

Created by T. Madas

MULTIVARIABLE INTEGRATION

(SPHERICAL POLAR COORDINATES)

Created by T. Madas

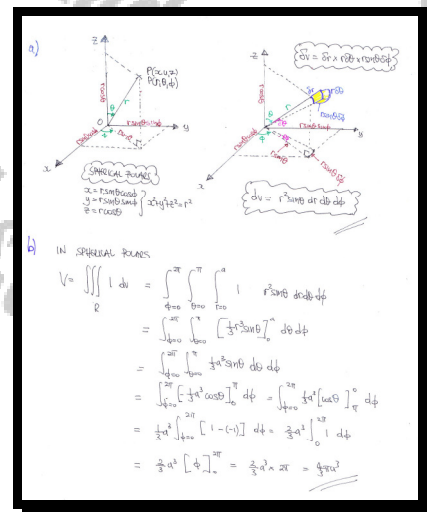
Question 1

- a) Determine with the aid of a diagram an expression for the volume element in spherical polar coordinates, (r, θ, ϕ) .

[You may not use Jacobians in this part]

- b) Use spherical polar coordinates to obtain the standard formula for the volume of a sphere of radius a .

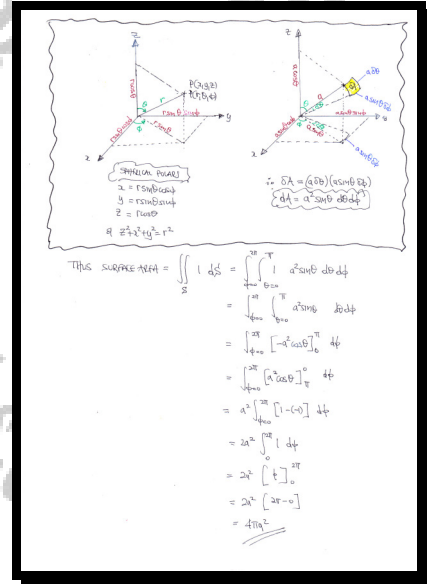
$$dv = r^2 \sin \theta dr d\theta d\phi$$



Question 2

Use spherical polar coordinates, (r, θ, ϕ) , to obtain the standard formula for the surface area of a sphere of radius a .

$$dA = a^2 \sin \theta \, d\theta \, d\phi \Rightarrow A = 4\pi a^2$$



Question 3

Determine an exact simplified value for

$$\int_R \frac{1}{(x^2 + y^2 + z^2)^2} dx dy dz,$$

where R is the region **outside** the sphere with equation

$$x^2 + y^2 + z^2 = 1.$$

4π

$$\begin{aligned}
 z &= r(\cos \theta + j \sin \theta) \\
 y &= r \sin \theta \cos \theta \\
 z &= r \cos \theta \\
 x &= r \sin \theta
 \end{aligned}$$

$$dxdy = r^2 \sin \theta \, d\theta \, dr$$

Th. 5.2

$$\begin{aligned}
 &\iint_{\text{disk of radius } r} \frac{1}{(\sqrt{x^2+y^2}+z^2)^2} \, dx \, dy \, dz \\
 &= \int_0^{2\pi} \int_0^\pi \int_0^r \frac{1}{(r^2+z^2)^2} r^2 \sin \theta \, dr \, d\theta \, dz \\
 &= \int_0^{2\pi} \int_0^\pi \frac{1}{12} r^2 \sin \theta \, d\theta \, dz \\
 &= \int_0^{2\pi} \int_0^\pi \left[-\frac{1}{12} \cos \theta \right]_0^\pi \, dz \, d\theta \\
 &= \int_0^{2\pi} \int_0^\pi 0 - (-\cos \theta) \, dz \, d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{2\pi} \left[\sin \theta \right]_0^\pi \, dz \, d\theta \\
 &= \int_0^{2\pi} \left[-\cos \theta \right]_0^\pi \, dz \, d\theta \\
 &= \int_0^{2\pi} \left[\cos \theta \right]_0^\pi \, dz \, d\theta \\
 &= \int_0^{2\pi} (-1 - (-1)) \, dz \, d\theta \\
 &= \int_0^{2\pi} 2 \, dz \, d\theta \\
 &= 2 \times 2\pi = 4\pi
 \end{aligned}$$

Question 4

The finite R region is defined as

$$1 \leq x^2 + y^2 + z^2 \leq 4.$$

Determine an exact simplified value for

$$\int_R \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dx dy dz.$$

$$4\pi \ln 2$$

• R & A: SPHERICAL COORDINATES WITH DEFINITION 1.0.2.

$x = \rho \sin \phi \cos \theta$
 $y = \rho \sin \phi \sin \theta$
 $z = \rho \cos \phi$

$dxdydz = r^2 \sin \phi \, d\rho \, d\phi \, d\theta$

• TASK 10: SPHERICAL COORDINATES

$$\int_V \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy \, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{2}} \frac{1}{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^2}{2} \right]_0^{\sqrt{2}} \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \left[\sin \phi \right]_0^{\pi} \frac{1}{2} \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta = \left[\frac{\theta}{2} \right]_0^{2\pi} = \frac{1}{2} [2\pi - 0] = \pi$$

Question 5

Determine an exact simplified value for

$$\int_R 15z^2 \, dx \, dy \, dz,$$

where R is the region between the spheres with equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4.$$

$$124\pi$$

SPHERICAL COORDINATES: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ & $dx \, dy \, dz = r^2 \sin \theta \, dr \, d\theta \, d\phi$
 $x^2 + y^2 + z^2 = 1 \Rightarrow r = 1$
 $x^2 + y^2 + z^2 = 4 \Rightarrow r = 2$
 Thus $\iiint_R 15z^2 \, dx \, dy \, dz = \dots$ Switch into SPHERICAL COORDINATES
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=1}^2 15(r \cos \theta)^2 (r^2 \sin \theta \, dr \, d\theta \, d\phi)$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=1}^2 15r^4 \cos^2 \theta \sin \theta \, dr \, d\theta \, d\phi$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[3r^5 \cos^2 \theta \sin \theta \right]_{r=1}^2 d\theta \, d\phi$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (96 \cos^2 \theta \sin \theta - 3 \cos^2 \theta \sin \theta) d\theta \, d\phi$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 93 \cos^2 \theta \sin \theta \, d\theta \, d\phi$
 $= \int_{\phi=0}^{2\pi} \left[-31 \cos^3 \theta \right]_{\theta=0}^{\pi} d\phi$
 $= \int_{\phi=0}^{2\pi} (31 - (-31)) d\phi$
 $= \int_{\phi=0}^{2\pi} 62 \, d\phi$
 $= 62 \left[\phi \right]_0^{2\pi}$
 $= 124\pi$

Question 6

The finite R region is defined as the interior of the sphere with equation

$$x^2 + y^2 + z^2 = 1.$$

Determine an exact simplified value for

$$\int_R \sqrt{x^2 + y^2 + z^2} e^{-(x^2 + y^2 + z^2)} dx dy dz.$$

$$2\pi \left[1 - \frac{2}{e} \right]$$

Handwritten solution for Question 6:

$$\iiint_R \sqrt{x^2 + y^2 + z^2} e^{-(x^2 + y^2 + z^2)} dx dy dz = \dots$$

Spherical coordinates: $x = r \cos \phi \sin \theta$, $y = r \sin \phi \sin \theta$, $z = r \cos \theta$.
 $dx dy dz = r^2 \sin \theta dr d\theta d\phi$.

