MULTIVARIABLE INTEGRATION

(PLANE & CYLINDRICAL POLAR COORDINATES)

PLANE POLAR COORDINATES

Question 1

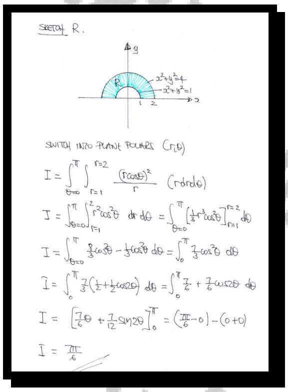
The finite region on the x-y plane satisfies

$$1 \le x^2 + y^2 \le 4, \quad y \ge 0.$$

Find, in terms of π, the value of I.

$$I = \int_R \frac{x^2}{\sqrt{x^2+y^2}} \, dx\, dy.$$

$\boxed{\frac{7}{6}\pi}$


SKETCH R.

SWITCH INTO PLANE POLARS (r,θ)

$I = \int_{\theta=0}^{\pi} \int_{r=1}^{r=2} \frac{(r\cos\theta)^2}{r} (r\,dr\,d\theta)$

$I = \int_{\theta=0}^{\pi} \int_{r=1}^{2} r^2\cos^2\theta \, dr\, d\theta = \int_{\theta=0}^{\pi} \left[\frac{1}{3}r^3\cos^2\theta\right]_{r=1}^{r=2} d\theta$

$I = \int_{\theta=0}^{\pi} \frac{8}{3}\cos^2\theta - \frac{1}{3}\cos^2\theta \, d\theta = \int_0^{\pi} \frac{7}{3}\cos^2\theta \, d\theta$

$I = \int_0^{\pi} \frac{7}{3}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta = \int_0^{\pi} \frac{7}{6} + \frac{7}{6}\cos 2\theta \, d\theta$

$I = \left[\frac{7}{6}\theta + \frac{7}{12}\sin 2\theta\right]_0^{\pi} = \left(\frac{7\pi}{6} - 0\right) - (0+0)$

$I = \frac{7\pi}{6}$

Question 2

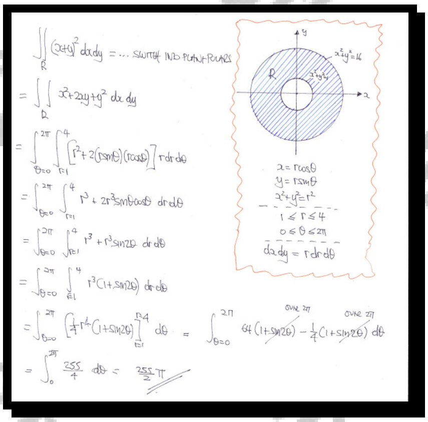
The finite region on the x-y plane satisfies

$$1 \le x^2 + y^2 \le 16, \quad y \ge 0 .$$

Find, in terms of π, the value of I.

$$I = \int_R (x+y)^2 \; dx\,dy .$$

$$\boxed{\frac{255\pi}{2}}$$

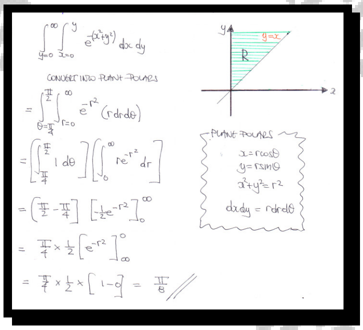


Question 3

Find the exact simplified value for the following integral.

$$\int_0^{\infty}\int_0^{y} e^{-\left(x^2+y^2\right)}\ dx\,dy\,.$$

$$\boxed{\frac{\pi}{8}}$$

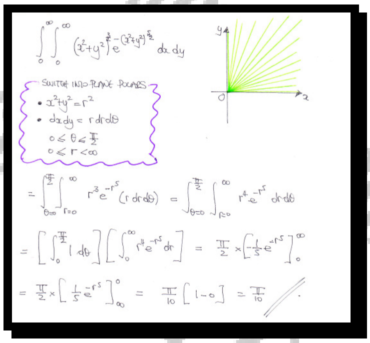


Question 4

Find the exact simplified value for the following integral.

$$\int_0^{\infty}\int_0^{\infty} \left(x^2+y^2\right)^{\frac{3}{2}} e^{-\left(x^2+y^2\right)^{\frac{5}{2}}}\ dx\,dy\,.$$

$$\boxed{\frac{\pi}{10}}$$



Question 5

The finite region on the x-y plane satisfies

$$4 \le x^2 + y^2 \le 4x, \quad y \ge 0 .$$

Find the value of I.

$$I = \int_R xy \, dx\, dy .$$

$\boxed{9}$

SKETCH THE REGION FIRST

- $x^2+y^2 \geqslant 4$

"OUTSIDE" THE CIRCLE, $y \geqslant 0$

POLARS: $\boxed{r \geqslant 2}$

- $x^2+y^2 \leqslant 4x$

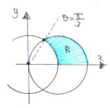
$x^2-4x+y^2 \leqslant 0$

$(x-2)^2+y^2 \leqslant 4$

"INSIDE" THE CIRCLE, $y \geqslant 0$

POLARS: $r^2 \leqslant 4r\cos\theta$

$\boxed{r \leqslant 4\cos\theta}$



FIND INTERSECTION

$2 = 4\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

THUS $\int_R xy \, dx\,dy = \int_{\theta=0}^{\frac{\pi}{3}} \int_{r=2}^{4\cos\theta} (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$

$= \int_{\theta=0}^{\frac{\pi}{3}} \int_{r=2}^{4\cos\theta} r^3\cos\theta\sin\theta \, dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{3}} \left[\frac{1}{4}r^4\cos\theta\sin\theta\right]_2^{4\cos\theta} d\theta$

$= \int_{\theta=0}^{\frac{\pi}{3}} 64\cos^5\theta\sin\theta - 4\cos\theta\sin\theta \, d\theta = \left[-\frac{64}{6}\cos^6\theta - 2\sin^2\theta\right]_0^{\frac{\pi}{3}}$

$= \left[\frac{32}{3}\cos^6\theta + 2\sin^2\theta\right]_{\frac{\pi}{3}}^{0} = \left(\frac{32}{3}+0\right) - \left(\frac{1}{6}+\frac{3}{2}\right) = 9$

Question 6

The points A and B have Cartesian coordinates $(1,0)$ and $(1,1)$, respectively.

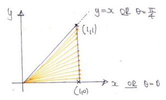
The finite region R is defined as the triangle OAB, where O is the origin.

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R \frac{x^3}{x^2+y^2}\,dx\,dy\,.$$

[*No credit will be given for workings in other coordinate systems*.]

$$\boxed{\frac{\pi}{12}}$$



EQUATION $x=1$
$r\cos\theta=1$
$r=\sec\theta$

THUS $0 \le r \le \sec\theta$
$0 \le \theta \le \frac{\pi}{4}$

$$\int_R \frac{x^3}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{r=\sec\theta} \frac{(r\cos\theta)^3}{r^2}\,(r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{r=\sec\theta} r^2\cos^3\theta\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{4}}\left[\frac{1}{3}r^3\cos^3\theta\right]_{r=0}^{r=\sec\theta}$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \frac{1}{3}\sec^3\theta\cos^3\theta\,d\theta = \int_0^{\frac{\pi}{4}}\frac{1}{3}\,d\theta = \left[\frac{1}{3}\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{12}$$

Question 7

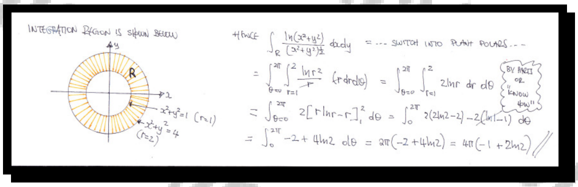
The finite region R is defined as

$$1 \le x^2 + y^2 \le 4 .$$

Determine an exact simplified value for

$$\int_R \frac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}} \, dx\,dy .$$

$$\boxed{4\pi\left(-1+2\ln 2\right)}$$



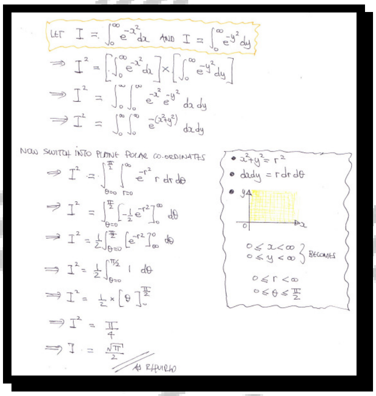
Question 8

$$I = \int_{0}^{\infty} e^{-x^2} \, dx \quad \text{and} \quad I = \int_{0}^{\infty} e^{-y^2} \, dy .$$

By considering an expression for I^2 and the use of plane polar coordinates, show clearly that

$$I = \frac{1}{2}\sqrt{\pi} .$$

$\boxed{\text{proof}}$


LET $I = \int_0^\infty e^{-x^2} dx$ AND $I = \int_0^\infty e^{-y^2} dy$

$\Rightarrow I^2 = \left[\int_0^\infty e^{-x^2} dx\right] \times \left[\int_0^\infty e^{-y^2} dy\right]$

$\Rightarrow I^2 = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dx \, dy$

$\Rightarrow I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$

NOW SWITCH INTO PLANE POLAR CO-ORDINATES

$\Rightarrow I^2 = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r \, dr \, d\theta$

$\Rightarrow I^2 = \int_{\theta=0}^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2}\right]_0^\infty d\theta$

$\Rightarrow I^2 = \frac{1}{2} \int_{\theta=0}^{\frac{\pi}{2}} \left[e^{-r^2}\right]_\infty^0 d\theta$

$\Rightarrow I^2 = \frac{1}{2} \int_{\theta=0}^{\pi/2} 1 \, d\theta$

$\Rightarrow I^2 = \frac{1}{2} \times \left[\theta\right]_0^{\frac{\pi}{2}}$

$\Rightarrow I^2 = \frac{\pi}{4}$

$\Rightarrow I = \frac{\sqrt{\pi}}{2}$

AS REQUIRED

- $x^2 + y^2 = r^2$
- $dx\,dy = r\,dr\,d\theta$

$\left.\begin{array}{l} 0 \le x < \infty \\ 0 \le y < \infty \end{array}\right\}$ BECOMES

$0 \le r < \infty$
$0 \le \theta \le \frac{\pi}{2}$

Question 9

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\ dx\,dy\ .$$

a) Use plane polar coordinates (r,θ), to find the exact simplified value of the above integral.

b) Hence evaluate

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx\ .$$

$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\ dx\,dy=\pi}$, $\boxed{\int_{-\infty}^{\infty} e^{-x^2}\ dx=\sqrt{\pi}}$

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \dots$ SWITCH INTO PLANE POLARS OVER THE ENTIRE xy PLANE

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}(r\,dr\,d\theta)$

$= \left[\int_{\theta=0}^{2\pi} 1\,d\theta\right]\left[\int_{r=0}^{\infty} re^{-r^2}\,dr\right]$

$= 2\pi \times \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = 2\pi \times \left[0-\left(-\frac{1}{2}\right)\right]$

$= \pi$

b) NOW $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \pi$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \pi$

$\left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right] = \pi$

$\left[\int_{-\infty}^{\infty} e^{-u^2}\,du\right]\left[\int_{-\infty}^{\infty} e^{-u^2}\,du\right] = \pi$

$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$

$\therefore \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Question 10

The points A and B have Cartesian coordinates $(0,1)$ and $(1,1)$, respectively.

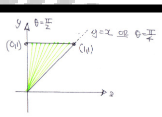
The finite region R is defined as the triangle OAB, where O is the origin.

