e^{4x}+1}{e^{2x}} \right) + D$
 $\Rightarrow y = \frac{1}{4} \ln (2e^{4x}+1) - \frac{1}{2} x + D$
APPLY CONDITION $x=0, y=0$
 $0 = \frac{1}{4} \ln 3 + D$
 $D = -\frac{1}{4} \ln 3$
 $\therefore y = \frac{1}{4} \ln (2e^{4x}+1) - \frac{1}{2} x - \frac{1}{4} \ln 3$
 $y = \frac{1}{4} \ln \left(\frac{2e^{4x}+1}{3} \right) - \frac{1}{2} x$

Question 4 (***)

$$\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 1.$$

Given that $y = \frac{dy}{dx} = 0$ at $x = 0$, show that

$$y = -x + \ln \left[\frac{1}{2} (1 + e^{2x}) \right].$$

proof

Handwritten Solution:

Step 1: Reduce the O.D.E. into a first order A.S. equation

$$\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 1 \quad \text{subject to } y = \frac{dy}{dx} = 0 \text{ at } x=0$$

$$\Rightarrow \frac{d}{dx} \left[\frac{dy}{dx} \right] + \left[\frac{dy}{dx} \right]^2 = 1$$

$$\Rightarrow \frac{dp}{dx} + p^2 = 1 \quad \text{let } p = \frac{dy}{dx}$$

$$\Rightarrow \frac{dp}{dx} = 1 - p^2$$

Step 2: Separation of variables & integrate subject to $x=0, \frac{dy}{dx}=p=0$

$$\Rightarrow \frac{1}{1-p^2} dp = 1 dx$$

$$\Rightarrow \int_0^p \frac{1}{1-p^2} dp = \int_0^x 1 dx$$

$$\Rightarrow \int_0^p \frac{1}{(1+p)(1-p)} dp = \int_0^x 1 dx$$

Step 3: Partial fractions by inspection (look up) on the LHS

$$\Rightarrow \int_0^p \frac{\frac{1}{2}}{1+p} + \frac{\frac{1}{2}}{1-p} dp = \int_0^x 1 dx$$

$$\Rightarrow \int_0^p \frac{1}{1+p} + \frac{1}{1-p} dp = \int_0^x 2 dx$$

$$\Rightarrow [\ln|1+p| + \ln|1-p|]_0^p = [2x]_0^x$$

$$\Rightarrow [\ln|1+p| + \ln|1-p|] - [\ln|1+0| + \ln|1-0|] = 2x$$

Step 4: Substitution

$$\Rightarrow \ln \left(\frac{1+p}{1-p} \right) = 2x$$

$$\Rightarrow \frac{1+p}{1-p} = e^{2x}$$

$$\Rightarrow 1+p = e^{2x} - p e^{2x}$$

$$\Rightarrow p e^{2x} + p = e^{2x} - 1$$

$$\Rightarrow p(e^{2x} + 1) = e^{2x} - 1$$

$$\Rightarrow p = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow \int_0^y dy = \int_0^x \frac{e^{2x} - 1}{e^{2x} + 1} dx$$

$$\Rightarrow [y]_0^y = \int_0^x \frac{e^{2x} - 1}{e^{2x} + 1} dx$$

$$\Rightarrow y = \int_0^x \frac{e^{2x} - 1}{e^{2x} + 1} dx$$

Step 5: Variation of constants

$$\Rightarrow y = \ln(e^{2x} + 1) - \ln 2 - x$$

$$\Rightarrow y = \ln(e^{2x} + 1) - \ln 2 - x$$

$$\Rightarrow y = \ln \left[\frac{e^{2x} + 1}{2} \right] - x$$

Step 6: Variation of constants

$$\Rightarrow y = \ln \left[\frac{1}{2} (1 + e^{2x}) \right] - x$$

Step 7: Variation of constants

$$\Rightarrow y = \ln \left[\frac{1}{2} (1 + e^{2x}) \right] - x$$

Step 8: Variation of constants

$$\Rightarrow y = \ln \left[\frac{1}{2} (1 + e^{2x}) \right] - x$$

Question 5 (***)

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{2x-1} \left(1 - \frac{dy}{dx} \right), \quad y(0) = 1, \quad \frac{dy}{dx}(0) = -1.$$

Solve the above differential equation giving the answer in the form $y = f(x)$.