Limits: r from 0 to 1, θ from 0 to π , ϕ from 0 to 2π .

The integral becomes:

$$\int_0^{2\pi} \int_0^\pi \int_0^1 r^3 e^{-r^2} \sin \theta dr d\theta d\phi$$

Integrate with respect to r :

$$= \int_0^{2\pi} \int_0^\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^1 \sin \theta d\theta d\phi$$

Integrate with respect to θ :

$$= \int_0^{2\pi} \left[\frac{1}{2} e^{-r^2} \cos \theta \right]_0^\pi d\phi$$

Integrate with respect to ϕ :

$$= \left[\frac{1}{2} e^{-r^2} \cos \theta \right]_0^\pi \Big|_0^{2\pi}$$

Final result:

$$= 2\pi \left[1 - \frac{2}{e} \right]$$

Question 7

Use spherical polar coordinates to find an exact value for the following integral.

$$\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)} dx dy dz.$$

MM2F

 $\pi\sqrt{\pi}$

TRANSFORM THE INTEGRAL

$$\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)} dx dy dz$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^\infty e^{-r^2} (r^2 \sin \theta) dr d\theta d\phi$$

WE CAN PUT THE INTEGRAL INTO 3

$$= \left[\int_0^\pi 1 d\theta \right] \left[\int_0^{2\pi} \sin \theta d\theta \right] \left[\int_0^\infty r^2 e^{-r^2} dr \right]$$

$$= 2\pi \times \left[-\cos \theta \right]_0^\pi \times \int_0^\infty r^2 e^{-r^2} dr$$

$$= 2\pi \times \left[\cos \theta \right]_0^\pi \times \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty + \frac{1}{2} \int_0^\infty e^{-r^2} dr$$

INTEGRATION BY PARTS

$u = r^2$	$dv = e^{-r^2}$
$du = 2r dr$	$v = \frac{1}{2} \sqrt{\pi}$

$$= 2\pi \times (1 - (-1)) \times \frac{1}{2} \int_0^\infty e^{-r^2} dr$$

$$= 2\pi \int_0^\infty e^{-r^2} dr$$

$$= 2\pi \times \frac{\sqrt{\pi}}{2}$$

$$= \pi \sqrt{\pi}$$

Question 8

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

The air bubble density $\rho(z)$, in m^{-3} , is given by

$$\rho(z) = kz,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geq 0$, determine the total number of air bubbles in the solid.

$$\boxed{\frac{1}{4} \pi k a^4}$$

IF THE "BUBBLE DENSITY" IS GIVEN BY $\rho(z) = kz$, THEN

TOTAL BUBBLES = $\int_V \rho(z) \, dv$

$$= \int_{\text{Hemisphere}} kz \, dv$$

WORKING IN SPHERICAL COORDINATES

$$= \int_0^{2\pi} \int_0^\pi \int_0^a k(r \cos \theta) (r^2 \sin \theta \, dr \, d\theta \, d\phi)$$

$$= \int_0^{2\pi} \int_0^\pi k r^3 \cos \theta \sin \theta \, dr \, d\theta \, d\phi$$

$$= \left[\int_0^{2\pi} d\phi \right] \left[\int_0^\pi \cos \theta \sin \theta \, d\theta \right] \left[\int_0^a r^3 \, dr \right]$$

$$= 2\pi k \times \left[\frac{1}{2} \sin^2 \theta \right]_0^\pi \times \left[\frac{1}{4} r^4 \right]_0^a$$

$$= 2\pi k \times \frac{1}{2} \times \frac{1}{4} a^4$$

$$= \frac{1}{4} \pi k a^4$$

$x = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$
 $x^2 + y^2 + z^2 = r^2$
 FOR THE HEMISPHERE
 $0 \leq r \leq a$
 $0 \leq \theta \leq \frac{\pi}{2}$
 $0 \leq \phi \leq 2\pi$
 $dv = r^2 \sin \theta \, dr \, d\theta \, d\phi$

Question 9

A uniform solid has equation

$$x^2 + y^2 + z^2 = a^2,$$

with $x > 0$, $y > 0$, $z > 0$, $a > 0$.

Use integration in spherical polar coordinates, (r, θ, ϕ) , to find in Cartesian form the coordinates of the centre of mass of the solid.

$$\left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a\right)$$

The solid is $\frac{1}{8}$ of a sphere
 so it has 3-way symmetry
 the solid has one coordinate
 and that will suffice as all
 3 will be the same

Total mass = $\frac{1}{8} \times \frac{4}{3}\pi \rho a^3 = \frac{1}{6}\pi \rho a^3$
 where ρ is mass density

- Infinitesimal 'volume' dV has mass ρdV
- Moment of infinitesimal about the z -axis (z fixed) is $(\rho dV) \times z$ (see diagram)

\therefore Summing up a TENS of UNITS

$$M\bar{z} = \iiint \rho z dV$$

Get sum

Switch into spherical POLARS

$$M\bar{z} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \rho (r \cos \theta) r^2 \sin \theta dr d\theta d\phi$$

$r^2 \cos \theta$ is $\frac{dz}{d\theta}$

$$\Rightarrow \frac{1}{6}\pi \rho a^3 \bar{z} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} \rho r^3 \cos \theta \right]_0^a d\theta d\phi$$

$$\Rightarrow \frac{1}{6}\pi \rho a^3 \bar{z} = \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} \rho a^3 \sin \theta \right]_0^{\frac{\pi}{2}} d\phi$$

$$\Rightarrow \frac{1}{6}\pi \rho a^3 \bar{z} = \int_0^{\frac{\pi}{2}} \frac{1}{3} \rho a^3 d\phi$$

$$\Rightarrow \frac{1}{6}\pi \rho a^3 \bar{z} = \left[\frac{1}{3} \rho a^3 \phi \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{1}{6}\pi \rho a^3 \bar{z} = \frac{1}{6}\pi \rho a^3 \frac{\pi}{2}$$

$$\therefore \bar{z} = \frac{\pi}{4}$$

$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a\right)$

Question 10

A solid sphere has equation

$$x^2 + y^2 + z^2 = a^2.$$

The density, ρ , at the point of the sphere with coordinates (x_1, y_1, z_1) is given by

$$\rho = \sqrt{x_1^2 + y_1^2}.$$

Determine the average density of the sphere.

$$\boxed{\bar{\rho} = \frac{3}{16}\pi a}$$

• SETTING BY THE DEFINITION OF MASS FOR VARIABLE DENSITY

$$\text{MASS} = \int_V \rho(x,y,z) \, dV = \int_V \sqrt{x^2+y^2} \, d_3x \, d_3y \, d_3z$$