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R y^2 \; dx\,dy .$$

[*No credit will be given for workings in other coordinate systems.*]

$\boxed{\frac{1}{4}}$



EQUATION $y=1$

$r\sin\theta=1$

$r=\operatorname{cosec}\theta$

THUS $0 \le r \le \operatorname{cosec}\theta$, $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\int_R y^2 \,dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{r=\operatorname{cosec}\theta} (r\sin\theta)^2 \, r\,dr\,d\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=\operatorname{cosec}\theta} r^3\sin^2\theta \; dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\sin^2\theta\right]_{r=0}^{r=\operatorname{cosec}\theta} d\theta = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\operatorname{cosec}^4\theta\sin^2\theta \; d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\operatorname{cosec}^2\theta \; d\theta = \left[-\frac{1}{4}\cot\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \left[\frac{1}{4}\cot\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[1-0\right] = \frac{1}{4}$$

Question 11

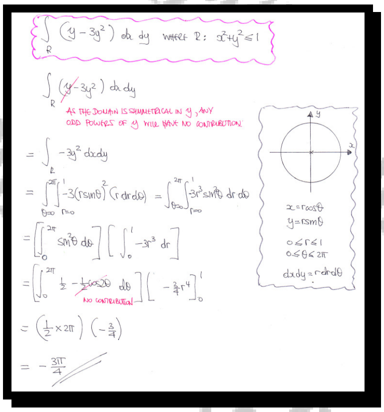
The finite region R, on the x-y plane, satisfies

$$x^2 + y^2 \le 1 .$$

Find, in terms of π, the value of

$$\int_R \left(y - 3y^2 \right) \, dx \, dy .$$

$$\boxed{-\frac{3\pi}{4}}$$



$$\boxed{\int_R \left(y - 3y^2 \right) dx\,dy \quad \text{WHERE } R:\ x^2+y^2 \le 1}$$

$$\int_R \left(\cancel{y} - 3y^2 \right) dx\,dy$$

AS THE DOMAIN IS SYMMETRICAL IN y, ANY ODD POWERS OF y WILL HAVE NO CONTRIBUTION

$$= \int_R -3y^2 \, dx\,dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} -3(r\sin\theta)^2 (r\,dr\,d\theta) = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} -3r^3 \sin^2\theta \, dr\,d\theta$$

$$= \left[\int_0^{2\pi} \sin^2\theta \, d\theta \right] \left[\int_0^1 -3r^3 \, dr \right]$$

$$= \left[\int_0^{2\pi} \frac{1}{2} - \cancel{\frac{1}{2}\cos 2\theta} \, d\theta \right] \left[-\frac{3}{4} r^4 \right]_0^1$$

NO CONTRIBUTION

$$= \left(\frac{1}{2} \times 2\pi \right) \left(-\frac{3}{4} \right)$$

$$= -\frac{3\pi}{4}$$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

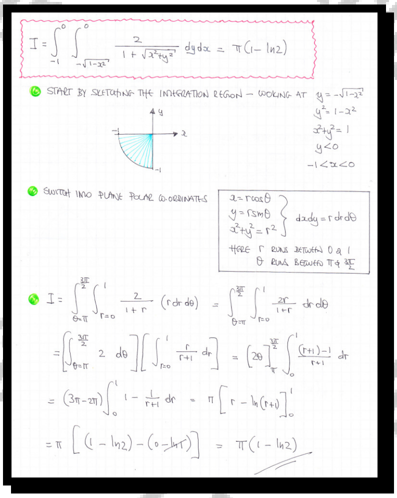
$dx\,dy = r\,dr\,d\theta$

Question 12

Find the exact simplified value for the following integral.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y}}\, dy\, dx \, .$$

$$\boxed{\pi(1-\ln 2)}$$



$$I = \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dy\, dx = \pi(1-\ln 2)$$

START BY SKETCHING THE INTEGRATION REGION – LOOKING AT $y = -\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2 = 1$

$y<0$

$-1<x<0$

SWITCH INTO PLANE POLAR CO-ORDINATES

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{array}\right\} \; dx\,dy = r\,dr\,d\theta$

HERE r RUNS BETWEEN 0 & 1

θ RUNS BETWEEN π & $\frac{3\pi}{2}$

$$I = \int_{\theta=\pi}^{\frac{3\pi}{2}}\int_{r=0}^{1} \frac{2}{1+r}\,(r\,dr\,d\theta) = \int_{\theta=\pi}^{\frac{3\pi}{2}}\int_{r=0}^{1} \frac{2r}{1+r}\,dr\,d\theta$$

$$= \left[\int_{\theta=\pi}^{\frac{3\pi}{2}} 2\, d\theta\right]\left[\int_{r=0}^{1} \frac{r}{r+1}\,dr\right] = \Big[2\theta\Big]_{\pi}^{\frac{3\pi}{2}} \int_{0}^{1} \frac{(r+1)-1}{r+1}\,dr$$

$$= (3\pi-2\pi)\int_{0}^{1} 1-\frac{1}{r+1}\,dr = \pi\Big[r-\ln(r+1)\Big]_{0}^{1}$$

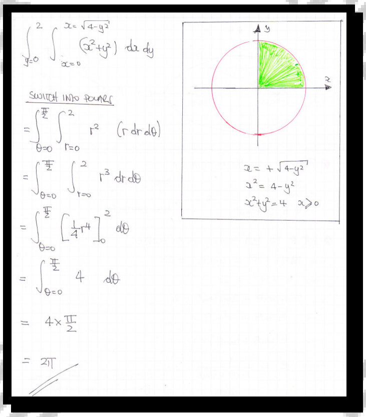
$$= \pi\Big[(1-\ln 2)-(0-\cancel{\ln 1})\Big] = \pi(1-\ln 2)$$

Question 13

Find the exact simplified value for the following integral.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} x^2 + y^2 \quad dy\,dx\,.$$

$\boxed{2\pi}$



Question 14

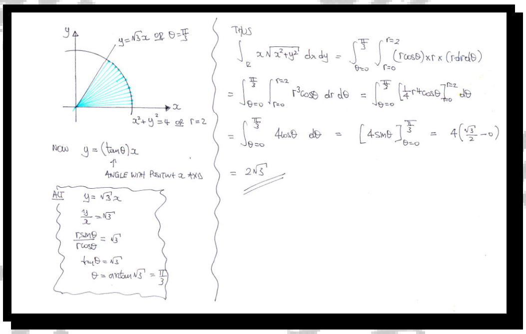
The finite region R is bounded by the straight lines and curves with the following equations.

$$y = 0, \; x = 0, \; x^2 + y^2 = 4 \;\text{ and }\; y = \sqrt{3}x.$$

Determine an exact simplified value for

$$\int_R x\sqrt{x^2+y^2}\; dx\,dy .$$

$\boxed{2\sqrt{3}}$

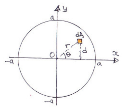


Question 15

A uniform circular lamina of mass M and radius a.

Use double integration to find the moment of inertia of the lamina, when the axis of rotation is a diameter.

$$\boxed{\frac{1}{4}Ma^2}$$



- $\rho = \frac{M}{\pi a^2}$ (MASS PER UNIT AREA)
- LET THE AXIS OF ROTATION BE THE x-AXIS
- INFINITESIMAL MOMENT OF INERTIA IS $(\rho\,dA)\,d^2$
- IN POLARS $d = r\sin\theta$

SUMMING UP ALL INFINITESIMALS IN PLANE POLARS

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \rho d^2\,(r\,dr\,d\theta) = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{M}{\pi a^2}(r^2\sin^2\theta)r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{M}{\pi a^2} r^3\sin^2\theta\,dr\,d\theta = \int_{\theta=0}^{2\pi}\left[\frac{M}{4\pi a^2} r^4\sin^2\theta\right]_{r=0}^{r=a} d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{Ma^2}{4\pi}\sin^2\theta\,d\theta = \frac{Ma^2}{4\pi}\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta\,d\theta$$

$$= \frac{Ma^2}{4\pi}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{Ma^2}{4\pi}\left[(\pi - 0) - (0 - 0)\right]$$

$$= \frac{1}{4}Ma^2$$

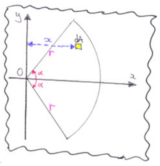
Question 16

A circular sector of radius r subtends an angle of 2α at its centre O. The position of the centre of mass of this sector lies at the point G, along its axis of symmetry.

Use calculus to show that

$$|OG| = \frac{2r\sin\alpha}{3\alpha}.$$

proof



- LET ρ = MASS PER UNIT AREA
- MASS OF INFINITESIMAL AREA IS $\rho\, dA$
- MOMENT OF INFINITESIMAL ABOUT THE y-AXIS IS $(\rho\, dA)x$
- TOTAL MASS OF THE SECTOR IS $\frac{1}{2}\rho r^2(2\alpha) = \alpha r^2\rho$
- SUMMING MOMENTS ABOUT THE y-AXIS
  $$M\bar{x} = \int_{\text{SECTOR}} \rho x\, dA$$
- $\bar{x}$ WILL ALSO BE THE POSITION OF THE CENTRE OF MASS BY SYMMETRY
- TAKING LIMITS
  $$\alpha r^2\rho\bar{x} = \int_{\text{SECTOR}} \rho x\, dA$$
- SWITCH INTO PLANE POLARS

$$\Rightarrow \underset{\text{CONSTANT}}{\alpha r^2\rho}\,\bar{x} = \int_{\theta=-\alpha}^{\alpha}\int_{r=0}^{r \leftarrow \text{CONSTANT}} \rho(r\cos\theta)\, r\, dr\, d\theta$$

$$\Rightarrow \underset{\text{CONSTANT}}{\alpha r^2\rho}\,\bar{x} = \rho\int_{-\alpha}^{\alpha}\int_{0}^{r} r^2\cos\theta\, dr\, d\theta$$

$$\Rightarrow \alpha r^2\bar{x} = \int_{-\alpha}^{\alpha}\left[\frac{1}{3}r^3\right]_0^r \cos\theta\, d\theta$$

$$\Rightarrow \alpha r^2\bar{x} = \frac{1}{3}r^3\int_{-\alpha}^{\alpha}\cos\theta\, d\theta$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \int_{-\alpha}^{\alpha}\cos\theta\, d\theta$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \left[\sin\theta\right]_{-\alpha}^{\alpha}$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \left[\sin\alpha - \sin(-\alpha)\right]$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \sin\alpha + \sin\alpha$$

$$\Rightarrow \frac{3\alpha\bar{x}}{r} = 2\sin\alpha$$

$$\Rightarrow \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Question 17

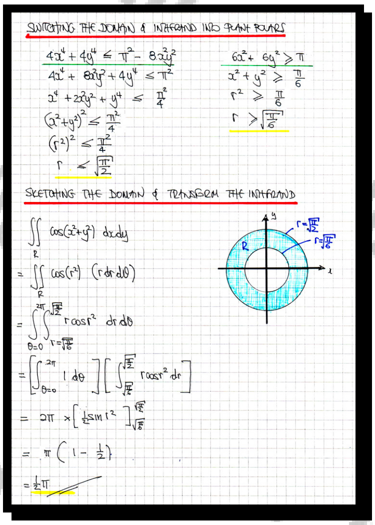
The finite region on the x-y plane satisfies

$$4x^4 + 4y^4 \le \pi^2 - 8x^2y^2 \quad \text{and} \quad 6x^2 + 6y^2 \ge \pi .$$

Find the value of the following integral.

$$\int_R \cos\left(x^2 + y^2\right)\, dx\, dy .$$

$\boxed{}$, $\boxed{\frac{1}{2}\pi}$



SWITCHING THE DOMAIN & INTEGRAND INTO PLANE POLARS

$4x^4 + 4y^4 \le \pi^2 - 8x^2y^2$
$4x^4 + 8x^2y^2 + 4y^4 \le \pi^2$
$x^4 + 2x^2y^2 + y^4 \le \frac{\pi^2}{4}$
$(x^2+y^2)^2 \le \frac{\pi^2}{4}$
$(r^2)^2 \le \frac{\pi^2}{4}$
$r \le \sqrt{\frac{\pi}{2}}$

$6x^2 + 6y^2 \ge \pi$
$x^2 + y^2 \ge \frac{\pi}{6}$
$r^2 \ge \frac{\pi}{6}$
$r \ge \sqrt{\frac{\pi}{6}}$

SKETCHING THE DOMAIN & TRANSFORM THE INTEGRAND

$\iint_R \cos(x^2+y^2)\,dx\,dy$
$= \iint_R \cos(r^2)\,(r\,dr\,d\theta)$
$= \int_{\theta=0}^{2\pi}\int_{r=\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} r\cos r^2\,dr\,d\theta$
$= \left[\int_{\theta=0}^{2\pi} 1\,d\theta\right]\left[\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} r\cos r^2\,dr\right]$
$= 2\pi \times \left[\frac{1}{2}\sin r^2\right]_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}}$
$= \pi\left(1 - \frac{1}{2}\right)$
$= \frac{1}{2}\pi$