$$y = x + 1 + \ln|2x-1|$$

• START BY REWRITING THE O.D.E AS FOLLOWS

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{2x-1} \left(1 - \frac{dy}{dx} \right)$$

$$\Rightarrow (2x-1) \frac{d^2y}{dx^2} = 2 \left(1 - \frac{dy}{dx} \right)$$

$$\Rightarrow (2x-1) \frac{d^2y}{dx^2} = 2 - 2 \frac{dy}{dx}$$

$$\Rightarrow (2x-1) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 2$$

• BY INSPECTION THE L.H.S IS A PERFECT DIFFERENTIAL

$$\Rightarrow \frac{d}{dx} \left[(2x-1) \frac{dy}{dx} \right] = 2$$

$$\Rightarrow (2x-1) \frac{dy}{dx} = 2x + A$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+A}{2x-1}$$

• Now $\frac{dy}{dx} = -1$ AT $x=0$, GIVES $A=1$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+1}{2x-1}$$

$$\Rightarrow y = \int \frac{2x+1}{2x-1} dx$$

$$\Rightarrow y = \int \frac{(2x-1)+2}{2x-1} dx$$

$$\Rightarrow y = \int 1 + \frac{2}{2x-1} dx$$

$$\Rightarrow y = x + \ln|2x-1| + B$$

Now if $x=0, y=1 \Rightarrow B=1$

$$\therefore y = x + \ln|2x-1| + 1$$

Question 6 (****)

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2) y = 0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n = \frac{1}{2}$, is

$$y = x^{-\frac{1}{2}} [A \cosh x + B \sinh x].$$

proof

The image shows three pages of handwritten work for Question 6. The first page (left) shows the differential equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2) y = 0$ and the assumption of a Frobenius series $y = \sum_{r=0}^{\infty} a_r x^{r+p}$. It then shows the substitution into the equation and the resulting indicial equation $p(p-1) + p - n^2 = 0$, which simplifies to $p^2 - n^2 = 0$. For $n = \frac{1}{2}$, the roots are $p = \pm \frac{1}{2}$. The second page (middle) shows the recurrence relation for the coefficients a_r and the calculation of the first few terms for $p = \frac{1}{2}$. It shows that $a_1 = 0$, $a_2 = \frac{1}{2!} a_0$, $a_3 = 0$, $a_4 = \frac{1}{4!} a_0$, etc., leading to the series $y = \frac{1}{\sqrt{x}} \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) = \frac{1}{\sqrt{x}} \cosh x$. The third page (right) shows the calculation of the first few terms for $p = -\frac{1}{2}$. It shows that $a_1 = 0$, $a_2 = \frac{1}{2!} a_0$, $a_3 = 0$, $a_4 = \frac{1}{4!} a_0$, etc., leading to the series $y = \frac{1}{\sqrt{x}} \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) = \frac{1}{\sqrt{x}} \sinh x$. The final answer is $y = x^{-\frac{1}{2}} [A \cosh x + B \sinh x]$.

Question 7 (***)

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (3 - 4x^2)y = 0.$$

Give the final answer in terms of elementary functions.

$$y = \sqrt{x} (A \cosh x + B \sinh x)$$

The image shows three pages of handwritten work for Question 7, using the Frobenius method to solve the differential equation $4x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (3 - 4x^2)y = 0$.

Page 1 (Left): Assumes a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+p}$. It calculates the first and second derivatives, substitutes them into the differential equation, and simplifies to get a recurrence relation: $4a_n(n+p)(n+p-1) - 4a_n(n+p) + (3-4n^2-4np)a_n = 0$. This simplifies to $(n^2 - 2np + p^2 - 1)a_n = 0$. For $n=0$, it finds $p = \frac{1}{2}$. For $n=1$, it finds $a_1 = 0$. For $n=2$, it finds $a_2 = 0$. It concludes that the two solutions are not different by an integer.

Page 2 (Middle): Solves the recurrence relation for a_n . It finds $a_0 = \frac{1}{2}$ and $a_1 = 0$. It then finds $a_2 = \frac{1}{8}$, $a_3 = 0$, $a_4 = \frac{1}{64}$, and $a_5 = 0$. It identifies the pattern as $a_n = \frac{1}{2^n n!}$ for even n and $a_n = 0$ for odd n . It then writes the series solution $y = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{n!} = \frac{1}{2} e^x x^{1/2}$.