Volume of sphere $\frac{4}{3}\pi a^3$

• SWITCH INTO SPHERICAL COORDINATES

$$\Rightarrow \text{MASS} = \int_0^{2\pi} \int_0^\pi \int_0^a \sqrt{(r \sin \theta \cos \phi)^2 + (r \sin \theta \sin \phi)^2} \, r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$\Rightarrow \text{MASS} = \int_0^{2\pi} \int_0^\pi \int_0^a r^2 \sin \theta \sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi)} \, dr \, d\theta \, d\phi$$

$$\Rightarrow \text{MASS} = \int_0^{2\pi} \int_0^\pi \int_0^a r^2 \sin \theta \, dr \, d\theta \, d\phi$$

• INTEGRATING WITH RESPECT TO r FIRST

$$\Rightarrow \text{MASS} = \int_0^{2\pi} \int_0^\pi \left[\frac{1}{3} r^3 \sin \theta \right]_0^a \, d\theta \, d\phi$$

$$\Rightarrow \text{MASS} = \int_0^{2\pi} \int_0^\pi \frac{1}{3} a^3 \sin \theta \, d\theta \, d\phi$$

$$\Rightarrow \text{MASS} = \frac{1}{3} a^3 \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi$$

• INTEGRATING WITH RESPECT TO ϕ NEXT

$$\text{MASS} = \frac{1}{3} a^3 \times 2\pi \times \int_0^\pi \sin \theta \, d\theta$$

$$\text{MASS} = \frac{1}{3} a^3 \times 2\pi \times \left[-\cos \theta \right]_0^\pi$$

$$\text{MASS} = \frac{1}{3} a^3 \times 2\pi \times 2$$

• NOW THE VOLUME OF A SPHERE OF RADIUS a IS $V = \frac{4}{3}\pi a^3$

$$\therefore \text{AVERAGE DENSITY} = \frac{\text{MASS}}{\text{VOLUME}} = \frac{\frac{1}{3} \pi^2 a^4}{\frac{4}{3} \pi a^3} = \frac{3}{16} \pi a$$

Question 11

A thin uniform spherical shell with equation

$$x^2 + y^2 + z^2 = a^2, \quad a > 0,$$

occupies the region in the first octant.

Use integration in spherical polar coordinates, (r, θ, ϕ) , to find in Cartesian form the coordinates of the centre of mass of the shell.

$$\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$$

Handwritten solution for Question 11:

Diagram: A spherical shell in the first octant of a 3D coordinate system (x, y, z). A small element of area dA is shown on the shell.

Equation: $dA = a^2 \sin\theta \, d\theta \, d\phi$

Steps:

- As the object is symmetrical we only need to find the position with respect to one of 3 axes.
- Total mass = $\frac{1}{2} \pi a^2 \rho = \frac{1}{2} \pi a^2$ (since ρ is mass per unit area)
- Consider infinitesimal mass $dm = \rho \, dA$ in spherical polar coordinates.
- It lies in the xy plane at $z = 0$ (if we consider infinitesimal).
- Due to symmetry $\bar{x} = \bar{y} = \bar{z}$.
- Summing up & taking limits: $M\bar{z} = \int \rho \, z \, dA$

Calculations:

$$M\bar{z} = \int \rho \, z \, dA = \rho \int_0^{\pi/2} \int_0^{\pi/2} a^2 \sin\theta \cos\theta \, d\theta \, d\phi$$

$$= \rho a^2 \int_0^{\pi/2} \left[-\frac{1}{2} \sin^2\theta \right]_0^{\pi/2} d\phi = -\frac{1}{2} \rho a^2 \int_0^{\pi/2} \sin^2\theta \, d\phi$$

$$= -\frac{1}{2} \rho a^2 \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} d\phi = -\frac{1}{4} \rho a^2 \int_0^{\pi/2} (1 - \cos 2\theta) d\phi$$

$$= -\frac{1}{4} \rho a^2 \left[\phi - \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = -\frac{1}{4} \rho a^2 \left[\frac{\pi}{2} - 0 \right] = -\frac{1}{8} \rho a^2 \pi$$

By symmetry $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

Question 12

The finite region Ω is defined as

$$x^2 + y^2 + z^2 \leq 1.$$

Use Spherical Polar Coordinates (r, θ, φ) , to evaluate the volume integral

$$\int_{\Omega} (r \sin \theta \cos \varphi)^4 dV.$$

$$\frac{4\pi}{35}$$

Handwritten solution for the volume integral:

$$\begin{aligned} \int_{\Omega} (r \sin \theta \cos \varphi)^4 dV &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^1 r^4 \sin^4 \theta \cos^4 \varphi (r^2 \sin \theta dr d\theta d\varphi) \\ &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^1 r^6 \sin^4 \theta \cos^4 \varphi dr d\theta d\varphi \\ &= \int_{\varphi=0}^{2\pi} \cos^4 \varphi d\varphi \left[\int_{\theta=0}^{\pi} \sin^4 \theta d\theta \right] \left[\int_{r=0}^1 r^6 dr \right] \\ &= \left[\int_{\varphi=0}^{2\pi} \cos^4 \varphi d\varphi \right] \left[\int_{\theta=0}^{\pi} \sin^4 \theta d\theta \right] \left[\frac{1}{7} r^7 \right]_0^1 \\ &= \left[\int_{\varphi=0}^{2\pi} 2 \cos^4 \varphi d\varphi \right] \left[\int_{\theta=0}^{\pi} 2 (\sin \theta)^2 (1 - \sin^2 \theta) d\theta \right] \times \frac{1}{7} \\ &= 2 \left(\frac{3\pi}{4} \right) \times \frac{8}{15} \times \frac{1}{7} \\ &= \frac{2 \left(\frac{3\pi}{4} \right) \left(\frac{8}{15} \right)}{7} = \frac{2 \left(\frac{3\pi}{4} \right) \left(\frac{8}{15} \right)}{7} \times \frac{1}{7} \\ &= \frac{2 \left(\frac{3\pi}{4} \right) \left(\frac{8}{15} \right)}{7} \times \frac{1}{7} \\ &= \frac{2 \left(\frac{3\pi}{4} \right) \left(\frac{8}{15} \right)}{7} \times \frac{1}{7} \\ &= \frac{4\pi}{35} \end{aligned}$$

Question 13

A solid sphere has equation

$$x^2 + y^2 + z^2 = 1.$$

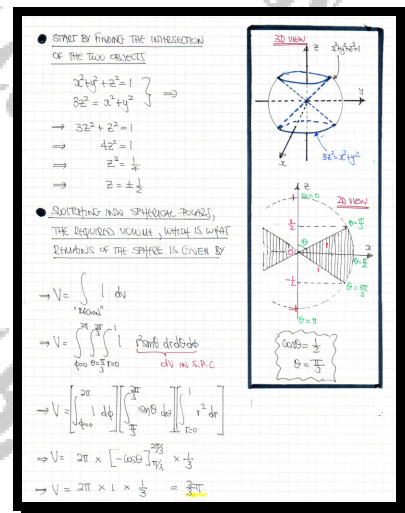
The region defined by the double cone with Cartesian equation

$$3z^2 \geq x^2 + y^2,$$

is bored out of the sphere.