Question 18

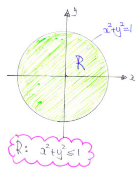
The finite region R, on the x-y plane, satisfies

$$x^2 + y^2 \le 1.$$

Find, in terms of π, the value of I.

$$I = \int_R \left(1 + 3xy + 4x - 2yx^2\right)\, dx\, dy.$$

$\boxed{\pi}$



$R: \ x^2+y^2 \le 1$

$$\iint_R 1 + \cancel{3xy} + \cancel{4x} - \cancel{2yx^2}\, dx\, dy$$

- AS THE INTEGRATION REGION IS CLEARLY A SYMMETRICAL DOMAIN IN x $[-1,1]$ ANY ODD POWERS OF x WILL HAVE NO CONTRIBUTION
- AS THE INTEGRATION REGION IS CLEARLY A SYMMETRICAL DOMAIN IN y $[-1,1]$ ANY ODD POWERS OF y WILL HAVE NO CONTRIBUTION

HENCE THE INTEGRAL REDUCES TO

$$\iint_R 1\, dx\, dy = \text{AREA OF UNIT CIRCLE} = \pi$$

ALTERNATIVE SWITCH INTO PLANE POLARS

$$\iint_R 1 + 3xy + 4x - 2yx^2\, dx\, dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left[1 + 3(r\cos\theta)(r\sin\theta) + 4r\cos\theta - 2(r\sin\theta)(r\cos\theta)^2\right]\left[r\, dr\, d\theta\right]$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left[r + \cancel{3r^3\cos\theta\sin\theta} + \cancel{4r^2\cos\theta} - \cancel{2r^4\sin\theta\cos^2\theta}\right] dr\, d\theta$$

NO CONTRIBUTION FROM THE θ INTEGRATION FOR LIMITS 0 to 2π

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\, dr\, d\theta = \int_{\theta=0}^{2\pi}\left[\tfrac{1}{2}r^2\right]_0^1 d\theta = \int_{\theta=0}^{2\pi} \tfrac{1}{2}\, d\theta$$

$$= \left[\tfrac{1}{2}\theta\right]_0^{2\pi} = \tfrac{1}{2}\times 2\pi = \pi$$

OR CONTINUE WITH THE POLAR INTEGRATION

$$\ldots = \int_0^{2\pi}\int_0^1 \left[r + 3r^3\cos\theta\sin\theta + 4r^2\cos\theta - 2r^4\sin\theta\cos^2\theta\right] dr\, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{1}{2}r^2 + \tfrac{3}{4}r^4\cos\theta\sin\theta + \tfrac{4}{3}r^3\cos\theta - \tfrac{2}{5}r^5\sin\theta\cos^2\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2} + \tfrac{3}{4}\cos\theta\sin\theta + \tfrac{4}{3}\cos\theta - \tfrac{2}{5}\sin\theta\cos^2\theta\, d\theta$$

$$= \left[\tfrac{1}{2}\theta + \tfrac{3}{8}\sin^2\theta + \tfrac{4}{3}\sin\theta + \tfrac{2}{15}\cos^3\theta\right]_0^{2\pi}$$

$$= \left(\pi + 0 + 0 + \cancel{\tfrac{2}{15}}\right) - \left(0 + 0 + 0 + \cancel{\tfrac{2}{15}}\right)$$

$$= \pi$$

AS BEFORE

Question 19

The finite region R is bounded by the straight lines with the following equations.

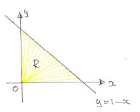
$$x = 0, \qquad y = 0 \quad \text{and} \quad y = 1 - x.$$

Use plane **polar** coordinates, (r, θ) to determine the value of

$$\int_R \frac{x+y}{x^2+y^2} \, dx\, dy.$$

[*No credit will be given for workings in other coordinate systems*.]

$$\boxed{\frac{\pi}{2}}$$



- $y = 1 - x$

$r\sin\theta = 1 - r\cos\theta$

$r(\sin\theta + \cos\theta) = 1$

- $r = \dfrac{1}{\sin\theta + \cos\theta}$

So $\begin{cases} 0 \leqslant r \leqslant \dfrac{1}{\sin\theta + \cos\theta} \\ 0 \leqslant \theta \leqslant \dfrac{\pi}{2} \end{cases}$

$$\int_R \frac{x+y}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{r=\frac{1}{\sin\theta+\cos\theta}} \frac{r\cos\theta + r\sin\theta}{r^2}\,(r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{r=\frac{1}{\sin\theta+\cos\theta}} (\cos\theta + \sin\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \Big[r(\cos\theta + \sin\theta) \Big]_{r=0}^{r=\frac{1}{\sin\theta+\cos\theta}} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 1 \, d\theta = \Big[\theta \Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Question 20

$$I = \int_{0}^{2} \int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}} \frac{2y}{x^2+y^2} \, dx \, dy \, .$$

Use polar coordinates to find an exact simplified answer for I .

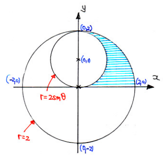
$\boxed{}$, $\boxed{4-\pi}$

AUXILIARIES FIRST INCLUDING A REGION OF INTEGRATION DIAGRAM

- $x = \sqrt{2y-y^2}$
  $x^2 = 2y - y^2$
  $x^2+y^2-2y = 0$
  $x^2 + (y-1)^2 = 1$
- $x = \sqrt{4-y^2}$
  $x^2 = 4-y^2$
  $x^2+y^2 = 4$

SWITCH INTO PLANE POLARS

- $x^2+y^2-2y = 0$
  $r^2 - 2r\sin\theta = 0$
  $r - 2\sin\theta = 0$
  $r = 2\sin\theta$
- $x^2+y^2 = 4$
  $r^2 = 4$
  $r = 2$



FINISHING OFF THE INTEGRATION

$$I = \int_0^2 \int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}} \frac{2y}{x^2+y^2} \, dx\,dy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=2\sin\theta}^{r=2} \frac{2(r\sin\theta)}{r^2} \underbrace{(r\,dr\,d\theta)}_{dx\,dy}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=2\sin\theta}^{r=2} \frac{2r^2\sin\theta}{r^2} \, dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=2\sin\theta}^{r=2} 2\sin\theta \, dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \Big[2r\sin\theta \Big]_{r=2\sin\theta}^{r=2} \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 2(2\sin\theta)\sin\theta \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 4\sin^2\theta \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 4\sin\theta - 2 + 2\cos 2\theta \, d\theta$$

$$= \Big[-4\cos\theta - 2\theta + \sin 2\theta \Big]_0^{\frac{\pi}{2}}$$

$$= (0 - \pi + 0) - (-4 - 0 + 0)$$

$$= 4 - \pi$$

Question 21

The finite region R is bounded by the straight lines with the following equations.

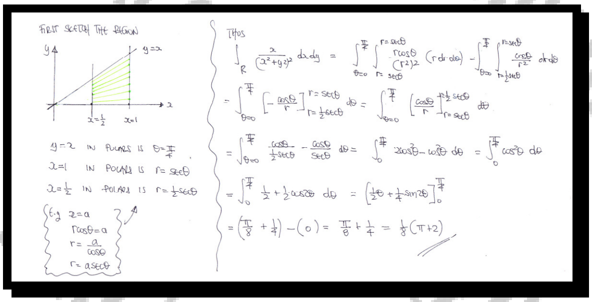
$$y=0, \quad x=\frac{1}{2}, \quad x=1 \quad \text{and} \quad y=x.$$

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R \frac{x}{\left(x^2+y^2\right)^2}\,dx\,dy.$$

[*No credit will be given for workings in other coordinate systems*.]

$$\boxed{\frac{1}{8}(\pi+2)}$$



Question 22

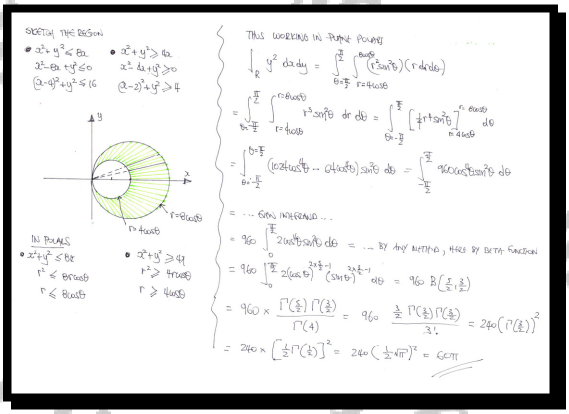
The finite region R is defined as

$$4x \le x^2 + y^2 \le 8x .$$

Determine the value of

$$\int_R y^2 \; dx\,dy .$$

$\boxed{60\pi}$



SKETCH THE REGION

- $x^2+y^2 \le 8x$
  $x^2-8x+y^2 \le 0$
  $(x-4)^2+y^2 \le 16$
- $x^2+y^2 \ge 4x$
  $x^2-4x+y^2 \ge 0$
  $(x-2)^2+y^2 \ge 4$

IN POLARS

- $x^2+y^2 \le 8x$
  $r^2 \le 8r\cos\theta$
  $r \le 8\cos\theta$
- $x^2+y^2 \ge 4x$
  $r^2 \ge 4r\cos\theta$
  $r \ge 4\cos\theta$

THUS WORKING IN PLANE POLARS

$$\int_R y^2 \, dx\,dy = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=4\cos\theta}^{8\cos\theta} (r^2\sin^2\theta)(r\,dr\,d\theta)$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=4\cos\theta}^{r=8\cos\theta} r^3\sin^2\theta \, dr\,d\theta = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tfrac{1}{4}r^4\sin^2\theta\right]_{r=4\cos\theta}^{r=8\cos\theta} d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} (1024\cos^4\theta - 64\cos^4\theta)\sin^2\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 960\cos^4\theta\sin^2\theta \, d\theta$$

$= \dots$ EVEN INTEGRAND $\dots$

$= 960 \int_0^{\frac{\pi}{2}} 2\cos^4\theta\sin^2\theta \, d\theta = \dots$ BY ANY METHOD, HERE BY BETA FUNCTION

$$= 960 \int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{5}{2}-1}(\sin\theta)^{2\times\frac{3}{2}-1} d\theta = 960\, B\left(\tfrac{5}{2},\tfrac{3}{2}\right)$$

$$= 960 \times \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)} = 960 \, \frac{\frac{3}{2}\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{3!} = 240\left(\Gamma(\tfrac{3}{2})\right)^2$$

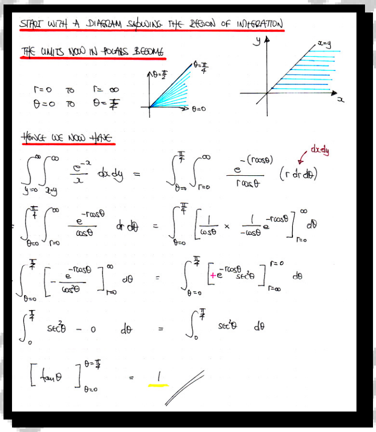
$$= 240 \times \left[\tfrac{1}{2}\Gamma(\tfrac{1}{2})\right]^2 = 240\left(\tfrac{1}{2}\sqrt{\pi}\right)^2 = 60\pi$$

Question 23

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty} \frac{e^{-x}}{x}\, dxdy \, .$$

, $\boxed{1}$


START WITH A DIAGRAM SHOWING THE REGION OF INTEGRATION

THE LIMITS NOW IN POLARS BECOME

$r=0$ TO $r=\infty$

$\theta=0$ TO $\theta=\frac{\pi}{4}$

HENCE WE NOW HAVE

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty} \frac{e^{-x}}{x}\,dxdy = \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\infty} \frac{e^{-(r\cos\theta)}}{r\cos\theta}\,(\overbrace{r\,dr\,d\theta}^{dxdy})$$

$$\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\infty} \frac{e^{-r\cos\theta}}{\cos\theta}\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{4}} \left[\frac{1}{\cos\theta} \times \frac{1}{-\cos\theta}e^{-r\cos\theta}\right]_{r=0}^{\infty} d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{4}} \left[-\frac{e^{-r\cos\theta}}{\cos^2\theta}\right]_{r=0}^{\infty} d\theta = \int_{\theta=0}^{\frac{\pi}{4}} \left[+e^{-r\cos\theta}\sec^2\theta\right]_{r=\infty}^{r=0} d\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^2\theta - 0 \quad d\theta = \int_0^{\frac{\pi}{4}} \sec^2\theta \quad d\theta$$

$$\left[\tan\theta\right]_{\theta=0}^{\theta=\frac{\pi}{4}} = 1$$

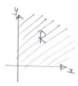
Question 24

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R e^{-(x+y)^2}\ dx\,dy\ ,$$

where R is the region in the first quadrant in a standard Cartesian coordinate system.