Page 3 (Right): Finds the second solution y_2 by assuming $y_2 = x^p \sum_{n=0}^{\infty} b_n x^n$. It finds $p = \frac{3}{2}$ and $b_0 = 1$. It then finds $b_1 = 0$, $b_2 = \frac{1}{8}$, $b_3 = 0$, $b_4 = \frac{1}{64}$, and $b_5 = 0$. It identifies the pattern as $b_n = \frac{1}{2^n n!}$ for even n and $b_n = 0$ for odd n . It then writes the series solution $y_2 = x^{3/2} \sum_{n=0}^{\infty} \frac{x^n}{2^n n!} = x^{3/2} e^x$. The final answer is $y = \sqrt{x} (A \cosh x + B \sinh x)$.

Question 8 (****)

Find the solution of following differential equation

$$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\frac{1}{2}\pi\right) = y'\left(-\frac{1}{2}\pi\right) = 0, \quad y''\left(-\frac{1}{2}\pi\right) = \frac{1}{2}.$$

Given the answer in the form $y = f(x)$.

$$\boxed{}, \quad y = 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{1}{4}\pi\right) \right|$$

BY SUBSTITUTION - LET $p = \frac{dy}{dx}$ & SEPARATE VARIABLES

$$\Rightarrow \frac{dp}{dx} \times \frac{dy}{dx} = \frac{d^3y}{dx^3}$$

$$\Rightarrow p \frac{dp}{dx} = \frac{d^2p}{dx^2}$$

$$\Rightarrow \int p \, dp = \int \frac{d^2p}{dx^2} \, dx$$

$$\Rightarrow \frac{1}{2}p^2 = \frac{d^2p}{dx^2} + A$$

$$\Rightarrow p^2 = 2\frac{d^2p}{dx^2} + A$$

APPLY CONDITION $x = -\frac{\pi}{2}$, $\frac{dy}{dx} = p = 0$, $\frac{d^2p}{dx^2} = \frac{d^3y}{dx^3} = \frac{1}{2}$

$$\Rightarrow 0 = 2 \times \frac{1}{2} + A$$

$$\Rightarrow A = -1$$

$$\Rightarrow p^2 = 2\frac{dp}{dx} - 1$$

REARRANGE & SEPARATE VARIABLES AGAIN

$$\Rightarrow p^2 + 1 = 2\frac{dp}{dx}$$

$$\Rightarrow 1 \, dx = \frac{2}{p^2+1} \, dp$$

$$\Rightarrow \int \frac{2}{p^2+1} \, dp = \int 1 \, dx$$

$$\rightarrow 2 \arctan p = x + B$$

$$\Rightarrow \arctan p = \frac{1}{2}x + B$$

$$\Rightarrow p = \tan\left(\frac{1}{2}x + B\right)$$

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + B\right)$$

APPLY THE BOUNDARY CONDITION $x = -\frac{\pi}{2}$, $\frac{dy}{dx} = 0$

$$\Rightarrow 0 = \tan\left(-\frac{\pi}{4} + B\right)$$

$$\Rightarrow B = \frac{\pi}{4} \quad \text{ONLY IN THIS WAY}$$

FACT CHECK FROM PREVIOUS

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$$

FINALLY WE HAVE BY DIRECT INTEGRATION

$$\frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$$

$$y = 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) \right| + C$$

APPLY THE LAST CONDITION, $x = -\frac{\pi}{2}$, $y = 0$

$$\Rightarrow 0 = 2 \ln \left| \sec\left(-\frac{\pi}{4} + \frac{\pi}{4}\right) \right| + C$$

$$\Rightarrow 0 = 2 \ln(\sec(0)) + C$$

$$\Rightarrow 0 = 2 \ln 1 + C$$

$$\Rightarrow C = 0$$

$$\therefore y = 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) \right|$$

Question 9 (****+)

A curve has a stationary point at $(-\frac{1}{2}, -\frac{1}{2})$.

The rate of change of the gradient function of the curve is given by

$$x + y + 2,$$

where $x + y + 2 > 0$.