Determine the volume of the remaining solid.

$$\frac{4\pi}{3}, \quad V = \frac{2}{3}\pi$$



Question 14

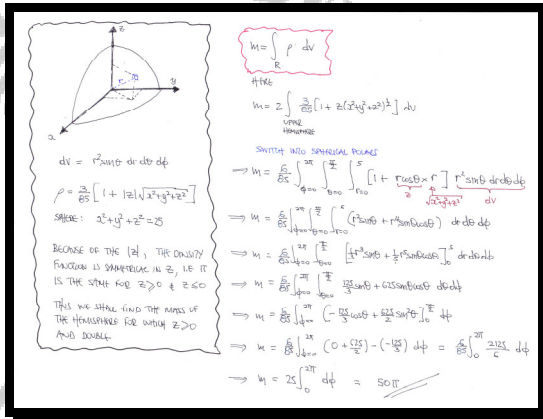
A solid sphere has radius 5 and is centred at the Cartesian origin O .

The density ρ at point $P(x_1, y_1, z_1)$ of the sphere satisfies

$$\rho = \frac{3}{85} \left[1 + |z_1| \sqrt{x_1^2 + y_1^2 + z_1^2} \right].$$

Use spherical polar coordinates, (r, θ, ϕ) , to find the mass of the sphere.

$$m = 50\pi$$



Handwritten solution for the mass of the sphere:

Diagram: A sphere of radius 5 centered at the origin O in a 3D coordinate system. A point P is shown on the sphere's surface.

Equations and steps:

- $M = \int_V \rho \, dv$
- $M = 2 \int_0^\pi \int_0^{2\pi} \int_0^5 \left[1 + z \sqrt{x^2 + y^2 + z^2} \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$
- SWITCH INTO SPHERICAL COORDINATES
- $\Rightarrow M = \int_0^\pi \int_0^{2\pi} \int_0^5 \left[1 + r \cos \theta \cdot r \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$
- $\Rightarrow M = \int_0^\pi \int_0^{2\pi} \left[\frac{1}{4} r^4 + \frac{1}{3} r^3 \cos \theta \right]_0^5 d\theta \, d\phi$
- $\Rightarrow M = \int_0^\pi \int_0^{2\pi} \left[\frac{625}{4} + \frac{125}{3} \cos \theta \right] d\theta \, d\phi$
- $\Rightarrow M = \int_0^\pi \left[\frac{625}{4} \theta + \frac{125}{3} \sin \theta \right]_0^{2\pi} d\phi$
- $\Rightarrow M = \int_0^{2\pi} \left(0 + \frac{625}{4} \right) d\phi = \int_0^{2\pi} \frac{625}{4} d\phi$
- $\Rightarrow M = \frac{625}{4} \cdot 2\pi = 312.5\pi$

BECAUSE OF THE $|z|$, THE DENSITY FUNCTION IS SYMMETRIC IN z , LET IT BE THE SPHERE FOR $z \geq 0$ & $z \leq 0$

THIS WE CAN FIND THE MASS OF THE HEMISPHERES FOR $z \geq 0$ AND DOUBLE

$\Rightarrow M = 2 \int_0^\pi \int_0^{2\pi} \int_0^5 \left[1 + r \cos \theta \cdot r \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$

$\Rightarrow M = 2 \int_0^\pi \int_0^{2\pi} \left[\frac{625}{4} + \frac{125}{3} \cos \theta \right] d\theta \, d\phi$

$\Rightarrow M = 2 \int_0^{2\pi} \left(0 + \frac{625}{4} \right) d\phi = 2 \int_0^{2\pi} \frac{625}{4} d\phi$

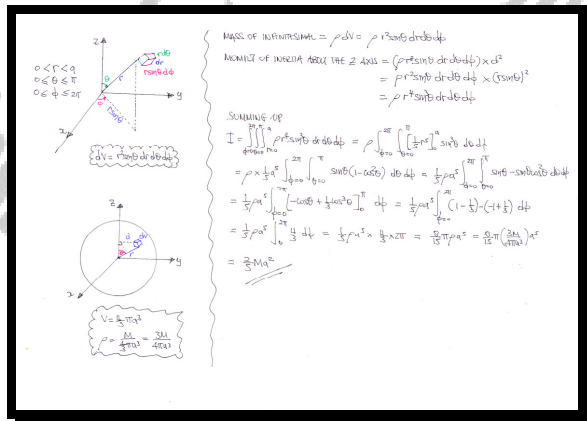
$\Rightarrow M = 2 \cdot \frac{625}{4} \cdot 2\pi = 312.5\pi$

Question 15

A solid uniform sphere has mass M and radius a .

Use spherical polar coordinates, (r, θ, ϕ) , to show that the moment of inertia of this sphere about one of its diameters is $\frac{2}{5}Ma^2$.

proof



MASS OF INFINITESIMAL $= \rho \, dV = \rho \, r^2 \sin \theta \, dr \, d\theta \, d\phi$

MOMENT OF INERTIA ABOUT THE Z AXIS $= (\text{perpendicular distance}) \times dV$
 $= \rho \, r^2 \sin^2 \theta \, dr \, d\theta \, d\phi \times (r \sin \theta)^2$
 $= \rho \, r^4 \sin^3 \theta \, dr \, d\theta \, d\phi$

SCALING UP

$$I = \int_0^a \int_0^\pi \int_0^{2\pi} \rho \, r^4 \sin^3 \theta \, d\phi \, d\theta \, dr$$

$$= \rho \times \frac{1}{5} a^5 \int_0^\pi \sin^3 \theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{5} \rho a^5 \int_0^\pi (-\cos \theta + \frac{1}{3} \cos^3 \theta) \Big|_0^\pi \, d\theta \times 2\pi$$

$$= \frac{1}{5} \rho a^5 \left[(-1) - (-1) \right] \times 2\pi = \frac{1}{5} \rho a^5 \times 2\pi$$

$$= \frac{2}{5} \rho \pi a^5$$

$V = \frac{4}{3} \pi a^3$
 $\rho = \frac{M}{V} = \frac{3M}{4\pi a^3}$

Substituting ρ into the equation for I :


$$I = \frac{2}{5} \left(\frac{3M}{4\pi a^3} \right) \pi a^5 = \frac{2}{5} M a^2$$

Question 16

A thin uniform spherical shell has mass m and radius a .

Use spherical polar coordinates, (r, θ, φ) , to show that the moment of inertia of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof



\bullet AREA ELEMENT IN SPHERICAL COORDINATES IS
 $dA = R^2 \sin\theta \, d\theta \, d\phi$

\bullet SURFACE AREA = $4\pi R^2$
 \bullet MAGNITUDE OF UNIT AREA
 $\hat{r} = \frac{\vec{r}}{R}$