$\boxed{\frac{1}{2}}$



$$\iint_R e^{-(x+y)^2}\,dx\,dy = \dots \text{ SWITCH INTO POLARS}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-(r\cos\theta + r\sin\theta)^2}\, r\,d\theta\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} r e^{-r^2(\cos\theta+\sin\theta)^2}\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} -\frac{1}{2(\cos\theta+\sin\theta)^2}\left[e^{-r^2(\cos\theta+\sin\theta)^2}\right]_{r=0}^{\infty} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} -\frac{1}{2(\cos\theta+\sin\theta)^2}\left[0-(1)\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2(\cos\theta+\sin\theta)^2}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\theta\left(1+\frac{\sin\theta}{\cos\theta}\right)^2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{(1+\tan\theta)^2}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sec^2\theta\,(1+\tan\theta)^{-2}\,d\theta$$

$$= \frac{1}{2}\left[-(1+\tan\theta)^{-1}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{1}{1+\tan\theta}\right]_{\frac{\pi}{2}}^{0}$$

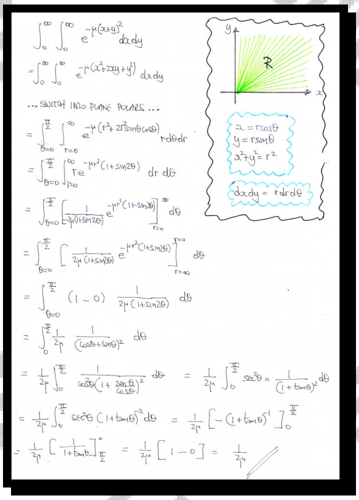
$$= \frac{1}{2}\left[1-0\right] = \frac{1}{2}$$

Question 25

Given that μ is a positive constant determine the value of

$$\int_0^{\infty}\int_0^{\infty} e^{-\mu(x+y)^2}\ dx\,dy\ ,$$

$$\boxed{\frac{1}{2\mu}}$$



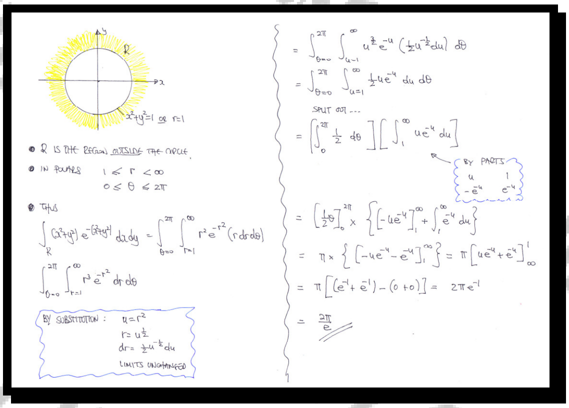
Question 26

Determine an exact simplified value for

$$\iint_R \left(x^2+y^2\right)e^{-\left(x^2+y^2\right)}\ dx\,dy\,,$$

where R is the region $x^2+y^2>1$

$$\boxed{\frac{2\pi}{e}}$$



Question 27

Find the exact simplified value for the following integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{\sqrt{16x^2+16y^2}}{x^2+y^2+1}\,dx\,dy\,.$$

$\boxed{}$, $\boxed{\pi(4-\pi)}$

LOOKING AT THE REGION OF INTEGRATION AND THE STRUCTURE OF THE INTEGRAND IT IS EVIDENT THAT PLANE POLAR COORDINATES ARE NEEDED

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{\sqrt{16x^2+16y^2}}{x^2+y^2+1}\,dx\,dy$$

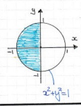
$$= \int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{r=0}^{1} \frac{\sqrt{16r^2}}{r^2+1}\,(r\,dr\,d\theta)$$

$$= \int_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{r=0}^{1} \frac{4r^2}{r^2+1}\,dr\,d\theta$$

$x = -\sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2=1,\ x\leq 0$



CARRY OUT THE INTEGRATION WITH RESPECT TO θ FIRST

$$= \left(\frac{3\pi}{2}-\frac{\pi}{2}\right)\int_{0}^{1} \frac{4r^2}{r^2+1}\,dr$$

$$= \pi\int_{0}^{1} \frac{4r^2}{r^2+1}\,dr$$

MANIPULATE THE INTEGRAND AS FOLLOWS

$$= \pi\int_{0}^{1} \frac{4(r^2+1)-4}{r^2+1}\,dr$$

$$= \pi\int_{0}^{1} 4-\frac{4}{r^2+1}\,dr$$

$$= \pi\Big[4r - 4\arctan r\Big]_{0}^{1}$$

$$= \pi\Big[(4-4\arctan 1)-(0)\Big]$$

$$= \pi\left(4-4\times\frac{\pi}{4}\right)$$

$$= \pi(4-\pi)$$

Question 28

The finite region R is bounded by the straight lines with the following equations.

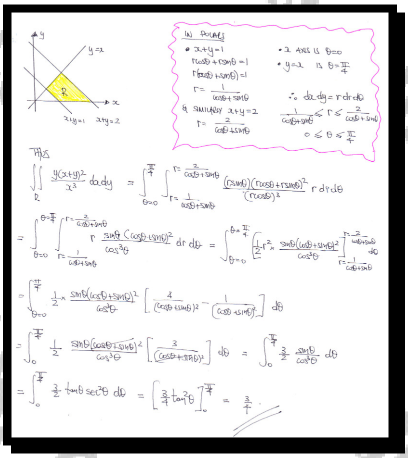
$$x+y=1, \quad x+y=2, \quad y=x \quad \text{and} \quad y=0.$$

Use plane polar coordinates (r,θ) to find the value of

$$\iint_R \frac{y(x+y)^2}{x^3}\,dx\,dy .$$

[*No credit will be given for workings in other coordinate systems*.]

$$\boxed{\frac{3}{4}}$$



Question 29

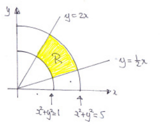
The region R on the x-y plane is defined by the inequalities

$$1 \le x^2 + y^2 \le 5 \qquad \text{and} \qquad \frac{1}{2}x \le y \le 2x .$$

Show clearly that

$$\int_R (x+y)\, dx\, dy = \frac{2}{15}\left(25 - \sqrt{5}\right).$$

$\boxed{\text{proof}}$



SWITCH INTO POLARS

$y = \frac{1}{2}x \implies y = \left(\arctan\frac{1}{2}\right)x$
$\implies \theta = \arctan\frac{1}{2} = \theta_1$

$y = 2x \implies y = (\arctan 2)x$
$\implies \theta = \arctan 2 = \theta_2$

$x^2+y^2=1 \implies r=1$
$x^2+y^2=5 \implies r=\sqrt{5}$

$\sin\theta_1 = \frac{1}{\sqrt{5}}, \ \cos\theta_1 = \frac{2}{\sqrt{5}}$
$\sin\theta_2 = \frac{2}{\sqrt{5}}, \ \cos\theta_2 = \frac{1}{\sqrt{5}}$

THUS IN PLANE POLARS:

$$\int_R (x+y)\,dx\,dy = \int_R (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta = \int_R r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_{\theta_1}^{\theta_2}\int_{r=1}^{\sqrt{5}} r^2(\cos\theta+\sin\theta)\,dr\,d\theta = \int_{\theta_1}^{\theta_2}\left[\frac{1}{3}r^3(\cos\theta+\sin\theta)\right]_{r=1}^{r=\sqrt{5}} d\theta$$

$$= \int_{\theta_1}^{\theta_2}\left(\frac{1}{3}\times 5\sqrt{5} - \frac{1}{3}\right)(\cos\theta+\sin\theta)\,d\theta = \frac{1}{3}(5\sqrt{5}-1)\int_{\theta_1}^{\theta_2}\cos\theta+\sin\theta\,d\theta$$

$$= \frac{1}{3}(5\sqrt{5}-1)\Big[\sin\theta - \cos\theta\Big]_{\theta_1}^{\theta_2} = \frac{1}{3}(5\sqrt{5}-1)\Big[(\sin\theta_2-\cos\theta_2)-(\sin\theta_1-\cos\theta_1)\Big]$$

$$= \frac{1}{3}(5\sqrt{5}-1)\left[\left(\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{5}}\right)-\left(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}\right)\right]$$

$$= \frac{1}{3}(5\sqrt{5}-1)\times\frac{2}{\sqrt{5}}$$

$$= \frac{2}{3}\left(\frac{5\sqrt{5}-1}{\sqrt{5}}\right) = \frac{2}{3}\left[\frac{(5\sqrt{5}-1)\sqrt{5}}{\sqrt{5}\sqrt{5}}\right] = \frac{2}{3}\left[\frac{25-\sqrt{5}}{5}\right]$$

$$= \frac{2}{3}\left(5 - \frac{\sqrt{5}}{5}\right) = \frac{2}{15}\left(25-\sqrt{5}\right)$$

AS REQUIRED

Question 30

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

It is asserted that

$$I = \int_{-\infty}^{\infty} e^{-x^2} \cos\left(2x^2\right)\, dx = \sqrt{\frac{\pi\phi}{5}}.$$

By considering the real part of a suitable function, use double integration in plane polar coordinates to prove the validity of the above result.

You may assume the principal value in any required complex evaluation.

$\boxed{}$, $\boxed{\text{proof}}$

CONSIDER THE FOLLOWING INTEGRAL

$$\Rightarrow I = \int_{-\infty}^{\infty} e^{-x^2} e^{i(2x^2)}\, dx = \int_{-\infty}^{\infty} e^{-x^2(1-2i)}\, dx$$

OUR INTEGRAL IS THE REAL PART OF "I"

$$\Rightarrow I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2(1-2i)}\, dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2(1-2i)}\, dy\right]$$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2(1-2i)} \times e^{-y^2(1-2i)}\, dx\, dy$$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(1-2i)(x^2+y^2)}\, dx\, dy$$

THE AREA OF INTEGRATION IS THE ENTIRE x-y PLANE SO VERY EASY TO TRANSFORM INTO POLARS

$$\Rightarrow I^2 = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-(1-2i)r^2} (r\, dr\, d\theta)$$

IN POLARS
$x^2+y^2 = r^2$
$dx\,dy = r\,dr\,d\theta$

CARRY OUT THE θ INTEGRATION FIRST

$$\Rightarrow I^2 = 2\pi \int_0^{\infty} r e^{-(1-2i)r^2}\, dr$$

$$\Rightarrow I^2 = 2\pi \left[-\frac{1}{2} \times \frac{1}{1-2i} \times e^{-(1-2i)r^2}\right]_0^{\infty}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[e^{-(1-2i)r^2}\right]_{\infty}^{0}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[e^{-r^2} \times e^{2r^2 i}\right]_{\infty}^{0}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[e^{-r^2}\left(\cos 2r^2 + i \sin 2r^2\right)\right]_{\infty}^{0}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[1(1+0i) - 0\left(\text{"}\cos\infty + i\sin\infty\text{"}\right)\right]$$

FINITE

$$\Rightarrow I^2 = \frac{\pi}{1-2i} = \frac{\pi(1+2i)}{5}$$

$$\Rightarrow I = \sqrt{\frac{\pi}{5}}\sqrt{1+2i}$$

& RECALL WE WANT THE REAL PART OF THIS AT THE END

NOW WE NEED THE PRINCIPAL VALUE OF $\sqrt{1+2i}$

$$\Rightarrow z^2 = 1+2i$$

$$\Rightarrow z^2 = |1+2i|\, e^{i\arg(1+2i)}$$

PRINCIPAL ONLY

$$\Rightarrow z^2 = \sqrt{5}\, e^{i\theta}$$

WHERE $\theta = \arctan 2$

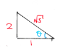
$$\Rightarrow z^2 = 5^{\frac{1}{2}} e^{i\theta}$$

$$\Rightarrow z = 5^{\frac{1}{4}} e^{i\frac{\theta}{2}}$$

RETURNING TO THE INTEGRAL

$$\Rightarrow I = \sqrt{\frac{\pi}{5}} \times 5^{\frac{1}{4}} e^{i\frac{\theta}{2}} \quad [\theta = \arctan 2]$$

