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LINE INTEGRALS

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LINE INTEGRALS

IN 2 DIMENSIONAL CARTESIAN COORDINATES

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Question 1

Evaluate the integral

$$\int_C (x+2y) \, dx,$$

where C is the path along the curve with equation $y = x^2 + 1$, from $(0,1)$ to $(6,37)$.

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Parametric Equations

$$\begin{aligned} \int_C (x+2y) \, dx &= \int_{x=0}^{x=6} x + 2(x^2+1) \, dx \\ &= \int_0^6 x + 2x^2 + 2 \, dx \\ &= \left[\frac{1}{2}x^2 + \frac{2}{3}x^3 + 2x \right]_0^6 \\ &= (18 + 144 + 12) - 0 \\ &= 174 \end{aligned}$$

Question 2

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x^2 - y^2)\mathbf{i} + (2xy)\mathbf{j}.$$

Evaluate the line integral

$$\int_{(-2,-1)}^{(4,2)} \mathbf{F} \cdot d\mathbf{r},$$

along a path joining directly the points with Cartesian coordinates $(-2, -1)$ and $(4, 2)$.

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Handwritten solution for the line integral problem:

$$\int_{(-2,-1)}^{(4,2)} \mathbf{F} \cdot d\mathbf{r} = \int_{(-2,-1)}^{(4,2)} (x^2 - y^2) dx + 2xy dy$$

Now direct path is straight line
 direction = $\frac{(4,2) - (-2,-1)}{4-(-2)} = \frac{(6,3)}{6} = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$
 eqn of line: $y - (-1) = \frac{1}{2}(x - (-2))$
 $y + 1 = \frac{1}{2}(x + 2)$
 $y = \frac{1}{2}x - 1$
 $dy = \frac{1}{2}dx$

$$\begin{aligned} &= \int_{-2}^4 (x^2 - (\frac{1}{2}x - 1)^2) dx + 2x(\frac{1}{2}x - 1) dx \\ &= \int_{-2}^4 (\frac{3}{4}x^2 - \frac{1}{2}x^2) dx \\ &= \int_{-2}^4 \frac{1}{4}x^2 dx \\ &= \left[\frac{1}{12}x^3 \right]_{-2}^4 \\ &= \frac{64}{3} - \left(-\frac{16}{3} \right) \\ &= 30 \end{aligned}$$

Question 3

The path along the straight line with equation $y = x + 2$, from $A(0, 2)$ to $B(3, 5)$, is denoted by C .

- a) Evaluate the integral

$$\int_C (x^3 + y) dx + (x - y^3) dy.$$

- b) Show that the integral is independent of the path chosen from A to B .
 c) Verify the independence of the path by evaluating the integral of part (a) along a different path from A to B .

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a) $\int_C (x^3 + y) dx + (x - y^3) dy = \int_{x=0}^{x=3} [x^3 + (x+2)] dx + \int_{y=2}^{y=5} [x - (x+2)^3] dy$
 $= \int_0^3 x^4 + 2x dx + \int_2^5 (-2x^3 - 6x - 8) dy$
 $= \left[\frac{x^5}{5} + x^2 \right]_0^3 + \left[-\frac{2x^3}{3} - 6xy - 8y \right]_2^5$
 $= \left[\frac{243}{5} + 9 \right] - \left[-\frac{26}{3} - 30 - 40 \right] = -117$

b) $\frac{\partial}{\partial x} (x - y^3) = 1$ and $\frac{\partial}{\partial y} (x^3 + y) = 1$
 Since these are equal, the integral is independent of the path.

c) $\int_C (x^3 + y) dx + (x - y^3) dy = \int_0^3 x^4 dx + \int_2^5 (x - y^3) dy$
 $= \left[\frac{x^5}{5} \right]_0^3 + \left[xy - \frac{y^4}{4} \right]_2^5$
 $= \frac{243}{5} + \left[5y - \frac{y^4}{4} \right]_2^5$
 $= \frac{243}{5} + \left(25 - \frac{625}{4} \right) - \left(10 - \frac{16}{4} \right)$
 $= -117$

Question 4

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C .