SUMMING UP AND TACKLING UNIT

$$I = \int \int \int \frac{\mu_0 q}{4\pi R^2} \sin\theta \, d\theta \, d\phi$$

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\mu_0 q}{4\pi R^2} \sin\theta \, d\theta \, d\phi$$

$$I = \frac{\mu_0 q}{4\pi R^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi$$

$$I = \frac{\mu_0 q}{4\pi R^2} \left[\int_{\phi=0}^{2\pi} \left[-\cos\theta \right]_{\theta=0}^{\pi} d\phi \right]$$

$$I = \frac{\mu_0 q}{4\pi R^2} \left[\int_{\phi=0}^{2\pi} \left[-(1 - (-1)) \right] d\phi \right]$$

$$I = \frac{\mu_0 q}{4\pi R^2} \left[\int_{\phi=0}^{2\pi} 2 \, d\phi \right]$$

$$I = \frac{\mu_0 q}{4\pi R^2} \left[2\phi \right]_0^{2\pi}$$

$$I = \frac{\mu_0 q}{4\pi R^2} \left[2(2\pi - 0) \right]$$

$$I = \frac{\mu_0 q}{4\pi R^2} \cdot 4\pi R^2$$

WITHOUT LOSS OF GENERALITY TAKE THE Z-AXIS AS THE SYMMETRY OF THE SHELL

- \bullet MAX OF INVARIANT IS $\rho \cos\theta$
- \bullet MOMENT OF INERTIA ABOUT THE Z-AXIS IS

$$(\rho \cos\theta)^2 \cdot dV = \rho^2 \cos^2\theta \cdot (4\pi R^2 \sin^2\theta \, d\theta)$$

$$= \frac{4\pi \rho^2 R^2}{4\pi} \int_0^\pi \sin^2\theta \cos^2\theta \, d\theta$$

$$= \frac{4\pi \rho^2 R^2}{4\pi} \int_0^\pi \sin\theta \cos\theta \, d\theta$$

$$= \frac{4\pi \rho^2 R^2}{4\pi} \left[\frac{1}{2} \sin^2\theta \right]_0^\pi$$

$$= \frac{4\pi \rho^2 R^2}{4\pi} \left[\frac{1}{2} (0 - 0) \right]$$

Question 17

The finite region R is defined as the region enclosed by the ellipsoid with Cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1.$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r, θ, ϕ) , find the value of

$$\iiint_R (x^2 + y^2 + z^2) \, dx \, dy \, dz.$$

800π

The handwritten solution shows the following steps:

- Start with the ellipsoid equation: $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$.
- Use the substitution: $x = 3u, y = 4v, z = 5w$, where $-3 \leq u \leq 3, -4 \leq v \leq 4, -5 \leq w \leq 5$.
- Transform the volume element: $dx \, dy \, dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du \, dv \, dw = 60 \, du \, dv \, dw$.
- Transform the integrand: $x^2 + y^2 + z^2 = 9u^2 + 16v^2 + 25w^2$.
- Set up the triple integral: $\iiint_R (x^2 + y^2 + z^2) \, dx \, dy \, dz = \int_{-5}^5 \int_{-4}^4 \int_{-3}^3 (9u^2 + 16v^2 + 25w^2) \cdot 60 \, du \, dv \, dw$.
- Integrate with respect to u : $\int_{-3}^3 (9u^2 + 16v^2 + 25w^2) \, du = \left[3u^3 + 16vu^2 + 25wu^2 \right]_{-3}^3 = 108 + 144v + 225w$.
- Integrate with respect to v : $\int_{-4}^4 (108 + 144v + 225w) \, dv = \left[108v + 72v^2 + 225wv \right]_{-4}^4 = 1080 + 1800w$.
- Integrate with respect to w : $\int_{-5}^5 (1080 + 1800w) \, dw = \left[1080w + 900w^2 \right]_{-5}^5 = 10800 + 22500 = 33300$.
- Final result: $33300 \cdot 60 = 1998000$.

Question 18

A solid uniform sphere of radius a , has variable density $\rho(r) = r$, where r is the radial distance of a given point from the centre of the sphere.

- a) Use spherical polar coordinates, (r, θ, ϕ) , to find the moment of inertia of this sphere I , about one of its diameters.
- b) Given that the total mass of the sphere is m , show that

$$I = \frac{4}{9}ma^2.$$

$$I = \frac{4}{9}\pi a^6$$

WORK IN SPHERICAL COORDINATES

- Smallest box of symmetry that the x axis is to be the symmetry axis.
- $\rho(r) = r$
- Mass of infinitesimal volume in spherical coordinates is given by $dm = \rho(r) dr d\theta d\phi = r^2 \sin\theta dr d\theta d\phi$
- Moment of inertia of the infinitesimal is $dI = (r^2 \sin\theta dr d\theta d\phi) \times d^2 = (r^2 \sin\theta dr d\theta d\phi) (r^2 \sin^2\theta)$
 $= r^4 \sin^3\theta dr d\theta d\phi$

Therefore $I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a r^4 \sin^3\theta dr d\theta d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{r^5}{5} \sin^3\theta \right]_{r=0}^a d\theta d\phi$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{5} a^5 \sin^3\theta d\theta d\phi = \frac{1}{5} a^5 \int_{\phi=0}^{2\pi} \left[-\cos\theta + \frac{1}{3} \cos^3\theta \right]_{\theta=0}^{\pi} d\phi$
 $= \frac{1}{5} a^5 \int_{\phi=0}^{2\pi} \left(\frac{1}{3} - (-1) \right) d\phi = \frac{1}{5} a^5 \int_{\phi=0}^{2\pi} \frac{4}{3} d\phi = \frac{4}{15} a^5 \int_{\phi=0}^{2\pi} d\phi = \frac{4}{15} a^5 [2\pi]_{\phi=0}^{2\pi} = \frac{8\pi}{15} a^5$

b) Now find mass $M = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a r^2 \sin\theta dr d\theta d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{r^3}{3} \sin\theta \right]_{r=0}^a d\theta d\phi$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{3} a^3 \sin\theta d\theta d\phi = \frac{1}{3} a^3 \int_{\phi=0}^{2\pi} \left[-\cos\theta \right]_{\theta=0}^{\pi} d\phi$
 $= \frac{1}{3} a^3 \int_{\phi=0}^{2\pi} (-(-1) - (-1)) d\phi = \frac{1}{3} a^3 \int_{\phi=0}^{2\pi} 2 d\phi = \frac{2}{3} a^3 [2\pi]_{\phi=0}^{2\pi} = \frac{4\pi}{3} a^3$

$\therefore I = \frac{8\pi}{15} a^5 = \frac{4\pi a^3}{3} \times \frac{2\pi}{3} = \frac{4\pi a^3}{3} \times \frac{2\pi}{3} = \frac{8\pi^2}{9} a^3$

Question 19

Evaluate the triple integral

$$\int_V 5x^2 \, dx \, dy \, dz,$$

where V is the finite region contained within the closed surface with equation

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1.$$

 8π

$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$
 Firstly use a "STRETCH" TRANSFORMATION to convert into SPHERICAL FORMS
 $\begin{matrix} x = X \\ y = 2Y \\ z = 3Z \end{matrix} \Rightarrow \begin{matrix} X = x \\ Y = \frac{y}{2} \\ Z = \frac{z}{3} \end{matrix} \quad \begin{matrix} dx = dX \\ dy = 2dY \\ dz = 3dZ \end{matrix} \quad \text{NOW} \quad X^2 + Y^2 + Z^2 = 1$