PROCEED AS FOLLOWS



$\tan\theta = 2$
$\cos\theta = \frac{1}{\sqrt{5}}$

$$\Rightarrow \cos 2A = 2\cos^2 A - 1$$

$$\Rightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow \frac{1}{\sqrt{5}} = 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow 1 + \frac{1}{\sqrt{5}} = 2\cos^2\frac{\theta}{2}$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{2}\left(\frac{\sqrt{5}+1}{\sqrt{5}}\right)$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{\sqrt{5}}\phi$$

$$\Rightarrow \cos\frac{\theta}{2} = +\sqrt{\frac{\phi}{\sqrt{5}}}$$

RETURN TO THE MAIN QUESTION

$$\Rightarrow I = \sqrt{\frac{\pi}{5}} \times 5^{\frac{1}{4}} \times e^{i\frac{\theta}{2}}$$

$$\Rightarrow I = \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}} \times \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\cos(2x^2)\, dx = \mathrm{Re}\left[\frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]\right]$$

$$= \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}} \times \cos\frac{\theta}{2}$$

$$= \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}} \times \sqrt{\frac{\phi}{\sqrt{5}}}$$

$$= \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times \cancel{5^{\frac{1}{4}}} \times \frac{\phi^{\frac{1}{2}}}{\cancel{5^{\frac{1}{4}}}}$$

$$= \frac{\pi^{\frac{1}{2}}\phi^{\frac{1}{2}}}{5^{\frac{1}{2}}}$$

$$= \sqrt{\frac{\pi\phi}{5}}$$

AS REQUIRED

CYLINDRICAL COORDINATES

Question 1

Find the value of

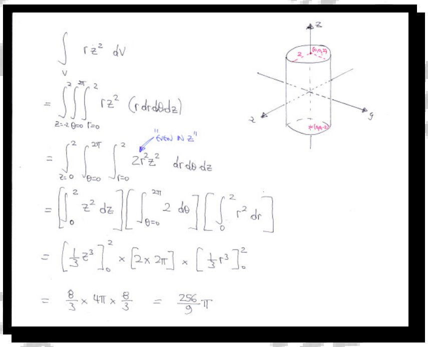
$$\int_{\Omega} rz^2 \ dV ,$$

where Ω is the region inside the cylinder with equation

$$x^2 + y^2 = 4 , \ -2 \le z \le 2$$

In this question use cylindrical polar coordinates (r, θ, z).

$$\boxed{\frac{256\pi}{9}}$$



Question 2

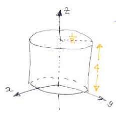
Find the value of

$$\int_V (x+y+z)\ dx\,dy\,dz\,,$$

where V is the region inside the cylinder with equation

$$x^2+y^2=\frac{1}{4},\ 0 \le z \le 4\,.$$

$\boxed{2\pi}$



$\int_V (x+y+z)\ dV$ = WORK IN CYLINDRICAL POLARS

$$\int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{\frac{1}{2}} (r\cos\theta + r\sin\theta + z)\ r\,dr\,d\theta\,dz$$

FROM THE θ LIMITS

$$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{\frac{1}{2}} \cancel{r^2\cos\theta} + \cancel{r^2\sin\theta} + rz\ dr\,d\theta\,dz$$

$$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi} \left[\frac{1}{2}r^2 z\right]_{r=0}^{\frac{1}{2}} d\theta\,dz$$

$$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi} \frac{1}{8}z\ d\theta\,dz$$

$$= \int_{z=0}^{4} \left[\frac{1}{8}z\theta\right]_{\theta=0}^{\theta=2\pi} dz$$

$$= \int_{z=0}^{4} \frac{1}{4}\pi z\ dz$$

$$= \left[\frac{1}{8}\pi z^2\right]_0^4$$

$$= 2\pi$$

Question 3

Find in exact form the volume enclosed by the cylinder with equation

$$x^2 + y^2 = 16, \; z \geq 0,$$

and the plane with equation

$$z = 12 - x.$$

$\boxed{192\pi}$

SET THE PROBLEM AS A VOLUME INTEGRAL <u>OR</u> AS $\int_R f(x,y)\,dx\,dy$ ...

$$\int_V 1\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=4}\int_{z=0}^{z=12-x} 1\; r\,dr\,d\theta\,dz \quad \text{(CYLINDRICAL POLARS)}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\int_{z=0}^{z=12-x} r\;dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\Big[rz\Big]_{z=0}^{z=12-x}\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} r(12-x)\;dr\,d\theta$$

SWITCH x INTO POLARS I.E $x = r\cos\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} r(12-r\cos\theta)\;dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} 12r - r^2\cos\theta\;dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi}\Big[6r^2 - \tfrac{1}{3}r^3\cos\theta\Big]_{r=0}^{r=4}\,d\theta$$

$$= \int_{\theta=0}^{2\pi} 96 - 8\cos\theta\;d\theta$$

$$= \Big[96\theta - 8\sin\theta\Big]_0^{2\pi}$$

$$= 192\pi$$

Question 4

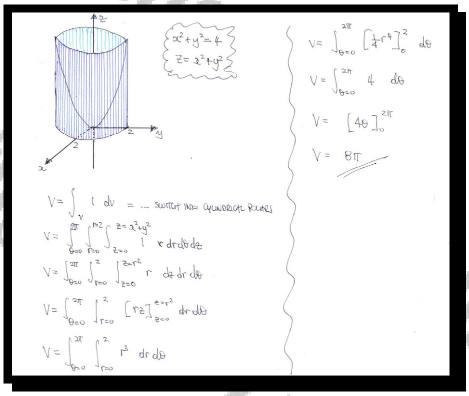
Find the volume of the region bounded by the cylinder with equation

$$x^2 + y^2 = 4,$$

and the surfaces with equations

$$z = x^2 + y^2 \quad \text{and} \quad z = 0.$$

$\boxed{8\pi}$

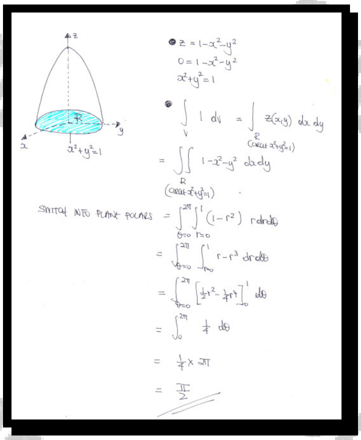


Question 5

Find the volume of the paraboloid with equation

$$z = 1 - x^2 - y^2, \; z \geq 0 .$$

$$\boxed{\frac{\pi}{2}}$$



- $z = 1 - x^2 - y^2$
  $0 = 1 - x^2 - y^2$
  $x^2 + y^2 = 1$
- $\displaystyle\int_V 1 \, dV = \int_{\substack{R \\ (\text{circle } x^2+y^2=1)}} z(x,y) \, dx \, dy$

$$= \iint_{\substack{R \\ (\text{circle } x^2+y^2=1)}} 1 - x^2 - y^2 \, dx \, dy$$

SWITCH INTO PLANE POLARS $= \displaystyle\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 - r^2) \, r \, dr \, d\theta$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r - r^3 \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4 \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4} \, d\theta$$

$$= \tfrac{1}{4} \times 2\pi$$

$$= \frac{\pi}{2}$$

Question 6

The finite region Ω is enclosed by the cylinder with Cartesian equation

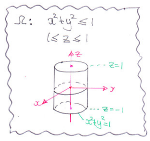
$$x^2 + y^2 = 1, \quad -1 \le z \le 1 .$$

Determine an exact simplified value for

$$\int_{\Omega} \left(1+z^3\right) e^{x^2+y^2} \, dx\,dy\,dz .$$

$$\boxed{2\pi[e-1]}$$

$$\int_{\Omega} (1+z^3) e^{x^2+y^2} \, dV$$



SWITCH INTO CYLINDRICAL POLARS

$$= \int_{\Omega} (1+z^3) e^{x^2+y^2} \, dx\,dy\,dz$$

$$= \int_{z=-1}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1+z^3) e^{r^2} (r \, dr \, d\theta \, dz)$$

$$= \int_{z=-1}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1+\cancel{z^3}) r e^{r^2} \, dr \, d\theta \, dz$$

ODD IN z, IN A SYMMETRICAL DOMAIN

$$= \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 2r e^{r^2} \, dr \, d\theta \, dz$$

$$= \left[\int_{z=0}^{1} 1 \, dz \right] \left[\int_{\theta=0}^{2\pi} 1 \, d\theta \right] \left[\int_{r=0}^{1} 2r e^{r^2} \, dr \right]$$

$$= 1 \times 2\pi \times \left[e^{r^2} \right]_0^1$$

$$= 2\pi [e-1]$$

Question 7

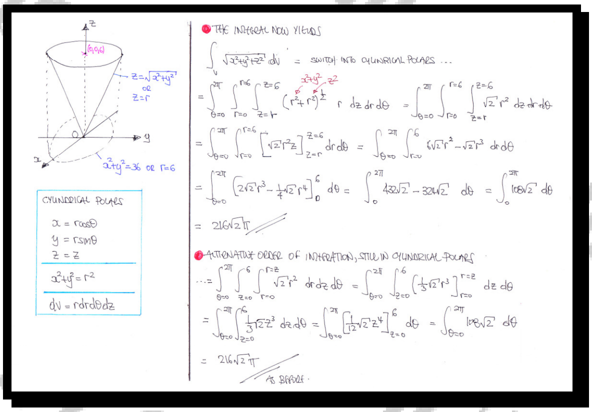
The finite region V is enclosed by the cone with Cartesian equation

$$z=\sqrt{x^2+y^2}\,,\quad 0\le z\le 6\,.$$

Determine an exact simplified value for

$$\int_V \sqrt{x^2+y^2+z^2}\;dx\,dy\,dz\,.$$

$$\boxed{216\sqrt{2}\,\pi}$$



Question 8

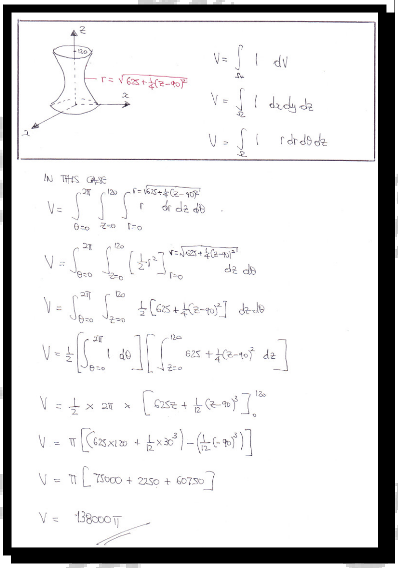
The height z, of a cooling tower, is 120 m.

The radius $r \text{ m}$, of any of the circular cross sections of the cooling tower is given by the equation

$$r = \sqrt{625 + \frac{1}{4}(z-90)}.$$

Use cylindrical polar coordinates (r, θ, z), to find the volume of the tower.

$\boxed{138000\pi}$



$V = \int_{\Omega} 1 \; dV$

$V = \int_{\Omega} 1 \; dx\,dy\,dz$

$V = \int_{\Omega} 1 \; r\,dr\,d\theta\,dz$

IN THIS CASE

$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{120} \int_{r=0}^{r=\sqrt{625+\frac{1}{4}(z-90)^2}} r \; dr\,dz\,d\theta$.