Evaluate the integral

$$\oint_C x^2 dx - 2xy dy.$$

$$\boxed{-\frac{1}{3}}$$

$C_1: y=0, dy=0, 0 \leq x \leq 1$
 $C_2: x=0, dx=0, 0 \leq y \leq 1$
 $C_3: y=1-x, dy=-dx, 1 \leq x \leq 0$

$$\begin{aligned} \oint_C x^2 dx - 2xy dy &= \int_{C_1} + \int_{C_2} + \int_{C_3} (x^2 dx - 2xy dy) \\ &= \int_0^1 x^2 dx + \int_0^1 0^2 dx - 2x(1-x)(-dx) + \int_1^0 0 dx - 0 dy \\ &= \int_0^1 x^2 dx + \int_0^1 -x^2 - 2x(1-x) dx \\ &= \int_0^1 x^2 - x^2 - 2x + 2x^2 dx \\ &= \left[\frac{2}{3}x^3 - x^2 \right]_0^1 \\ &= \left(\frac{2}{3} - 1 \right) - 0 \\ &= -\frac{1}{3} \end{aligned}$$

Question 5

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C .

Evaluate the integral

$$\oint_C (x^2 + x + y) \, dx + (x^2 y) \, dy.$$

$$-\frac{5}{12}$$

Handwritten solution for Question 5:

Diagram of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. The boundary is divided into three segments: C_1 (bottom), C_2 (left), and C_3 (hypotenuse).

Parameterizations and differentials:

- $C_1: y=0, dy=0, x \text{ from } 0 \text{ to } 1$
- $C_2: x=0, dx=0, y \text{ from } 1 \text{ to } 0$
- $C_3: x=y, dx=dy, y \text{ from } 0 \text{ to } 1$

The integral is evaluated as follows:

$$\oint_C (x^2 + x + y) \, dx + (x^2 y) \, dy = \int_{C_1} (x^2 + x) \, dx + \int_{C_2} (-y^3) \, dy + \int_{C_3} (x^2 + x + y) \, dx + (x^2 y) \, dy$$

$$= \int_0^1 (x^2 + x) \, dx + \int_1^0 (-y^3) \, dy + \int_0^1 (x^2 + x + y) \, dx + (x^2 y) \, dy$$

For the third integral, substitute $x=y$ and $dx=dy$:

$$= \int_0^1 (x^2 + x) \, dx + \int_1^0 (-y^3) \, dy + \int_0^1 (x^2 + x + x) \, dx + (x^2 x) \, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 + \left[-\frac{y^4}{4} \right]_1^0 + \left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{4+6-3}{12} = \frac{7}{12}$$

Wait, the final result in the image is $-\frac{5}{12}$. Let's re-evaluate the third integral:

$$\int_0^1 (x^2 + x + x) \, dx + (x^2 x) \, dx = \int_0^1 (2x^2 + x) \, dx + \int_0^1 x^3 \, dx$$

$$= \left[\frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1 + \left[\frac{x^4}{4} \right]_0^1 = \frac{2}{3} + \frac{1}{2} + \frac{1}{4} = \frac{8+6+3}{12} = \frac{17}{12}$$

Summing all parts:

$$\frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{17}{12} = \frac{4+6-3+17}{12} = \frac{24}{12} = 2$$

There is a discrepancy between the handwritten solution and the printed answer. The printed answer is $-\frac{5}{12}$.

Question 6

The functions F and G are defined as

$$F(x, y) = x^2 y \quad \text{and} \quad G(x, y) = (x + y)^2$$

The anticlockwise path along the perimeter of the triangle whose vertices are located at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C .

Evaluate the line integral

$$\int_C F dx + G dy.$$

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$F(x,y) = x^2 y$
 $G(x,y) = (x+y)^2$

$C_1: y=0, dy=0$ from $x=0$ to $x=1$
 $C_2: y=1-x, dy=-dx$ from $x=1$ to $x=0$
 $C_3: x=0, dx=0$ from $y=1$ to $y=0$