$\iiint_V 5x^2 \, dv = \iiint_V 5x^2 \, dx \, dy \, dz = \iiint_V 5X^2 \, dX \, (2dY) \, (3dZ)$
 $= \iiint_V 30X^2 \, dX \, dY \, dZ = \dots$ INTO SPHERICAL FORMS
 $= \int_0^{2\pi} \int_0^\pi \int_0^1 30(\cos\phi\cos\theta)^2 (r^2 \sin\theta) \, dr \, d\theta \, d\phi$
 $= \int_0^{2\pi} \int_0^\pi 30 \cos^2\phi \cos^2\theta \sin\theta \, d\theta \, d\phi$
 $= \int_0^{2\pi} \left[\cos^2\phi \int_0^\pi (1 + \cos 2\theta) \cos^2\theta \sin\theta \, d\theta \right] d\phi$
 $= \int_0^{2\pi} \left[\cos^2\phi \left(\frac{1}{3} \cos^3\theta + \frac{1}{5} \cos^5\theta \right) \right]_0^\pi d\phi$
 $= \int_0^{2\pi} \left[\cos^2\phi - 3\cos^2\phi \right]_0^\pi d\phi$
 $= \int_0^{2\pi} (-1 + 3) - (-1 + 3) d\phi$
 $= 8\pi$

$X = r \sin\theta \cos\phi$
 $dX \, dY \, dZ = r^2 \sin\theta \, dr \, d\theta \, d\phi$
 $0 \leq r \leq 1$
 $0 \leq \theta \leq \pi$
 $0 \leq \phi \leq 2\pi$

Question 20

The finite R region is defined as

$$x^2 + y^2 + z^2 \leq 2z.$$

Determine an exact simplified value for

$$\int_R \frac{z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz.$$

$$\frac{16\pi}{15}$$

Handwritten solution for Question 20:

The region R is defined by $x^2 + y^2 + z^2 \leq 2z$. This is a sphere of radius 1 centered at $(0, 0, 1)$ in the z -axis.

Using spherical coordinates, the region R is described by $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq \rho \leq 2\cos\phi$.

The integral is evaluated as follows:

$$\int_R \frac{z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz = \int_0^{2\pi} \int_0^\pi \int_0^{2\cos\phi} \frac{\rho \cos\phi}{\rho} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \int_0^{2\cos\phi} \rho \cos\phi \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \left[\frac{\rho^2}{2} \cos\phi \sin\phi \right]_0^{2\cos\phi} d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \frac{(2\cos\phi)^2}{2} \cos\phi \sin\phi d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi 2\cos^3\phi \sin\phi d\phi d\theta$$

$$= \int_0^{2\pi} \left[-\frac{2}{4} \cos^4\phi \right]_0^\pi d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} \cos^4\phi \right]_0^\pi d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} (1) - \left(-\frac{1}{2} (1) \right) \right] d\theta$$

$$= \int_0^{2\pi} 0 d\theta$$

$$= 0$$

However, the final result is $\frac{16\pi}{15}$, which suggests a different interpretation of the region or a different method.

Diagram of the region R (a sphere of radius 1 centered at $(0, 0, 1)$):

Using spherical coordinates, the region R is described by $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq \rho \leq 2\cos\phi$.

The integral is evaluated as follows:

$$\int_R \frac{z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz = \int_0^{2\pi} \int_0^\pi \int_0^{2\cos\phi} \frac{\rho \cos\phi}{\rho} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \int_0^{2\cos\phi} \rho \cos\phi \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \left[\frac{\rho^2}{2} \cos\phi \sin\phi \right]_0^{2\cos\phi} d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \frac{(2\cos\phi)^2}{2} \cos\phi \sin\phi d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi 2\cos^3\phi \sin\phi d\phi d\theta$$

$$= \int_0^{2\pi} \left[-\frac{2}{4} \cos^4\phi \right]_0^\pi d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} \cos^4\phi \right]_0^\pi d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} (1) - \left(-\frac{1}{2} (1) \right) \right] d\theta$$

$$= \int_0^{2\pi} 0 d\theta$$

$$= 0$$

However, the final result is $\frac{16\pi}{15}$, which suggests a different interpretation of the region or a different method.

Question 21

The finite R region is defined as

$$4z \leq x^2 + y^2 + z^2 \leq 16z.$$

Determine an exact simplified value for

$$\int_R \left(\frac{z}{8}\right)^3 dx dy dz.$$

,

THE INTERSECTION REGION SUPPORT SPHERICAL COORDINATES

- $4z = x^2 + y^2 + z^2$
 $x^2 + y^2 + z^2 - 4z = 0$
 $x^2 + y^2 + (z-2)^2 = 4$
 SPHERE OF RADIUS 2,
 CENTRE AT (0,0,2)
- $x^2 + y^2 + z^2 = 16z$
 $x^2 + y^2 + z^2 - 16z = 0$
 $x^2 + y^2 + (z-8)^2 = 64$
 SPHERE OF RADIUS 8,
 CENTRE AT (0,0,8)

SPHERE COORDINATES

$x = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$

TRANSFORM THE SPHERICAL COORDINATES

$x^2 + y^2 + z^2 = 4z$ $x^2 + y^2 + z^2 = 16z$
 $r^2 = 4r \cos \theta$ $r^2 = 16r \cos \theta$
 $r = 4 \cos \theta$ $r = 16 \cos \theta$

TRANSFORM THE LIMITS

$z > 0 \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$
 $0 \leq \phi \leq 2\pi$

VOLUME ELEMENT IN SPHERICAL COORDINATES

$dv = r^2 \sin \theta dr d\theta d\phi$

TRANSFORMING THE INTEGRAL

$$\iiint_R \left(\frac{z}{8}\right)^3 dx dy dz = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=4 \cos \theta}^{16 \cos \theta} \left(\frac{r \cos \theta}{8}\right)^3 (r^2 \sin \theta dr d\theta d\phi)$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=4 \cos \theta}^{16 \cos \theta} \frac{r^5 \cos^3 \theta}{512} dr d\theta d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^6 \cos^3 \theta}{312 \times 6} \right]_{r=4 \cos \theta}^{16 \cos \theta} d\theta d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \frac{16384 \cos^9 \theta}{312} - \frac{4 \cos^9 \theta}{312} d\theta d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} 52 \cos^8 \theta d\theta d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[\frac{52 \cos^7 \theta}{-7} \right]_0^{\frac{\pi}{2}} d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[\frac{52 \cos^7 \theta}{-7} \right]_{\frac{\pi}{2}}^0 d\phi$$

$$= \int_{\phi=0}^{2\pi} \frac{52}{7} d\phi$$

$$= \frac{1092\pi}{7}$$

A solid sphere has equation

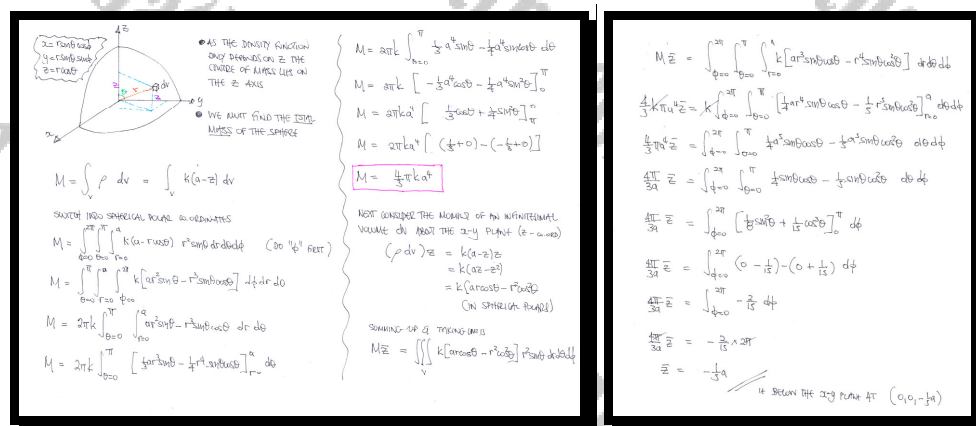
$$x^2 + y^2 + z^2 = a^2, \quad a > 0.$$

The sphere has variable density ρ , given by

$$\rho = k(a - z), \quad k > 0.$$

Use integration in spherical polar coordinates, (r, θ, φ) , to find in Cartesian form the coordinates of the centre of mass of the sphere.