$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{120} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\sqrt{625+\frac{1}{4}(z-90)^2}} dz\,d\theta$

$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{120} \frac{1}{2}\left[625 + \frac{1}{4}(z-90)^2\right] dz\,d\theta$

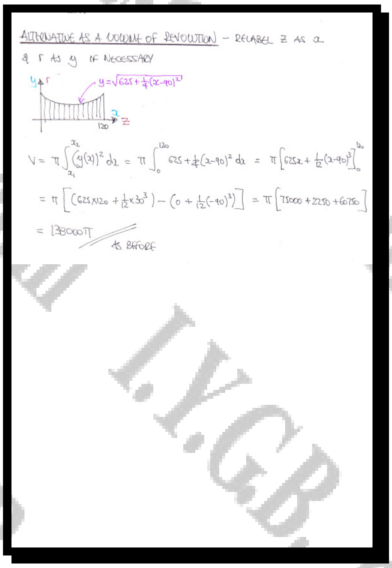
$V = \frac{1}{2}\left[\int_{\theta=0}^{2\pi} 1 \; d\theta\right]\left[\int_{z=0}^{120} 625 + \frac{1}{4}(z-90)^2 \; dz\right]$

$V = \frac{1}{2} \times 2\pi \times \left[625z + \frac{1}{12}(z-90)^3\right]_0^{120}$

$V = \pi\left[\left(625 \times 120 + \frac{1}{12} \times 30^3\right) - \left(\frac{1}{12}(-90)^3\right)\right]$

$V = \pi\left[75000 + 2250 + 60750\right]$

$V = 138000\pi$



ALTERNATIVE AS A VOLUME OF REVOLUTION – RELABEL z AS x & r AS y IF NECESSARY

$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^{120} 625 + \frac{1}{4}(x-90)^2 \, dx = \pi\left[625x + \frac{1}{12}(x-90)^3\right]_0^{120}$

$= \pi\left[\left(625 \times 120 + \frac{1}{12} \times 30^3\right) - \left(0 + \frac{1}{12}(-90)^3\right)\right] = \pi\left[75000 + 2250 + 60750\right]$

$= 138000\pi$

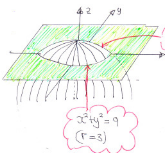
AS BEFORE

Question 9

Use cylindrical polar coordinates (r,θ,z) to find the volume of the region defined as

$$x^2+y^2+(z+4)^2 \le 25\,,\ z \ge 0\,.$$

$$\boxed{\frac{14}{3}\pi}$$



$r^2+(z+4)^2=25$

$(z+4)^2=25-r^2$

$z+4=+\sqrt{25-r^2}$

$z=-4+\sqrt{25-r^2}$

$$V=\int_V 1\,dV=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{z=-4+\sqrt{25-r^2}} \underbrace{r\,dz\,dr\,d\theta}_{dV} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\Big[rz\Big]_{z=0}^{z=-4+\sqrt{25-r^2}}\,dr\,d\theta$$

IN CYLINDRICAL POLARS

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\left(-4+\sqrt{25-r^2}\right)dr\,d\theta=\int_{\theta=0}^{2\pi}\int_{r=0}^{3} -4r+r(25-r^2)^{\frac{1}{2}}\,dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\left[-2r^2-\tfrac{1}{3}(25-r^2)^{\frac{3}{2}}\right]_{r=0}^{3}d\theta=\int_{\theta=0}^{2\pi}\left[2r^2+\tfrac{1}{3}(25-r^2)^{\frac{3}{2}}\right]_{3}^{0}d\theta$$

$$=\int_{\theta=0}^{2\pi}\left[0+\tfrac{1}{3}(25)^{\frac{3}{2}}\right]-\left[18+\tfrac{1}{3}(16)^{\frac{3}{2}}\right]d\theta$$

$$=\int_{\theta=0}^{2\pi}\left(\frac{125}{3}-18-\frac{64}{3}\right)d\theta=\int_{\theta=0}^{2\pi}\frac{7}{3}\,d\theta$$

$$=\frac{7}{3}\times 2\pi=\frac{14\pi}{3}$$

Question 10

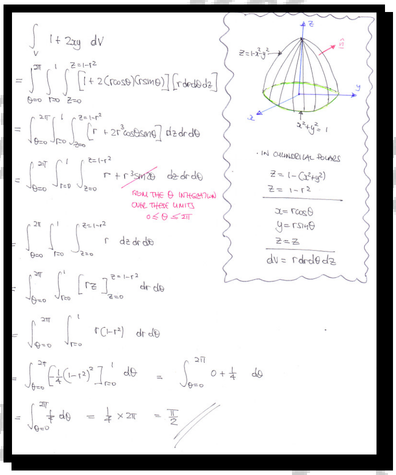
Find the value of

$$\int_V (1+2xy)\ dV\ ,$$

where V is the finite region enclosed by the surface with Cartesian equation

$$z = 1 - x^2 - y^2,\ z \ge 0 .$$

$$\boxed{\frac{\pi}{2}}$$

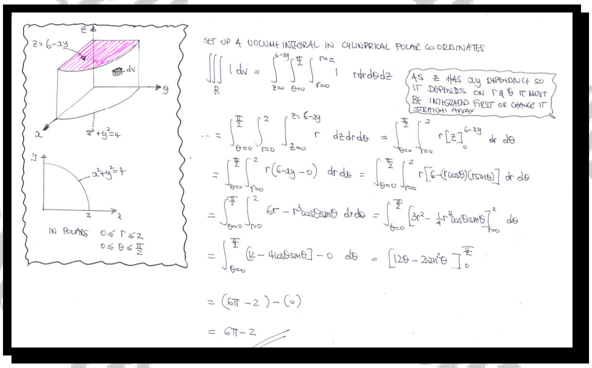


Question 11

Find in exact form the volume of the solid defined by the inequalities

$$x^2 + y^2 \le 4\,, \quad x \ge 0\,, \quad y \ge 0 \quad \text{and} \quad 0 \le z \le 6 - xy\,.$$

$$\boxed{6\pi - 2}$$

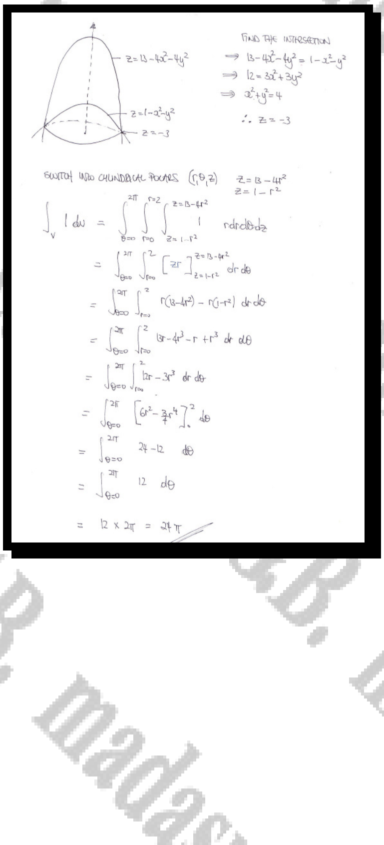


Question 12

Find the volume of the finite region bounded by the surfaces with Cartesian equations

$$z = 13 - 4x^2 - 4y^2 \quad \text{and} \quad z = 1 - x^2 - y^2$$

$$\boxed{V = 24\pi}$$



Question 13

A scalar field F exist inside the cylinder with equation

$$x^2 + y^2 = 1, \quad 0 \le z \le 4.$$

Given further that

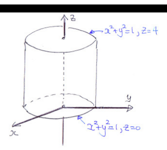
$$F(x, y, z) \equiv 2 + xy + 3yz^2,$$

evaluate the integral

$$\oiiint_V F \, dV,$$

where V denotes the region enclosed by the cylinder

$\boxed{14\pi}$



$$F(x,y,z) = 2 + xy + 3y^2z$$

$$\iiint_V 2 + \cancel{xy} + 3y^2z \, dV = \dots$$

THE x & y INTEGRATION IS IN A SYMMETRICAL DOMAIN, SO ANY ODD POWERS IN x OR y WILL HAVE NO CONTRIBUTION.

$$\dots = \iiint 2 + 3y^2z \, dx\,dy\,dz$$

SWITCH INTO CYLINDRICAL POLARS

$$= \int_{z=0}^{4} \int_{\theta=0}^{2\pi} \int_{r=0}^{r=1} \left[2 + 3(r\sin\theta)^2(z)\right](r\,dr\,d\theta\,dz)$$

$$= \int_{z=0}^{4} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (2r + 3r^3 z\sin^2\theta)\,dr\,d\theta\,dz = \int_{z=0}^{4} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(2r + 3r^3z\left(\tfrac{1}{2} - \cancel{\tfrac{1}{2}\cos 2\theta}\right)\right)dr\,d\theta\,dz$$

NO CONTRIBUTION IN θ FOR $0 \le \theta < 2\pi$

$$= \int_{z=0}^{4} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left[r + \tfrac{3}{2}r^3z\right] dr\,d\theta\,dz$$

CARRY THE θ INTEGRATION FIRST

$$= 2\pi \int_{z=0}^{4} \int_{r=0}^{1} \left[2r + \tfrac{3}{2}r^3z\right] dr\,dz = 2\pi \int_{z=0}^{4} \left[r^2 + \tfrac{3}{8}r^4z\right]_{r=0}^{1} dz$$

$$= 2\pi \int_{z=0}^{4} 1 + \tfrac{3}{8}z \, dz = 2\pi \left[z + \tfrac{3}{16}z^2\right]_0^4$$

$$= 2\pi\left[(4+3) - (0)\right] = 14\pi$$

Question 14

Use cylindrical polar coordinates (r, θ, z) to evaluate

$$\int_V \frac{5yz^2}{\sqrt{x^2+y^2}}\,dxdydz\,,$$

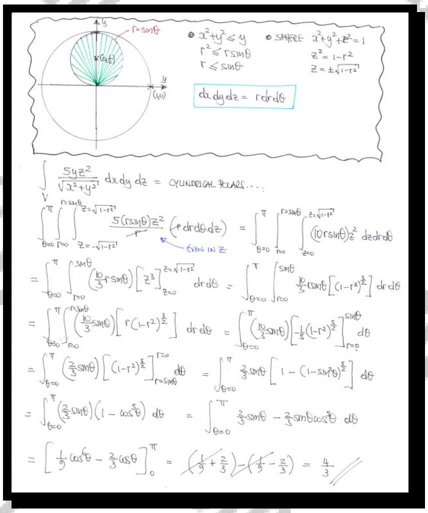
where V is the region defined as

$$x^2+y^2 \le y\,,$$

contained within the sphere with equation

$$x^2+y^2+z^2=1\,.$$

$\boxed{\frac{4}{3}}$



- $x^2+y^2 \le y$
  $r^2 \le r\sin\theta$
  $r \le \sin\theta$
- SPHERE $x^2+y^2+z^2=1$
  $z^2 = 1-r^2$
  $z = \pm\sqrt{1-r^2}$

$\boxed{dx\,dy\,dz = r\,dr\,d\theta}$

$\int_V \frac{5yz^2}{\sqrt{x^2+y^2}}\,dx\,dy\,dz$ = CYLINDRICAL POLARS . . .

$$\int_{\theta=0}^{\pi}\int_{r=0}^{r=\sin\theta}\int_{z=-\sqrt{1-r^2}}^{z=\sqrt{1-r^2}} \frac{5(r\sin\theta)z^2}{\cancel{r}}(\cancel{r}\,dr\,d\theta\,dz) = \int_{\theta=0}^{\pi}\int_{r=0}^{r=\sin\theta}\int_{z=0}^{z=\sqrt{1-r^2}} (10r\sin\theta)z^2\,dz\,dr\,d\theta$$

EVEN IN Z

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{\sin\theta}\left(\frac{10}{3}r\sin\theta\right)\left[z^3\right]_{z=0}^{z=\sqrt{1-r^2}}\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{\sin\theta}\frac{10}{3}r\sin\theta\left[(1-r^2)^{\frac{3}{2}}\right]dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{r=\sin\theta}\left(\frac{10}{3}\sin\theta\right)\left[r(1-r^2)^{\frac{3}{2}}\right]dr\,d\theta = \int_{\theta=0}^{\pi}\left(\frac{10}{3}\sin\theta\right)\left[-\frac{1}{5}(1-r^2)^{\frac{5}{2}}\right]_{r=0}^{\sin\theta}d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{2}{3}\sin\theta\right)\left[(1-r^2)^{\frac{5}{2}}\right]_{r=\sin\theta}^{r=0}d\theta = \int_{\theta=0}^{\pi}\frac{2}{3}\sin\theta\left[1-(1-\sin^2\theta)^{\frac{5}{2}}\right]d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{2}{3}\sin\theta\right)\left(1-\cos^5\theta\right)d\theta = \int_{\theta=0}^{\pi}\frac{2}{3}\sin\theta - \frac{2}{3}\sin\theta\cos^5\theta\,d\theta$$

$$= \left[\frac{1}{9}\cos^6\theta - \frac{2}{3}\cos\theta\right]_0^{\pi} = \left(\cancel{\frac{1}{9}}+\frac{2}{3}\right)-\left(\cancel{\frac{1}{9}}-\frac{2}{3}\right) = \frac{4}{3}$$