Determine the equation of the curve, giving the answer in the form $y = f(x)$.

$$\boxed{y = e^{x+\frac{1}{2}} - x - 2}$$

$\frac{dy}{dx} = x+y+2$ $\frac{dy}{dx} = 0$ at $x = -\frac{1}{2}$ $y = -\frac{1}{2}$
 $\frac{dy}{dx} > 0$

START WITH AN OBVIOUS SUBSTITUTION

$\Rightarrow V = x+y+2$ WITH $V=1$ AT $(-\frac{1}{2}, -\frac{1}{2})$ $\frac{dy}{dx} = 0$
 $\Rightarrow \frac{dy}{dx} = 1 + \frac{dy}{dx}$ WITH $\frac{dy}{dx} = 1$ AT $(-\frac{1}{2}, -\frac{1}{2})$ $\frac{dy}{dx} = 0$
 $\Rightarrow \frac{dy}{dx} = \frac{dy}{dx}$

HENCE THE O.D.E TRANSFORMS FOR $V = V(x)$

$\frac{dy}{dx} = V$

THIS O.D.E HAS THE INDEPENDENT VARIABLE MISSING, SO WE PROCEED WITH THE STANDARD SUBSTITUTION

$p = \frac{dy}{dx}$

DIFFERENTIATE WITH RESPECT TO V

$\frac{dp}{dV} = \frac{d}{dV} \left(\frac{dy}{dx} \right) = \frac{dy}{dx} \cdot \frac{dx}{dV} = \frac{dy}{dx} \cdot \frac{1}{p} \Rightarrow \frac{dp}{dV} = p \frac{dp}{dV}$

TRANSFORM THE 2ND ORDER O.D.E TO A FIRST ORDER SEPARABLE

$\Rightarrow p \frac{dp}{dV} = V$
 $\Rightarrow p dp = V dV$

$\Rightarrow \frac{1}{2} p^2 = \frac{1}{2} V^2 + C$
 $\Rightarrow p^2 = V^2 + C$ WITH $V=1$ $p = \frac{dy}{dx} = 0$
 $\Rightarrow p = \pm V$ $\leftarrow V = x+y+2 = \frac{dy}{dx} > 0$
 $\Rightarrow \frac{dy}{dx} = V$
 $\Rightarrow \frac{1}{V} dV = 1 dx$

INTEGRATE SUBJECT TO THE CONDITION $x = -\frac{1}{2}, V=1$

$\Rightarrow \int \frac{1}{V} dV = \int 1 dx$
 $\Rightarrow \ln|V| = x + \frac{1}{2}$
 $\Rightarrow V = e^{x+\frac{1}{2}}$
 $\Rightarrow x+y+2 = e^{x+\frac{1}{2}}$
 $\Rightarrow y = e^{x+\frac{1}{2}} - x - 2$

Question 10 (****+)

Solve the following differential equation

$$y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} = 0, \quad y(0) = 2, \quad \frac{dy}{dx}(0) = -\frac{1}{2}.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{}, \quad \boxed{y^2 = 3 + e^{-2x}}$$

$y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} = 0 \quad x=0, y=2, \frac{dy}{dx} = -\frac{1}{2}$
 BEFORE WE ATTEMPT A SUBSTITUTION, WE CHECK THAT THE FORM OF THIS O.D.E. RESEMBLES DIFFERENTIALS
 $\frac{d}{dx} \left(y \frac{dy}{dx} \right) = \frac{dy}{dx} \times \frac{dy}{dx} + y \times \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} \right)^2 + y \frac{d^2 y}{dx^2}$
 $\frac{d}{dx} \left(y^2 \right) = 2y \frac{dy}{dx}$
 THEREFORE WE MAY REWRITE AS
 $\Rightarrow \frac{d}{dx} \left[y \frac{dy}{dx} + y^2 \right] = 0$
 $\Rightarrow y \frac{dy}{dx} + y^2 = C$
 APPLY CONDITION $y=2, \frac{dy}{dx} = -\frac{1}{2}$
 $\Rightarrow 2 \left(-\frac{1}{2} \right) + 2^2 = C$
 $\Rightarrow C = 3$
 $\Rightarrow y \frac{dy}{dx} + y^2 = 3$
 PROCEED BY SEPARATION OF VARIABLES
 $\Rightarrow y \frac{dy}{dx} = 3 - y^2$
 $\Rightarrow \frac{dy}{dx} = \frac{3 - y^2}{y}$
 $\Rightarrow \frac{y}{3 - y^2} dy = \frac{1}{x} dx$
 $\Rightarrow \int \frac{y}{3 - y^2} dy = \int \frac{1}{x} dx$