$$\left(0, 0, -\frac{1}{5}a\right)$$



Question 23

A solid is defined in a Cartesian system of coordinates by

$$x^2 + y^2 = xz, \quad 0 \leq z \leq 2.$$

- Describe the solid with the aid of a sketch.
- Use standard elementary formulas to find the volume of the solid.
- Use spherical polar coordinates to verify the answer to part (b)

$$V = \frac{2}{3}\pi$$

a) $x^2 + y^2 = z^2, 0 \leq z \leq 2$

• Z is non-negative
Thus if $z < 0 \Rightarrow x^2 + y^2 < 0$
Which is impossible
 $\therefore z > 0$

• However, y can be negative


• Second's restriction only the xy plane
only lie in the first fourth quadrant!


• Next if $y > 0 \Rightarrow (x-z)$ plane

$z^2 = x^2$
 $z = \pm x$

• Hence m, z are different constraints

$x^2 + y^2 = k^2$
 $z^2 = k^2 + y^2 + 0$
 $(z - \frac{y}{2})^2 + y^2 = \frac{3y^2}{4}$
If circles at the same sign below

3d

Let θ range from $0 \leq \theta \leq 2\pi$

b) 

Base of the cone
 $x^2 + y^2 = 1$
 $z^2 - 2x + y^2 = 0$
 $(z-y)^2 + y^2 = 1$
 \therefore Base Area = $\pi \times 1^2 = \pi$
Height = 2
Volume = $\frac{1}{3} \pi \times 2 = \frac{2}{3} \pi$

9) cylindrical coordinates

$x = r \cos(\theta)$
 $y = r \sin(\theta)$
 $z = r \cos(\theta)$
 $x^2 + y^2 + z^2 = r^2$

• $x^2 + y^2 = z^2$
 $r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = (r \cos(\theta))^2$
 $r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = r^2 \cos^2(\theta)$
 $\sin^2(\theta) = \cos^2(\theta)$
 $\sin(\theta) = \cos(\theta)$
 $\theta = \cos(\theta)$

c) $0 \leq \theta \leq \arctan(\cos(\theta))$

• Now $z = 2$
 $r \cos(\theta) = 2$
 $r = \frac{2}{\cos(\theta)}$

$\therefore 0 \leq r \leq \frac{2}{\cos(\theta)}$

• $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
(use 4th quadrant)

Thus
 $V = \int_{-\pi/2}^{\pi/2} \int_0^{2/\cos(\theta)} \int_0^{2/\cos(\theta)} r^2 \sin(\theta) dr d\theta$
 $V = \int_{-\pi/2}^{\pi/2} \int_0^{2/\cos(\theta)} \frac{r^3}{3} \sin(\theta) dr d\theta$
 $V = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{12} \sin(\theta) \right]_0^{2/\cos(\theta)} d\theta$
 $V = \int_{-\pi/2}^{\pi/2} \left(\frac{16}{12} \sin(\theta) \right) d\theta$
 $V = \frac{4}{3} \int_{-\pi/2}^{\pi/2} \sin(\theta) d\theta$
 $V = \frac{4}{3} [-\cos(\theta)]_{-\pi/2}^{\pi/2}$
 $V = \frac{4}{3} [-\cos(\pi/2) + \cos(-\pi/2)]$
 $V = \frac{4}{3} [0 + 0] = 0$

d) $0 \leq \theta \leq \arctan(\cos(\theta))$

• Now $z = 2$
 $r \cos(\theta) = 2$
 $r = \frac{2}{\cos(\theta)}$

$\therefore 0 \leq r \leq \frac{2}{\cos(\theta)}$

• $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
(use 4th quadrant)

Thus
 $V = \int_{-\pi/2}^{\pi/2} \int_0^{2/\cos(\theta)} \int_0^{2/\cos(\theta)} r^2 \sin(\theta) dr d\theta$
 $V = \int_{-\pi/2}^{\pi/2} \int_0^{2/\cos(\theta)} \frac{r^3}{3} \sin(\theta) dr d\theta$
 $V = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{12} \sin(\theta) \right]_0^{2/\cos(\theta)} d\theta$
 $V = \int_{-\pi/2}^{\pi/2} \left(\frac{16}{12} \sin(\theta) \right) d\theta$
 $V = \frac{4}{3} \int_{-\pi/2}^{\pi/2} \sin(\theta) d\theta$
 $V = \frac{4}{3} [-\cos(\theta)]_{-\pi/2}^{\pi/2}$
 $V = \frac{4}{3} [-\cos(\pi/2) + \cos(-\pi/2)]$
 $V = \frac{4}{3} [0 + 0] = 0$

Question 24

A solid sphere has radius a and mass m .

The density ρ at any point in the sphere is inversely proportional to the distance of this point from the centre of the sphere

Show that the moment of inertia of this sphere about one of its diameters is $\frac{1}{3}ma^2$

proof

Firstly we need the mass — use spherical polar coordinates (r, θ, ϕ)

$$\rho(r, \theta, \phi) = \frac{k}{r}$$


$$\text{Mass} = \int \rho \, dV = \int_0^{2\pi} \int_0^\pi \int_0^a \frac{k}{r} (r^2 \sin \theta \, dr \, d\theta \, d\phi)$$

$$= \int_0^{2\pi} \int_0^\pi k r \sin \theta \, dr \, d\theta \, d\phi = \int_0^{2\pi} \int_0^\pi \left[\frac{1}{2} k r^2 \sin \theta \right]_0^a d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \frac{1}{2} k a^2 \sin \theta \, d\theta \, d\phi = \int_0^{2\pi} \left[-\frac{1}{2} k a^2 \cos \theta \right]_0^\pi d\phi$$

$$= \int_0^{2\pi} \left[\frac{1}{2} k a^2 - \left(-\frac{1}{2} k a^2 \right) \right] d\phi = k a^2 \int_0^{2\pi} 1 \, d\phi = 2\pi k a^2$$