Question 15

The finite region Ω is defined by the inequalities

$$x^2 + y^2 \le 1 \quad \text{and} \quad |z| \le \sqrt{x^2 + y^2} .$$

Use cylindrical polar coordinates to evaluate

$$\int_{\Omega} 6x^2 \ \ dx\,dy\,dz .$$

$$\boxed{\frac{12\pi}{5}}$$

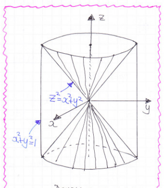
THE INTEGRATION REGION IS BETWEEN THE CYLINDER & THE DOUBLE CONE, BETWEEN $z=-1$ & $z=1$

$$\int_{\Omega} 6x^2 \ dV$$

SWITCH INTO CYLINDRICAL POLARS

$$2\int_{\theta=0}^{2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=r} 6x^2 \ \ r\,dr\,d\theta\,dz$$

USE SYMMETRY ABOVE & BELOW THE x–y PLANE



DOUBLE CONE IN C.P.C

$z^2 = x^2+y^2$

$z^2 = r^2$

$z = r$

CYLINDER IN C.P.C

$r=1$

$$= 12\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r} (r\cos\theta)^2 \ dr\,d\theta\,dz$$

$$= 12\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r} r^3\cos^2\theta \ dr\,d\theta\,dz$$

$$= 12\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\Big[r^3 z\cos^2\theta\Big]_{z=0}^{z=r} dr\,d\theta = 12\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^4\cos^2\theta \ dr\,d\theta$$

$$= 12\int_{\theta=0}^{2\pi}\Big[\tfrac{1}{5}r^5\cos^2\theta\Big]_{r=0}^{1} d\theta = 12\int_{\theta=0}^{2\pi}\tfrac{1}{5}\cos^2\theta \ d\theta$$

$$= 12\int_{\theta=0}^{2\pi}\tfrac{1}{5}\Big[\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\Big] d\theta = \frac{6}{5}\int_{0}^{2\pi} 1+\cancel{\cos 2\theta} \ d\theta$$

$$= \frac{6}{5}\times 2\pi = \frac{12\pi}{5}$$

Question 16

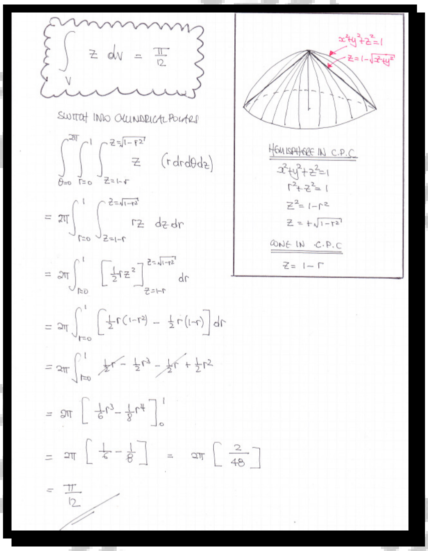
The finite region V is defined by the inequalities

$$x^2 + y^2 + z^2 \le 1 \qquad \text{and} \qquad z \ge 1 - \sqrt{x^2 + y^2}\,.$$

Use cylindrical polar coordinates to evaluate

$$\int_V z \ \, dx\,dy\,dz\,.$$

$$\boxed{\frac{\pi}{12}}$$



$$\boxed{\int_V z \, dV = \frac{\pi}{12}}$$

SWITCH INTO CYLINDRICAL POLARS

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=1-r}^{z=\sqrt{1-r^2}} z \quad (r\,dr\,d\theta\,dz)$$

$$= 2\pi\int_{r=0}^{1}\int_{z=1-r}^{z=\sqrt{1-r^2}} rz \ \ dz\,dr$$

$$= 2\pi\int_{r=0}^{1}\left[\frac{1}{2}rz^2\right]_{z=1-r}^{z=\sqrt{1-r^2}} dr$$

$$= 2\pi\int_{r=0}^{1}\left[\frac{1}{2}r(1-r^2) - \frac{1}{2}r(1-r)\right]dr$$

$$= 2\pi\int_{r=0}^{1} \cancel{\frac{1}{2}r} - \frac{1}{2}r^3 - \cancel{\frac{1}{2}r} + \frac{1}{2}r^2$$

$$= 2\pi\left[\frac{1}{6}r^3 - \frac{1}{8}r^4\right]_0^1$$

$$= 2\pi\left[\frac{1}{6} - \frac{1}{8}\right] = 2\pi\left[\frac{2}{48}\right]$$

$$= \frac{\pi}{12}$$

HEMISPHERE IN C.P.C

$x^2+y^2+z^2=1$

$r^2+z^2=1$

$z^2=1-r^2$

$z=+\sqrt{1-r^2}$

CONE IN C.P.C

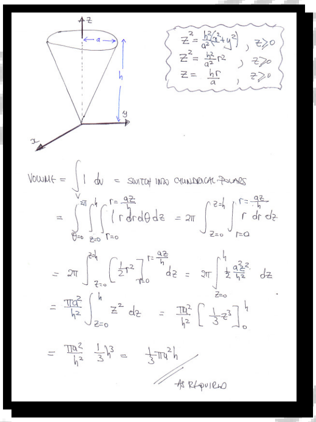
$z=1-r$

Question 17

Use cylindrical polar coordinates (r, θ, z) to show that the volume of a right circular cone of height h and base radius a is

$$\frac{1}{3}\pi a^2 h .$$

$\boxed{\text{proof}}$



$z^2 = \frac{h^2}{a^2}(x^2+y^2)$, $z \geqslant 0$

$z^2 = \frac{h^2}{a^2}r^2$, $z \geqslant 0$

$z = \frac{hr}{a}$, $z \geqslant 0$

VOLUME $= \int_V 1 \, dv$ = SWITCH INTO CYLINDRICAL POLARS

$= \int_{\theta=0}^{2\pi} \int_{z=0}^{h} \int_{r=0}^{r=\frac{az}{h}} 1 \, r \, dr \, d\theta \, dz = 2\pi \int_{z=0}^{z=h} \int_{r=0}^{r=\frac{az}{h}} r \, dr \, dz$

$= 2\pi \int_{z=0}^{z=h} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\frac{az}{h}} dz = 2\pi \int_{z=0}^{h} \frac{1}{2}\frac{a^2z^2}{h^2} \, dz$

$= \frac{\pi a^2}{h^2} \int_{z=0}^{h} z^2 \, dz = \frac{\pi a^2}{h^2}\left[\frac{1}{3}z^3\right]_0^h$

$= \frac{\pi a^2}{h^2} \frac{1}{3}h^3 = \frac{1}{3}\pi a^2 h$

AS REQUIRED

Question 18

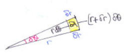
a) Determine with the aid of a diagram or a Jacobian matrix an expression for the area element in plane polar coordinates, (r,θ).

A cylinder of radius $\frac{1}{2}a$ is cut out of a sphere of radius a.

b) Find a simplified expression for the volume of the cylinder, given that one of its generators passes through the centre of the sphere

$$\boxed{dxdy = r\,dr\,d\theta}, \quad \boxed{V = \frac{2}{9}[3\pi - 4]}$$

a) METHOD A



- $\delta A \approx \delta r \times r\delta\theta \simeq r\,\delta r\,\delta\theta$

OR

- $\delta A = \frac{1}{2}(r+\delta r)^2\delta\theta - \frac{1}{2}r^2\delta\theta$

$= \frac{1}{2}\delta\theta\left[(r+\delta r)^2 - r^2\right]$

$= \frac{1}{2}\delta\theta\left[r+\delta r - r\right]\left[r+\delta r+r\right]$

$= \frac{1}{2}\delta\theta \times \delta r \times (2r+\delta r)$

$= r\,\delta r\,\delta\theta + \frac{1}{2}\delta r^2\delta\theta$

TAKING LIMITS WITH EITHER EXPRESSION GIVES $dA = r\,dr\,d\theta$

METHOD B (JACOBIAN)

$dxdy = \frac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta$ $\quad \left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$

$dxdy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}\,dr\,d\theta$

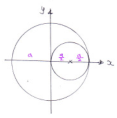
$dxdy = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}\,dr\,d\theta$

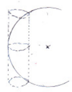
$dxdy = \left[r\cos^2\theta - (-r\sin^2\theta)\right]dr\,d\theta$

$dxdy = r\left[\cos^2\theta + \sin^2\theta\right]dr\,d\theta$

$\therefore dA = r\,dr\,d\theta$

b)





- PLACE CENTRE OF THE SPHERE AT A CARTESIAN ORIGIN
- USING SYMMETRY: THERE IS EQUAL VOLUME ABOVE & BELOW THE xy PLANE

THERE IS EQUAL VOLUME FOR $x \geq 0$ AND $x \leq 0$

SO CALCULATE THE VOLUME FOR $x \geq 0$, $z \geq 0$ AND MULTIPLY THE ANSWER BY 4

THE REQUIRED VOLUME IS "UNDER" THE SURFACE WITH EQUATION

$z^2 + y^2 + x^2 = a^2, \; z \geq 0$

AND ITS PROJECTION ON THE xy PLANE IS THE CIRCLE

$(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}, \; y \geq 0$

$\therefore V = 4\iint_R z\,dx\,dy = 4\iint_R +\sqrt{a^2-x^2-y^2}\,dx\,dy$

$V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=a\cos\theta}\sqrt{a^2-r^2}\,r\,d\theta\,d\theta$

$V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a\cos\theta} r(a^2-r^2)^{\frac{1}{2}}\,dr\,d\theta$

$V = 4\int_{\theta=0}^{\frac{\pi}{2}}\left[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_{r=0}^{r=a\cos\theta}d\theta$

$V = -\frac{4}{3}\int_0^{\frac{\pi}{2}}(a^2-a^2\cos^2\theta)^{\frac{3}{2}} - a^3\,d\theta$

$V = -\frac{4}{3}\int_0^{\frac{\pi}{2}}a^3(1-\cos^2\theta)^{\frac{3}{2}} - a^3\,d\theta$

$V = -\frac{4}{3}a^3\int_0^{\frac{\pi}{2}}\sin^3\theta - 1\,d\theta$

$V = -\frac{4}{3}a^3\int_0^{\frac{\pi}{2}}\sin\theta(1-\cos^2\theta) - 1\,d\theta$

$V = -\frac{4}{3}a^3\int_0^{\frac{\pi}{2}}\sin\theta - \sin\theta\cos^2\theta - 1\,d\theta$

$V = -\frac{4}{3}a^3\left[-\cos\theta + \frac{1}{3}\cos^3\theta - \theta\right]_0^{\frac{\pi}{2}}$

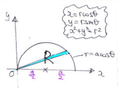
$V = -\frac{4}{3}a^3\left[\left(0 - 0 - \frac{\pi}{2}\right) - \left(-1 + \frac{1}{3}\right)\right]$

$V = -\frac{4}{3}a^3\left[-\frac{\pi}{2} + \frac{2}{3}\right]$

$V = \frac{4}{3}a^3\left[\frac{\pi}{2} - \frac{2}{3}\right]$

$V = \frac{4}{18}a^3\left[3\pi - 4\right]$

$V = \frac{2}{9}a^3\left[3\pi - 4\right]$



$(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$

$x^2 - ax + \cancel{\frac{a^2}{4}} + y^2 = \cancel{\frac{a^2}{4}}$

$x^2 + y^2 = ax$

$r^2 = a(r\cos\theta)$

$r = a\cos\theta$

LIMITS

$r: \; 0 \leq r \leq a\cos\theta$

$\theta: \; 0 \leq \theta \leq \frac{\pi}{2}$

Question 19

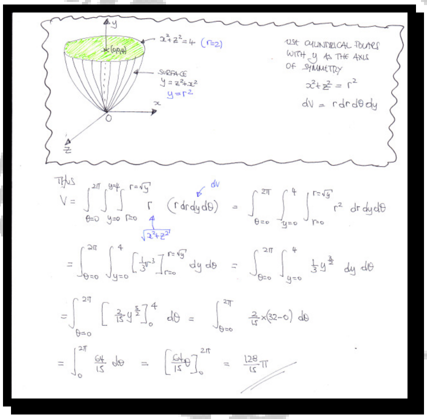
The region V is contained by the paraboloid with Cartesian equation