$\Rightarrow \ln|3 - y^2| = -2x + A$
 $\Rightarrow 3 - y^2 = e^{-2x + A}$
 $\Rightarrow 3 - y^2 = e^{-2x} e^A$
 $\Rightarrow 3 - y^2 = B e^{-2x}$
 $\Rightarrow y^2 = 3 + B e^{-2x}$
 APPLY FINAL CONDITION, $x=0, y=2$
 $\Rightarrow 4 = 3 + B e^0$
 $\Rightarrow B = 1$
 $\therefore y^2 = 3 + e^{-2x}$
 NOTE THAT THE SUBSTITUTION $y = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y \frac{dy}{dx}$ YIELDS
 $y \frac{dy}{dx} + y^2 + 2y \frac{dy}{dx} = 0$
 $\frac{dy}{dx} + \frac{y}{y} = -2$
 BY INTEGRATING FACTOR $y=2 \quad p = \frac{dy}{dx} = -\frac{1}{2}$ WE OBTAIN
 $\frac{dy}{dx} = -y + \frac{3}{y}$
 $\frac{dy}{dx} = \frac{3 - y^2}{y}$
 WHICH MATCHES WITH THE SOLUTION

Question 11 (****+)

By writing $\frac{dy}{dx} = p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \left(\frac{x^2 - y^2}{xy} \right) + 1.$$

Give the solution in the form $F(x, y)G(x, y) = 0$.

$$(xy + A)(x^2 - y^2 + B) = 0$$

Handwritten solution for Question 11:

Left side:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx} \left(\frac{x^2 - y^2}{xy} \right) + 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{dy}{dx} \left(\frac{x^2 - y^2}{xy} \right) + 1 \\ \Rightarrow p^2 &= \frac{x^2 - y^2}{xy} p + 1 \\ \Rightarrow p^2 - \frac{x^2 - y^2}{xy} p - 1 &= 0 \\ \Rightarrow xy p^2 - (x^2 - y^2)p - xy &= 0 \\ \Rightarrow (yp - x)(xp + y) &= 0 \\ \Rightarrow p &= \frac{x}{y} \text{ or } p = -\frac{y}{x} \\ \Rightarrow \frac{dy}{dx} &= \frac{x}{y} \text{ or } \frac{dy}{dx} = -\frac{y}{x} \\ \text{Thus } \frac{y dy}{y^2} &= \frac{x dx}{x^2} \\ \frac{1}{2} dy^2 &= \frac{1}{2} dx^2 \end{aligned}$$

Right side:

Solve $\frac{1}{2} y^2 = \frac{1}{2} x^2 + C_1$
 $y^2 - x^2 = C_1$
 $\ln y = -\ln x + \ln C_2$
 $\ln y = \ln \frac{C_2}{x}$
 $y = \frac{C_2}{x}$
 $yx = C_2$

$\therefore (xy + A)(y^2 - x^2 + B) = 0$
 is the general solution

Question 12 (****+)

By writing $\frac{dy}{dx} = p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 + y \frac{dy}{dx} = x^2 + xy.$$

Give the solution in the form $F(x, y)G(x, y) = 0$.

$$(2y - x^2 + A)(x + y - 1 + B e^{-x}) = 0$$

$\left(\frac{dy}{dx}\right)^2 + y \frac{dy}{dx} = x^2 + xy$
 • let $p = \frac{dy}{dx}$
 $p^2 + yp = x^2 + xy$
 $p^2 - x^2 + yp - xy = 0$
 $(p-x)(p+x) + y(p-x) = 0$
 $(p-x)[p+x+y] = 0$
 SPLIT INTO 2 DIFFERENT CASES
 • $\frac{dy}{dx} - x = 0$
 $\frac{dy}{dx} = x$
 $dy = x dx$
 $y = \frac{1}{2}x^2 + C_1$
 $2y - x^2 = C_1$
 • $\frac{dy}{dx} + x + y = 0$
 $\frac{dy}{dx} + y = -x$
 I.F. $e^{\int 1 dx} = e^x$
 $\frac{d}{dx}(ye^x) = -xe^x$
 $ye^x = \int -xe^x dx$
 BY PARTS
 $ye^x = -xe^x + e^x + C_2$
 $y = -x + 1 + C_2 e^{-x}$
 ∴ GENERAL SOLUTION
 $(2y - x^2 + A)(x + y - 1 + B e^{-x}) = 0$