$$\begin{aligned} \oint_C F dx + G dy &= \int_{C_1} F dx + G dy + \int_{C_2} F dx + G dy + \int_{C_3} F dx + G dy \\ &= \int_0^1 x^2 y dx + (x+y)^2 dy + \int_1^0 x^2 y dx + (x+y)^2 dy + \int_1^0 x^2 y dx + (x+y)^2 dy \\ &= \int_0^1 x^2(1-x) dx + (x+1-x)^2(-dx) + \int_1^0 x^2(1-x) dx + (x+1-x)^2(-dx) + \int_1^0 x^2(1-x) dx + (x+1-x)^2(-dx) \\ &= \int_0^1 x^2(1-x) dx - \int_1^0 x^2(1-x) dx - \int_1^0 x^2(1-x) dx + \int_1^0 x^2(1-x) dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 + x^2 \right]_0^1 - \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 + x^2 \right]_1^0 \\ &= \left(\frac{1}{3} - \frac{1}{4} + 1 \right) - \left(\frac{1}{3} - \frac{1}{4} + 1 \right) \\ &= \frac{1}{3} - \frac{1}{4} + 1 \\ &= \frac{3-1+12}{12} \\ &= \frac{14}{12} \\ &= \frac{7}{6} \end{aligned}$$

ALTERNATIVE

$F(x,y) = x^2 y$
 $G(x,y) = (x+y)^2$

By Green's theorem on the plane

$$\begin{aligned} \oint_C F dx + G dy &= \iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy \\ &= \iint_R (2(x+y) - x^2) dx dy \\ &= \int_0^1 \int_0^{1-x} (2x+2y-x^2) dy dx \\ &= \int_0^1 \left[2xy + y^2 - x^2 y \right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 (2x(1-x) + (1-x)^2 - x^2(1-x)) dx \\ &= \int_0^1 (2x-2x^2 + 1-2x+x^2 - x^2+x^3) dx \\ &= \int_0^1 (x^3 - x^2 + 1) dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + x \right]_0^1 = \frac{1}{4} - \frac{1}{3} + 1 \\ &= \frac{3-1+12}{12} = \frac{14}{12} = \frac{7}{6} \end{aligned}$$

Question 7

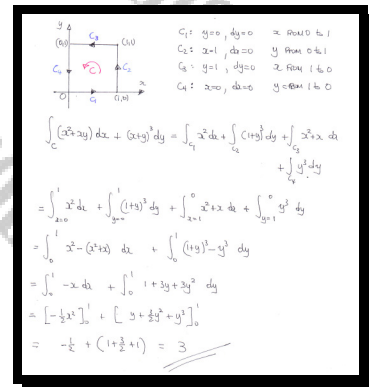
The anticlockwise path along the perimeter of the square whose vertices are located at the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, is denoted by C .

Evaluate the line integral

$$\int_C (x^2 + xy) dx + (x + y)^3 dy.$$

You may not use Green's theorem in this question.

3



Question 8

Evaluate the integral

$$\int_{(-1,7)}^{(5,0)} (3y) dx + (3x+2y) dy,$$

along a path joining the points with Cartesian coordinates $(-1,7)$ and $(5,0)$.

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$$I = \int_{(-1,7)}^{(5,0)} 3y dx + (3x+2y) dy$$

$$dx = \frac{\partial x}{\partial x} dx + \frac{\partial x}{\partial y} dy$$

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial x}{\partial y} = 0$$

$$\frac{\partial y}{\partial x} = 0, \quad \frac{\partial y}{\partial y} = 1$$

$$\therefore \text{INDEPENDENT OF THE PATH}$$

METHOD A GREEN'S THEOREM - A STRAIGHT LINE

GREENSET: $\frac{\partial}{\partial x}(3x+2y) - \frac{\partial}{\partial y}(3y) = 3 - 3 = 0$
 \therefore NOT A GREENSET
 \therefore NOT A STRAIGHT LINE

METHOD B OR USE 2 STRAIGHT LINES

$C_1: x=5 \Rightarrow dx=0$
 $C_2: y=0 \Rightarrow dy=0$
 $I = \int_{y=7}^{y=0} 3y dy + \int_{x=-1}^{x=5} 3x dx = 0 + \frac{3}{2}(5^2 - (-1)^2) = 36$

$$= \left[-3y + y^2 \right]_7^0 = \left[-3y + y^2 \right]_7^0 = (0 - 49) - 0$$

$$= -49$$

METHOD C

AS THIS IS AN EXACT DIFFERENTIAL $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$
 $3y dx + (3x+2y) dy$
 $\therefore f(x,y) = 3xy + y^2$ (BY INSPECTION)

$$I = \int_{(-1,7)}^{(5,0)} 3y dx + (3x+2y) dy = \left[3xy + y^2 \right]_{(-1,7)}^{(5,0)}$$

$$= (0 + 0) - (-21 + 49) = -28$$

Question 9

Evaluate the integral

$$\int_{(1,1)}^{(3,4)} (3x^2y^2) dx + (2x^3y) dy,$$

along a path joining the points with Cartesian coordinates (1,1) and (3,4).