Now the moment of inertia — take the diameter to be the z axis



$$dI = (r \sin \theta)^2 \, dm$$

$$I = \int \int \int (r \sin \theta)^2 \rho(r, \theta, \phi) \, r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$I = \int_0^{2\pi} \int_0^\pi \int_0^a k r^4 \sin^3 \theta \, dr \, d\theta \, d\phi$$

$$I = \int_0^{2\pi} \int_0^\pi \left[\frac{1}{5} k r^5 \sin^3 \theta \right]_0^a d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \frac{1}{5} k a^5 \sin^3 \theta \, d\theta \, d\phi = \int_0^{2\pi} \frac{1}{5} k a^5 \left[-\cos \theta + \frac{1}{3} \cos^3 \theta \right]_0^\pi d\phi$$

$$= \int_0^{2\pi} \frac{1}{5} k a^5 \left[(-1 + \frac{1}{3}) - (1 - \frac{1}{3}) \right] d\phi = \int_0^{2\pi} \frac{1}{5} k a^5 \left[-\frac{4}{3} \right] d\phi$$

$$= \frac{1}{5} k a^5 \left[-\frac{4}{3} \right] \int_0^{2\pi} 1 \, d\phi = \frac{1}{5} k a^5 \left[-\frac{4}{3} \right] 2\pi = -\frac{4\pi}{15} k a^5$$

(Equation)

Question 25

The finite R region is defined as

$$1 \leq x^2 + y^2 + z^2 \leq 2z.$$

Determine an exact simplified value for

$$\int_R z \, dx \, dy \, dz.$$

$$\frac{9\pi}{8}$$

Handwritten solution for Question 25:

Region R is defined by $1 \leq x^2 + y^2 + z^2 \leq 2z$.

Convert to spherical coordinates:

- $x = r \sin \theta \cos \phi$
- $y = r \sin \theta \sin \phi$
- $z = r \cos \theta$

Volume element $dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$.

Region R is defined by:

- $1 \leq r^2 \leq 2r \cos \theta$
- $0 \leq \theta \leq \pi$
- $0 \leq \phi \leq 2\pi$

Integrate z over R :

$$\int_R z \, dx \, dy \, dz = \int_0^{2\pi} \int_0^\pi \int_1^{2 \cos \theta} r^2 \cos \theta \, r^2 \sin \theta \, dr \, d\theta \, d\phi$$

First integrate with respect to r :

$$= \int_0^{2\pi} \int_0^\pi \left[\frac{r^3}{3} \cos \theta \sin \theta \right]_1^{2 \cos \theta} d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \left(\frac{8 \cos^3 \theta \sin \theta}{3} - \frac{\sin \theta}{3} \right) d\theta \, d\phi$$

Integrate with respect to θ :

$$= \int_0^{2\pi} \left[-\frac{2 \cos^2 \theta}{3} + \frac{\cos \theta}{3} \right]_0^\pi d\phi$$

$$= \int_0^{2\pi} \left(-\frac{2}{3} + \frac{1}{3} \right) d\phi = \int_0^{2\pi} -\frac{1}{3} d\phi$$

$$= -\frac{1}{3} \times 2\pi = -\frac{2\pi}{3}$$

Since the region is symmetric about the z -axis, the volume is positive. The final answer is $\frac{9\pi}{8}$.

Diagram: A circle in the xy -plane with radius 1, centered at the origin. The region R is the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Question 26

The finite R region is defined as

$$x^2 + y^2 + z^2 \leq 4z \quad \text{and} \quad z \geq 2.$$

Determine an exact simplified value for

$$\int_R \frac{z^2}{\sqrt{x^2 + y^2 + z^2}} dx dy dz.$$

$$\frac{44\pi}{3}$$

Handwritten solution for Question 26:

Given: $x^2 + y^2 + z^2 \leq 4z$ and $z \geq 2$.

Convert to spherical coordinates: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

The region R is defined by $r^2 \leq 4r \cos \theta$ and $r \cos \theta \geq 2$.

From $r^2 \leq 4r \cos \theta$, we get $r \leq 4 \cos \theta$.

From $r \cos \theta \geq 2$, we get $r \geq \frac{2}{\cos \theta}$.

Since $r \leq 4 \cos \theta$ and $r \geq \frac{2}{\cos \theta}$, we have $\frac{2}{\cos \theta} \leq 4 \cos \theta$, which simplifies to $2 \leq 4 \cos^2 \theta$, or $\cos^2 \theta \geq \frac{1}{2}$, or $\cos \theta \geq \frac{1}{\sqrt{2}}$.

Since $\cos \theta \geq \frac{1}{\sqrt{2}}$, we have $\theta \leq \frac{\pi}{4}$.

The limits for θ are $0 \leq \theta \leq \frac{\pi}{4}$.

The limits for ϕ are $0 \leq \phi \leq 2\pi$.

The limits for r are $\frac{2}{\cos \theta} \leq r \leq 4 \cos \theta$.

The integrand is $\frac{z^2}{\sqrt{x^2 + y^2 + z^2}} = \frac{(r \cos \theta)^2}{r} = r \cos^2 \theta$.

The volume element is $dx dy dz = r^2 \sin \theta dr d\theta d\phi$.

The integral is:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\frac{2}{\cos \theta}}^{4 \cos \theta} r \cos^2 \theta \cdot r^2 \sin \theta dr d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4} \cos^2 \theta \sin \theta \right]_{\frac{2}{\cos \theta}}^{4 \cos \theta} d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left(\frac{(4 \cos \theta)^4}{4} \cos^2 \theta \sin \theta - \frac{(\frac{2}{\cos \theta})^4}{4} \cos^2 \theta \sin \theta \right) d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left(64 \cos^6 \theta \sin \theta - \frac{4}{\cos^2 \theta} \cos^2 \theta \sin \theta \right) d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} (64 \cos^6 \theta \sin \theta - 4 \sin \theta) d\theta d\phi$$

$$= \int_0^{2\pi} \left[-\frac{64}{7} \cos^7 \theta + 4 \cos \theta \right]_0^{\frac{\pi}{4}} d\phi$$

$$= \int_0^{2\pi} \left(-\frac{64}{7} \cos^7 \frac{\pi}{4} + 4 \cos \frac{\pi}{4} + \frac{64}{7} - 4 \right) d\phi$$

$$= \int_0^{2\pi} \left(-\frac{64}{7} \left(\frac{\sqrt{2}}{2} \right)^7 + 4 \left(\frac{\sqrt{2}}{2} \right) + \frac{64}{7} - 4 \right) d\phi$$

$$= \int_0^{2\pi} \left(-\frac{64}{7} \frac{\sqrt{2}}{128} + 2\sqrt{2} + \frac{64}{7} - 4 \right) d\phi$$

$$= \int_0^{2\pi} \left(-\frac{\sqrt{2}}{2} + 2\sqrt{2} + \frac{64}{7} - 4 \right) d\phi$$

$$= \int_0^{2\pi} \left(\frac{3\sqrt{2}}{2} + \frac{64}{7} - 4 \right) d\phi$$

$$= \left(\frac{3\sqrt{2}}{2} + \frac{64}{7} - 4 \right) \int_0^{2\pi} d\phi$$

$$= \left(\frac{3\sqrt{2}}{2} + \frac{64}{7} - 4 \right) 2\pi$$

$$= \frac{44\pi}{3}$$

A **non right** circular cone has Cartesian equation

$$x^2 + y^2 = xz, \quad 0 \leq z \leq 2.$$

Use spherical polar coordinates to find an exact simplified value for

$$\int_R xz \, dV,$$

where R is the interior of the cone.

$$\frac{4\pi}{5}$$