Question 13 (*****)

A curve C is described implicitly by the equation

$$xy^2 = e^y.$$

- a) Show, by a detailed method, that

$$(y^2 - 2y) \frac{d^2 y}{dx^2} + (y^2 - 2) \left(\frac{dy}{dx} \right)^2 - 4y^3 \frac{dy}{dx} e^{-y} = 0.$$

- b) Use an analytical method, with suitable boundary conditions, to obtain the equation of C by solving the above differential equation.

V, , proof

DIFFERENTIATE WITH RESPECT TO x

$$\rightarrow \frac{d}{dx}(xy^2) = \frac{d}{dx}(e^y)$$

$$\rightarrow 1y^2 + 2(y \frac{dy}{dx}) = e^y \frac{dy}{dx}$$

$$\rightarrow y^2 + 2y \frac{dy}{dx} = e^y \frac{dy}{dx}$$

DIFFERENTIATE AGAIN WITH RESPECT TO x - THREE-PEPPER RULE IS NEEDED

$$\rightarrow 2y \frac{dy}{dx} + 2y \frac{d}{dx} \left(\frac{dy}{dx} \right) + 2y \frac{dy}{dx} \frac{dy}{dx} = e^y \frac{d^2 y}{dx^2} + e^y \left(\frac{dy}{dx} \right)^2$$

$$\rightarrow 4y \frac{dy}{dx} + 2 \left(\frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} \frac{dy}{dx} = e^y \left(\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right)$$

$$\rightarrow 4y \frac{dy}{dx} + 2y^2 \left[\frac{1}{y^2} \left(\frac{dy}{dx} \right)^2 + \frac{1}{y} \frac{dy}{dx} \right] = e^y \left[\left(\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right) \right]$$

$$\rightarrow 4y \frac{dy}{dx} + 2y^2 \left[\frac{1}{y^2} \left(\frac{dy}{dx} \right)^2 + \frac{1}{y} \frac{dy}{dx} \right] = e^y \left[\left(\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right) \right]$$

MOVING THROUGH BY y^2 & REARRANGE

$$\Rightarrow 4y^3 \frac{dy}{dx} + 2 \left(\frac{dy}{dx} \right)^2 + 2y^3 \frac{dy}{dx} = y^2 \left(\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right)$$

$$\Rightarrow 0 = (y^2 - 2y) \frac{d^2 y}{dx^2} + (y^2 - 2) \left(\frac{dy}{dx} \right)^2 - 4y^3 \frac{dy}{dx} e^{-y}$$

USE THE BOUNDARY CONDITIONS BASED ON THE EQUATION OF THE CURVE

$x=0, y=1 \rightarrow \frac{dy}{dx} = -\frac{1}{e}$

• $x=0, y=1 \rightarrow x=0$
• $y^2 + 2y \frac{dy}{dx} = e^y \frac{dy}{dx}$
 $1 + 2(-\frac{1}{e}) = e \frac{dy}{dx}$
 $\frac{dy}{dx} = -\frac{1}{e}$

AT THE O.D.E. HAS NO 2. PLEASE TRY THE STANDARD SUBSTITUTION

$$p = \frac{dy}{dx}$$

$$\frac{dp}{dx} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} p$$

TRANSFORM THE O.D.E

$$\Rightarrow (y^2 - 2y) \frac{dp}{dy} p + (y^2 - 2) p^2 - 4y^3 p e^{-y} = 0$$

$$\Rightarrow (y^2 - 2y) \frac{dp}{dy} + (y^2 - 2) p - 4y^3 e^{-y} = 0$$

$$\Rightarrow (y^2 - 2y) \frac{dp}{dy} + (y^2 - 2) p - 4y^3 e^{-y} = 0$$

$$\Rightarrow \frac{dp}{dy} + \frac{y^2 - 2}{y^2 - 2y} p = \frac{4y^3 e^{-y}}{y^2 - 2y}$$