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$$I = \int_{(1,1)}^{(3,4)} 3x^2y^2 dx + 2x^3y dy$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial f}{\partial x} = 6xy^2 \quad \frac{\partial f}{\partial y} = 2x^3$$

$$\frac{\partial}{\partial y} (6xy^2) = 6x \quad \frac{\partial}{\partial x} (2x^3) = 6x^2$$

$$6x \neq 6x^2 \Rightarrow \text{NOT AN EXACT DIFFERENTIAL}$$

METHOD A

Path from (1,1) to (3,4) via (3,1)

$$I = \int_{(1,1)}^{(3,1)} 3x^2y^2 dx + 2x^3y dy + \int_{(3,1)}^{(3,4)} 3x^2y^2 dx + 2x^3y dy$$

$$I = \int_{(1,1)}^{(3,1)} 3x^2(1)^2 dx + 2x^3(1) dy + \int_{(3,1)}^{(3,4)} 3x^2y^2 dx + 2x^3y dy$$

$$I = \int_{(1,1)}^{(3,1)} 3x^2 dx + \int_{(3,1)}^{(3,4)} 2x^3 dy$$

$$I = (x^3)_{(1,1)}^{(3,1)} + (x^3y)_{(3,1)}^{(3,4)}$$

$$I = (27 - 1) + (81 - 27)$$

$$I = 42$$

METHOD B

As it is an EXACT DIFFERENTIAL

$$I = \int_{(1,1)}^{(3,4)} df = [F(x,y)]_{(1,1)}^{(3,4)} = (x^3y^2)_{(1,1)}^{(3,4)}$$

$$= (27 \times 16) - (1 \times 1) = 431$$

NOTE

$$\frac{\partial f}{\partial x} = 6xy^2 \quad F = 2x^3y + A(y)$$

$$\frac{\partial f}{\partial y} = 2x^3 \quad F = 2x^3y + B(x) \quad \therefore A(y) = B(x) = C$$

$$\therefore F(x,y) = x^3y^2 + C$$

Question 10

$$\mathbf{F}(x, y) \equiv (2xy^2 + \cos x)\mathbf{i} + (2x^2y - \sin y)\mathbf{j}.$$

Show that the vector field \mathbf{F} is conservative, and hence evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the arc of the circle with equation

$$x^2 + y^2 = \frac{\pi^2}{4}, \quad y \geq 0,$$

from $A\left(-\frac{\pi}{2}, 0\right)$ to $B\left(\frac{\pi}{2}, 0\right)$.

2

Handwritten solution for Question 10:

Given $\mathbf{F} = (2xy^2 + \cos x)\mathbf{i} + (2x^2y - \sin y)\mathbf{j}$

Check if \mathbf{F} is conservative:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + \cos x & 2x^2y - \sin y & 0 \end{vmatrix} = (0 - 0, 0 - 0, 4xy - 4xy) = \mathbf{0}$$

$\therefore \mathbf{F}$ is conservative

As \mathbf{F} is conservative it is easier to integrate from A to B along the x-axis, since the integral is independent of the path.

Along the x-axis $y=0$

Let $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-\pi/2}^{\pi/2} (2xy^2 + \cos x) \cdot (dx, dy)$

At $y=0$, $dy=0$

$$= \int_{-\pi/2}^{\pi/2} (\cos x, 0) \cdot (dx, 0)$$

$$= \int_{-\pi/2}^{\pi/2} 2\cos x \, dx$$

$$= [2\sin x]_{-\pi/2}^{\pi/2}$$

$$= 2$$

Question 11

In this question α , β and γ are positive constants.

$$\mathbf{F} = -\alpha y^2 \mathbf{e}_x + \beta x^2 \mathbf{e}_y.$$

A particle of mass m is moving on the x - y plane, under the action of \mathbf{F} .