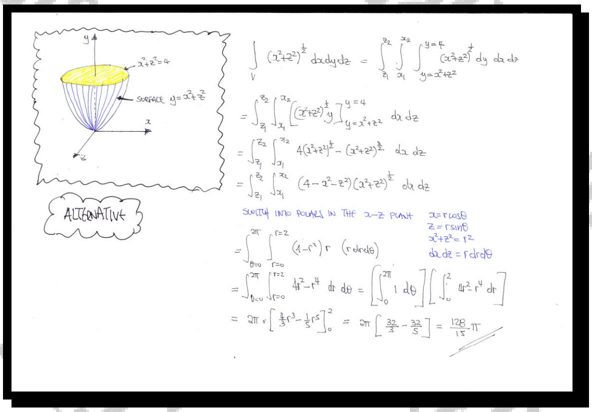
$$y = x^2 + z^2, \quad 0 \le y \le 4 .$$

Determine an exact simplified value for

$$\int_V \sqrt{x^2 + z^2} \; dx\,dy\,dz .$$

$$\boxed{\frac{128}{15}\pi}$$



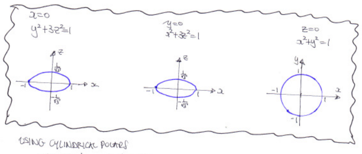


Question 20

Use cylindrical polar coordinates, (r,θ,z), to find the exact volume of the ellipsoid with Cartesian equation

$$x^2+y^2+3z^2=1.$$

$$\boxed{V=\frac{4\pi}{3\sqrt{3}}}$$



$r^2+3z^2=1$ OR $x^2+y^2+3z^2=1$

$x=0$, $y^2+3z^2=1$; $y=0$, $x^2+3z^2=1$; $z=0$, $x^2+y^2=1$

USING CYLINDRICAL POLARS

$$V=\int_{\theta=0}^{2\pi}\int_{z=-\frac{1}{\sqrt{3}}}^{z=\frac{1}{\sqrt{3}}}\int_{r=0}^{r=\sqrt{1-3z^2}} 1\ (r\,dr\,dz\,d\theta)$$

$$V=\int_{\theta=0}^{2\pi}\int_{z=-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}\int_{r=0}^{r=\sqrt{1-3z^2}} r\ dr\,dz\,d\theta$$

$$V=\int_{\theta=0}^{2\pi}\int_{z=-\frac{1}{\sqrt{3}}}^{z=\frac{1}{\sqrt{3}}}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\sqrt{1-3z^2}} dz\,d\theta$$

$$V=\int_{\theta=0}^{2\pi}\int_{z=-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}\frac{1}{2}(1-3z^2)\ dz\,d\theta$$

EVEN

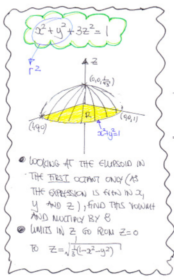
SEPARATE INTEGRALS

$$V=\left[\int_{\theta=0}^{2\pi}1\ d\theta\right]\left[\int_0^{\frac{1}{\sqrt{3}}}1-3z^2\ dz\right]$$

$$V=2\pi\times\left[z-z^3\right]_0^{\frac{1}{\sqrt{3}}}$$

$$V=2\pi\times\left[\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}\right]$$

$$V=2\pi\times\frac{2}{3\sqrt{3}}=\frac{4\pi}{3\sqrt{3}}$$



$x^2+y^2+3z^2=1$

- LOOKING AT THE ELLIPSOID IN THE FIRST OCTANT ONLY (AS THE EXPRESSION IS EVEN IN x, y AND z), FIND THIS VOLUME AND MULTIPLY BY 8
- LIMITS IN z GO FROM $z=0$ TO $z=\sqrt{\frac{1}{3}(1-x^2-y^2)}$

ALTERNATIVE

THUS

$$V=8\int_V 1\ dV$$

$$\Rightarrow V=8\int_R\int_{z=0}^{z=\sqrt{\frac{1}{3}(1-x^2-y^2)}}1\ dz\,dx\,dy$$

$$\Rightarrow V=8\int_R\left[z\right]_{z=0}^{z=\sqrt{\frac{1}{3}(1-x^2-y^2)}}dx\,dy$$

$$\Rightarrow V=8\int_R\sqrt{\frac{1}{3}(1-x^2-y^2)}\ dx\,dy$$

SWITCH INTO PLANE POLARS

$$\Rightarrow V=8\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1}\sqrt{\frac{1}{3}(1-r^2)}\ r\,dr\,d\theta$$

$$\Rightarrow V=8\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1}\frac{1}{\sqrt{3}}r(1-r^2)^{\frac{1}{2}}\ dr\,d\theta$$

SPLIT INTEGRAL

$$\Rightarrow V=\left[\frac{8}{\sqrt{3}}\int_{\theta=0}^{\frac{\pi}{2}}1\ d\theta\right]\left[\int_{r=0}^{1}r(1-r^2)^{\frac{1}{2}}\ dr\right]$$

$$\Rightarrow V=\frac{8}{\sqrt{3}}\times\left[\theta\right]_0^{\frac{\pi}{2}}\times\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1$$

$$\Rightarrow V=\frac{8}{\sqrt{3}}\times\frac{\pi}{2}\times\frac{1}{3}\left[(1-r^2)^{\frac{3}{2}}\right]_1^0$$

$$\Rightarrow V=\frac{4\pi}{3\sqrt{3}}\left[1-0\right]$$

$$\Rightarrow V=\frac{4\pi}{3\sqrt{3}}$$

Question 21

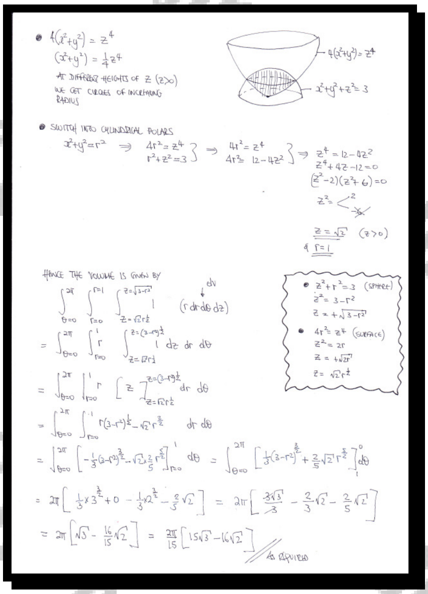
The finite region V is bounded by surfaces with Cartesian equations

$$z^4 = 4\left(x^2 + y^2\right), \ z \geq 0 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 3\,, z \geq 0\,.$$

Use cylindrical polar coordinates (r, θ, z) to show that the volume of V is

$$\frac{2\pi}{15}\left(15\sqrt{3} - 16\sqrt{2}\right).$$

proof



- $4(x^2+y^2) = z^4$
  $(x^2+y^2) = \frac{1}{4}z^4$
  AT DIFFERENT HEIGHTS OF z $(z>0)$ WE GET CIRCLES OF INCREASING RADIUS

$4(x^2+y^2)=z^4$

$x^2+y^2+z^2=3$

- SWITCH INTO CYLINDRICAL POLARS

$x^2+y^2=r^2 \Rightarrow \left.\begin{array}{l}4r^2=z^4\\ r^2+z^2=3\end{array}\right\} \Rightarrow \left.\begin{array}{l}4r^2=z^4\\ 4r^2=12-4z^2\end{array}\right\} \Rightarrow z^4=12-4z^2$

$z^4+4z-12=0$

$(z^2-2)(z^2+6)=0$

$z^2 = \begin{cases} 2 \\ \cancel{-6} \end{cases}$

$z=\sqrt{2}$ $(z>0)$

& $r=1$

HENCE THE VOLUME IS GIVEN BY

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{r=1}\int_{z=\sqrt{2}r^{\frac{1}{2}}}^{z=\sqrt{3-r^2}} 1 \ \overset{dV}{\downarrow}(r\,dr\,d\theta\,dz)$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r \int_{z=\sqrt{2}r^{\frac{1}{2}}}^{z=(3-r^2)^{\frac{1}{2}}} 1 \ dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\left[z\right]_{z=\sqrt{2}r^{\frac{1}{2}}}^{z=(3-r^2)^{\frac{1}{2}}} dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r(3-r^2)^{\frac{1}{2}} - \sqrt{2}r^{\frac{3}{2}} \ dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[-\frac{1}{3}(3-r^2)^{\frac{3}{2}} - \sqrt{2}\times\frac{2}{5}r^{\frac{5}{2}}\right]_{r=0}^{1} d\theta = \int_{\theta=0}^{2\pi}\left[\frac{1}{3}(3-r^2)^{\frac{3}{2}} + \frac{2}{5}\sqrt{2}r^{\frac{5}{2}}\right]_{1}^{0} d\theta$$

$$= 2\pi\left[\frac{1}{3}\times 3^{\frac{3}{2}} + 0 - \frac{1}{3}\times 2^{\frac{3}{2}} - \frac{2}{5}\sqrt{2}\right] = 2\pi\left[\frac{\cancel{3}\sqrt{3}}{\cancel{3}} - \frac{2}{3}\sqrt{2} - \frac{2}{5}\sqrt{2}\right]$$

$$= 2\pi\left[\sqrt{3} - \frac{16}{15}\sqrt{2}\right] = \frac{2\pi}{15}\left[15\sqrt{3} - 16\sqrt{2}\right]$$

// AS REQUIRED

- $z^2+r^2=3$ (SPHERE)
  $z^2=3-r^2$
  $z=+\sqrt{3-r^2}$
- $4r^2=z^4$ (SURFACE)
  $z^2=2r$
  $z=+\sqrt{2r}$
  $z=\sqrt{2}r^{\frac{1}{2}}$

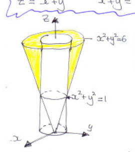
Question 22

Use cylindrical polar coordinates, (r,θ,z), to find the exact volume of the region defined by the following Cartesian inequalities

$$z \le x^2 + y^2, \quad x^2 + y^2 \ge 1 \quad \text{and} \quad z \le 6.$$

$$\boxed{V = \frac{200\pi}{3}}$$

$z = x^2+y^2 \quad x^2+y^2=1 \quad , \quad z \le 6$



SKETCH CURVES INTO CYLINDRICAL POLARS

$z=r$ & $r=1$

$$V = \int_{\theta=0}^{2\pi} \int_{z=1}^{z=} \int_{r=1}^{r=z} 1 \ (r\,dr\,d\theta\,dz)$$

Thus $V = \int_{\theta=0}^{2\pi} \int_{z=1}^{6} \int_{r=1}^{z} r \ dr\,dz\,d\theta$

$$V = \int_{\theta=0}^{2\pi} \int_{z=1}^{6} \left[\tfrac{1}{2}r^2\right]_{r=1}^{z} dz\,d\theta$$

$$V = \int_{\theta=0}^{2\pi} \int_{z=1}^{6} \tfrac{1}{2}z^2 - \tfrac{1}{2} \ dz\,d\theta$$

$$V = \int_{\theta=0}^{2\pi} \left[\tfrac{1}{6}z^3 - \tfrac{1}{2}z\right]_{z=1}^{6} d\theta$$

$$V = \int_{\theta=0}^{2\pi} (36-3) - \left(\tfrac{1}{6} - \tfrac{1}{2}\right) d\theta$$

$$V = \int_{0}^{2\pi} \frac{100}{3} \ d\theta$$

$$V = 2\pi \times \frac{100}{3}$$

$$V = \frac{200}{3}\pi$$

Question 23

Use cylindrical polar coordinates, (r, θ, z), to find the volume of the region defined by the following Cartesian inequalities

$$z \geq 4 - x^2 - y^2, \quad z \leq 4 + x^2 + y^2 \quad \text{and} \quad x^2 + y^2 \leq 4.$$

$\boxed{V = 16\pi}$