NEXT LOOK FOR AN INTEGRATING FACTOR - PARTIAL FRACTIONS BY INSPECTION

$$e^{\int \frac{y^2 - 2}{y^2 - 2y} dy} = e^{\int \frac{y-2}{y(y-2)} dy} = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

$$= e^{\ln y} = y$$

$$= e^{\ln y} = y$$

$$\Rightarrow \frac{1}{y} \left[p \frac{dy}{dx} \right] = \frac{4y^3 e^{-y}}{y^2 - 2y} \times e^{\ln y}$$

$$\Rightarrow \frac{1}{y} \left[p \frac{dy}{dx} \right] = 4y^2 e^{-y}$$

$$\Rightarrow p e^{\ln y} = \int 4y^2 dy$$

$$\Rightarrow p e^{\ln y} = y^3 + A$$

$$\Rightarrow p = \frac{y^3 + A}{y^2} e^{-y}$$

$$\Rightarrow \frac{dp}{dy} = \frac{(y^3 + A) e^{-y}}{y^2}$$

APPLY THE CONDITION $y=1, \frac{dy}{dx} = -\frac{1}{e}$

$$-\frac{1}{e} = \frac{(1^3 + A) e^{-1}}{1^2}$$

$$-1 = -(1+A)$$

$$-1 = -1 - A$$

$$A = 0$$

$$\therefore \frac{dp}{dy} = \frac{y^3 e^{-y}}{y^2}$$

SEPARATING VARIABLES

$$\rightarrow \left(\frac{y^2 - 2y}{y^2} \right) dy = 1 dx$$

$$\rightarrow \left(\frac{y^2 - 2y}{y^2} \right) dy = 1 dx$$

$$\rightarrow \int \left(\frac{y^2 - 2y}{y^2} \right) dy = \int 1 dx$$

$$\rightarrow \int \left(\frac{1}{y} - \frac{2}{y} \right) dy = \int 1 dx$$

$$\rightarrow \int \frac{1}{y} dy - \int \frac{2}{y} dy = x + B$$

NOW INTEGRATION BY PARTS - ONLY ONE OF THE INTEGRALS OF THE L.H.S. SAY THE FIRST ONE

$$\Rightarrow \left[\frac{1}{y} - \frac{2}{y} \right] dy = x + B$$

$$\Rightarrow \frac{1}{y} - \frac{2}{y} = x + B$$

APPLY THE FINAL CONDITION $x=0, y=1$

$$\Rightarrow \frac{1}{1} - \frac{2}{1} = 0 + B$$

$$\Rightarrow -1 = B$$

$$\Rightarrow B = -1$$

$\therefore \frac{1}{y} - \frac{2}{y} = x - 1$

$$2y^2 = e^y$$

NO SPECTRUM

Question 14 (****)

Find a general solution of the following differential equation.

$$y = x \frac{dy}{dx} + e^{\frac{dy}{dx}}.$$

$$\boxed{}, \quad \boxed{(y + Ax + B)(y - x \ln x + Cx) = 0}$$

Method 1: Differentiating the ODE

Start by differentiating the O.D.E. with respect to x

$$\frac{dy}{dx} = \left[x \frac{dy}{dx} + e^{\frac{dy}{dx}} \right] + e^{\frac{dy}{dx}} \times \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dx} + x \frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} \frac{dy}{dx}$$

$$0 = \frac{d^2y}{dx^2} \left[x + e^{\frac{dy}{dx}} \right]$$

Now rearranging the original O.D.E

$$0 = \frac{d^2y}{dx^2} \left[x + \left(y - x \frac{dy}{dx} \right) \right]$$

Thus we have two separate O.D.E to solve

- $\frac{d^2y}{dx^2} = 0 \Rightarrow y = Ax + B$
- $x + y - x \frac{dy}{dx} = 0$

$$\Rightarrow x \frac{dy}{dx} - y = x$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 1$$

Integrating factor = $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

Method 2: Using the ansatz $y = Ax + B$

$$\frac{dy}{dx} \left(\frac{y}{x} \right) = \frac{1}{x}$$

$$\Rightarrow \frac{y}{x} = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{y}{x} = \ln|x| + C$$

$$\Rightarrow y = x \ln x + Cx$$

Combining the solutions we have

$$y = \begin{cases} Ax + B \\ x \ln x + Cx \end{cases}$$

This can be written as

$$\Rightarrow (y - Ax - B)(y - x \ln x - Cx) = 0$$

$$\Rightarrow (y + Bx + Q)(y - x \ln x + R) = 0$$