Find the work done by \mathbf{F} on the particle in moving it from the Cartesian origin O to the point $(1,1)$, in each of the following cases.

- Directly from O to $(1,0)$, then directly from $(1,0)$ to $(1,1)$.
- Directly from O to $(0,1)$, then directly from $(0,1)$ to $(1,1)$.
- Moving the particle with velocity $\mathbf{v} = \gamma(\mathbf{e}_x + \mathbf{e}_y)$.

$$W_1 = \beta, \quad W_2 = -\alpha, \quad W_3 = \frac{1}{3}(\beta - \alpha)$$

Handwritten solution for Question 11:

Given $\mathbf{F} = -\alpha y^2 \mathbf{e}_x + \beta x^2 \mathbf{e}_y$.

Work done $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-\alpha y^2 dx + \beta x^2 dy)$.

a) Path 1: Directly from O to $(1,0)$, then directly from $(1,0)$ to $(1,1)$.

On C_1 (from O to $(1,0)$): $y=0, dy=0, x$ from 0 to 1 .
 $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -\alpha(0)^2 dx = 0$.

On C_2 (from $(1,0)$ to $(1,1)$): $x=1, dx=0, y$ from 0 to 1 .
 $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \beta(1)^2 dy = \beta$.

Total work $W_1 = 0 + \beta = \beta$.

b) Path 2: Directly from O to $(0,1)$, then directly from $(0,1)$ to $(1,1)$.

On C_3 (from O to $(0,1)$): $x=0, dx=0, y$ from 0 to 1 .
 $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -\alpha y^2 dx = 0$.

On C_4 (from $(0,1)$ to $(1,1)$): $y=1, dy=0, x$ from 0 to 1 .
 $\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \beta x^2 dx = \frac{\beta}{3}$.

Total work $W_2 = 0 + \frac{\beta}{3} = \frac{\beta}{3}$.

c) Path 3: Moving the particle with velocity $\mathbf{v} = \gamma(\mathbf{e}_x + \mathbf{e}_y)$.

On C_5 (from O to $(1,1)$): $y=x, dy=dx, x$ from 0 to 1 .
 $\int_{C_5} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (-\alpha x^2 + \beta x^2) dx = \frac{1}{3}(\beta - \alpha)$.

Total work $W_3 = \frac{1}{3}(\beta - \alpha)$.

NOTE: Velocity $\mathbf{v} = \gamma(\mathbf{e}_x + \mathbf{e}_y)$ implies the direction is 45° .

Question 12

Evaluate the line integral

$$\oint_C \left[y(x+1)e^x dx + x(e^x+1) dy \right],$$

where C is a circle of radius 1, centre at the origin O , traced anticlockwise. π

Handwritten solution for the line integral problem:

$$\int_C y(x+1)e^x dx + x(e^x+1) dy$$

where $C: x^2 + y^2 = 1$

$$\oint_C P dx + Q dy = \oint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\oint_C \left(y(x+1)e^x dx + x(e^x+1) dy \right)$$

$$= \oint_C (x^2 + e^x) - (x e^x + e^x) dx dy$$

$$= \oint_C x^2 - x e^x dx dy$$

$$= \oint_C 1 dx dy$$

= Area of the circle $x^2 + y^2 = 1$

$$= \pi$$

Question 13

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (\sin x^3 - xy)\mathbf{i} + (x + y^3 \sin y)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the ellipse with Cartesian equation

$$2x^2 + 3y^2 = 2y.$$

$$\frac{\pi}{3\sqrt{6}}$$

Handwritten solution for Question 13:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (\sin x^3 - xy) dx + (x + y^3 \sin y + 2) dy$$

By Green's Theorem on the plane:

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

...

$$= \iint_D (1 + 2x) dx dy = \iint_D 1 dx dy \quad \text{SINCE } 2x \text{ IS AN ODD POWER IN A SYMMETRIC REGION IN } x$$

2: $2x^2 + 3y^2 = 2y$

$$\frac{2}{3}x^2 + y^2 = \frac{2}{3}y$$

$$\frac{2}{3}x^2 + y^2 - \frac{2}{3}y = 0$$

$$\frac{2}{3}x^2 + (y - \frac{1}{3})^2 - \frac{1}{9} = 0$$

$$\frac{2}{3}x^2 + (y - \frac{1}{3})^2 = \frac{1}{9}$$

$$\frac{6x^2}{9} + \frac{(y - \frac{1}{3})^2}{\frac{1}{9}} = 1$$

$$\frac{x^2}{\frac{3}{2}} + \frac{(y - \frac{1}{3})^2}{\frac{1}{9}} = 1$$

Diagram of the ellipse with semi-major axis $\sqrt{\frac{3}{2}}$ and semi-minor axis $\frac{1}{3}$.

Area of the ellipse = $\pi \times \sqrt{\frac{3}{2}} \times \frac{1}{3} = \frac{\pi}{3\sqrt{6}}$

Question 14

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = [x \cos x] \mathbf{i} + [15xy + \ln(1 + y^3)] \mathbf{j}.$$

Evaluate the line integral

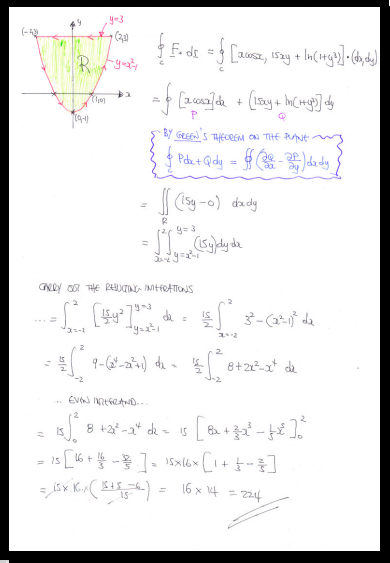
$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the curve

$$\{(x, y) : y = 3, -2 \leq x \leq 2\} \cup \{(x, y) : y = x^2 - 1, -2 \leq x \leq 2\},$$

traced in an anticlockwise direction.

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$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C [x \cos x, 15xy + \ln(1+y^3)] \cdot (dx, dy)$$

$$= \oint_C [x \cos x] dx + [15xy + \ln(1+y^3)] dy$$

by Green's theorem on the region

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$= \iint_R (15y - 0) dx dy$$

$$= \int_{x=-2}^2 \int_{y=x^2-1}^3 (15y) dy dx$$

OR by the double integrations

$$= \int_{x=-2}^2 \left[\frac{15}{2} y^2 \right]_{y=x^2-1}^3 dx = \frac{15}{2} \int_{x=-2}^2 (9 - (x^2-1)^2) dx$$

$$= \frac{15}{2} \int_{x=-2}^2 (9 - (x^4 - 2x^2 + 1)) dx = \frac{15}{2} \int_{x=-2}^2 (8 + 2x^2 - x^4) dx$$

... EVALUATE...

$$= \frac{15}{2} \left[8x + \frac{2}{3} x^3 - \frac{1}{5} x^5 \right]_0^2$$

$$= \frac{15}{2} \left[16 + \frac{16}{3} - \frac{32}{5} \right] = 15 \times 16 \times \left[1 + \frac{1}{3} - \frac{2}{5} \right]$$

$$= 15 \times 16 \times \left(\frac{8+4-6}{15} \right) = 16 \times 14 = 224$$

LINE INTEGRALS

2 DIMENSIONAL PARAMETERIZATIONS

Question 1

The path along the semicircle with equation

$$x^2 + y^2 = 1, \quad x \geq 0$$

from $A(0,1)$ to $B(0,-1)$, is denoted by C .

Evaluate the integral

$$\int_C (x^3 + y^3) \, dx.$$

$$\frac{3}{8}\pi$$

$$\int_C (x^3 + y^3) \, dx = \int_{\pi/2}^{3\pi/2} (x^3 + y^3) \, dx$$

$$= \int_{\pi/2}^{3\pi/2} (x^3 + (-x)^3) \, dx$$

$$= \int_{\pi/2}^{3\pi/2} 2(-x^3)^3 \, dx$$

... BY SUBSTITUTION ...

$$x = \cos \theta$$

$$dx = -\sin \theta \, d\theta$$

$$\frac{dx}{x} = \frac{-\sin \theta \, d\theta}{\cos \theta}$$

$$= \int_{\pi/2}^{3\pi/2} 2(-\cos \theta)^3 \sin \theta \, d\theta$$

$$= \int_{\pi/2}^{3\pi/2} 2 \cos^3 \theta \sin \theta \, d\theta$$

... BY THE IDENTITIES OF BETA FUNCTION ...

$$= \int_{\pi/2}^{3\pi/2} 2 \cos^3 \theta \sin \theta \, d\theta$$

$$= B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}{\Gamma(2)}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2} \times \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{2!}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \sqrt{\pi}}{2}$$

$$= \frac{3}{8}\pi$$

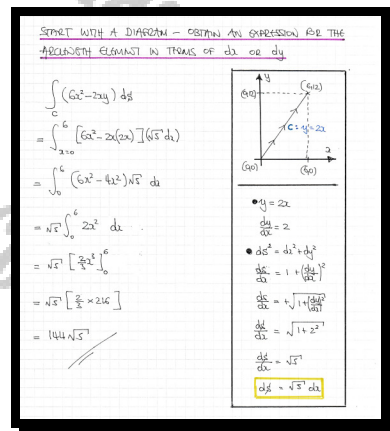
Question 2

Evaluate the integral

$$\int_{(0,0)}^{(6,12)} (6x^2 - 2xy) \, ds,$$

where s is the arclength along the straight line segment from $(0,0)$ to $(6,12)$.

$$\boxed{}, \boxed{144\sqrt{5}}$$



Question 3

Evaluate the integral

$$\int_{(1,-1)}^{(3,3)} (y+x) dx + (y-x) dy,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1.$$

 , 10

SOLUTION: THE INTEGRAL (C) PARAMETRIC

$$\begin{aligned} x &= 2t^2 - 3t + 1 & y &= t^2 - 1 \\ dx &= (4t - 3) dt & dy &= 2t dt \\ 0 &\leq t \leq 2 & (\text{Re direction}) \end{aligned}$$

Find the line

$$\begin{aligned} &\int_{(1,-1)}^{(3,3)} (y+x) dx + (y-x) dy \\ &= \int_0^2 (t^2 - 1 + 2t^2 - 3t + 1) (4t - 3) dt + (t^2 - 1 - 2t^2 + 3t - 1) (2t) dt \\ &= \int_0^2 [(3t^2 - 3t) (4t - 3) + 2t(-t^2 + 3t - 2)] dt \\ &= \int_0^2 [12t^3 - 9t^2 - 12t^2 + 9t - 2t^3 + 6t^2 - 4t] dt \\ &= \int_0^2 [10t^3 - 3t^2 + 5t] dt \\ &= \left[\frac{10}{4} t^4 - t^3 + \frac{5}{2} t^2 \right]_0^2 \\ &= \left(\frac{10}{4} \cdot 16 - 8 + \frac{5}{2} \cdot 4 \right) - 0 \\ &= 10 \end{aligned}$$

Question 4

Evaluate the line integral

$$\int_{(5,0)}^{(0,5)} (2x + y) \, ds,$$

where s is the arclength along the quarter circle with equation

$$x^2 + y^2 = 25.$$

75

Circle in parametric: $x = 5 \cos \theta$, $y = 5 \sin \theta$ \Rightarrow $\frac{dx}{d\theta} = -5 \sin \theta$, $\frac{dy}{d\theta} = 5 \cos \theta$ $0 < \theta < \frac{\pi}{2}$
 $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$
 $ds = \sqrt{25 \sin^2 \theta + 25 \cos^2 \theta} d\theta$
 $ds = 5 d\theta$
 Then $\int_{(5,0)}^{(0,5)} (2x + y) \, ds = \int_0^{\frac{\pi}{2}} (10 \cos \theta + 5 \sin \theta) \cdot 5 \, d\theta = \int_0^{\frac{\pi}{2}} 50 \cos \theta + 25 \sin \theta \, d\theta$
 $= [50 \sin \theta - 25 \cos \theta]_0^{\frac{\pi}{2}} = (50 - 0) - (0 - 25) = 75$

Question 5

Evaluate the line integral

$$\oint_C y^5 dx,$$

where C is a circle of radius 2, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

$$\boxed{-40\pi}$$

The handwritten solution shows two methods to evaluate the line integral $\oint_C y^5 dx$ over a circle C of radius 2 centered at the origin.

Method 1: Parametric Equations

The circle is parametrized as $x = 2\cos\theta$ and $y = 2\sin\theta$, where θ ranges from 0 to 2π . The differential dx is $dx = -2\sin\theta d\theta$. Substituting into the integral:

$$\oint_C y^5 dx = \int_0^{2\pi} (2\sin\theta)^5 (-2\sin\theta d\theta) = -128 \int_0^{2\pi} \sin^6\theta d\theta$$

Using the identity $\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$, the integral is evaluated as:

$$\begin{aligned} \int_0^{2\pi} \sin^6\theta d\theta &= \int_0^{2\pi} \left(\frac{1 - \cos(2\theta)}{2}\right)^3 d\theta \\ &= \frac{1}{8} \int_0^{2\pi} (1 - 3\cos(2\theta) + 3\cos^2(2\theta) - \cos^3(2\theta)) d\theta \\ &= \frac{1}{8} \left[\theta - \frac{3}{2}\sin(2\theta) + \frac{3}{4}\theta + \frac{3}{8}\sin(4\theta) - \frac{1}{4}\cos(4\theta) \right]_0^{2\pi} \\ &= \frac{1}{8} \left[\frac{7}{4}\theta \right]_0^{2\pi} = \frac{7}{4}\pi \end{aligned}$$

Therefore, the final result is $-128 \times \frac{7}{4}\pi = -40\pi$.

Method 2: Polar Coordinates (Green's Theorem)

Using Green's theorem, the line integral is converted to a double integral over the region R inside the circle:

$$\oint_C y^5 dx = - \iint_R \frac{\partial}{\partial y} (y^5) dy dx = - \iint_R 5y^4 dy dx$$

Converting to polar coordinates, $y = r\sin\theta$ and $dy dx = r dr d\theta$. The integral becomes:

$$- \int_0^{2\pi} \int_0^2 5(r\sin\theta)^4 r dr d\theta = - \frac{5}{4} \int_0^{2\pi} \sin^4\theta d\theta$$

Using the identity $\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$, the integral is evaluated as:

$$\begin{aligned} \int_0^{2\pi} \sin^4\theta d\theta &= \int_0^{2\pi} \left(\frac{1 - \cos(2\theta)}{2}\right)^2 d\theta \\ &= \frac{1}{4} \int_0^{2\pi} (1 - 2\cos(2\theta) + \cos^2(2\theta)) d\theta \\ &= \frac{1}{4} \left[\theta - \sin(2\theta) + \frac{1}{4}\theta + \frac{1}{8}\sin(4\theta) \right]_0^{2\pi} \\ &= \frac{1}{4} \left[\frac{5}{4}\theta \right]_0^{2\pi} = \frac{5}{4}\pi \end{aligned}$$

Therefore, the final result is $-\frac{5}{4} \times \frac{5}{4}\pi = -40\pi$.

Question 6

Evaluate the line integral

$$\oint_C \left[y^3 dx + (xy) dy \right],$$

where C is a circle of radius 1, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

$$\boxed{}, \quad -\frac{3\pi}{4}$$

The image shows two handwritten solutions for the line integral problem. The left page, titled 'PROCESSED BY PARAMETERIZING THE CIRCULAR PATH', uses the parametrization $x = \cos \theta$, $y = \sin \theta$ for $0 \leq \theta < 2\pi$. It calculates the line integral by substituting these into the expression $y^3 dx + (xy) dy$ and integrating from 0 to 2π . The right page, titled 'THAT IDIO GAMMA FUNCTIONS', uses the Gamma function property $\Gamma(x)\Gamma(y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)}$ to evaluate the integral, resulting in $-\frac{3\pi}{4}$. Both solutions conclude with the final answer $-\frac{3\pi}{4}$.

Question 7

Evaluate the line integral

$$\oint_C [y \, dx + x(2+y) \, dy] ,$$

where C is a circle of radius 1, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

 π

Handwritten solution for the line integral problem:

$$\begin{aligned} \oint_C y \, dx + x(2+y) \, dy &= \int_0^{2\pi} y \, dx + (2x+xy) \, dy \\ \text{PARAMETRIZE } x &= \cos \theta & dx &= -\sin \theta \, d\theta \\ y &= \sin \theta & dy &= \cos \theta \, d\theta \\ 0 &\leq \theta \leq 2\pi \quad (\text{anticlockwise}) \\ &= \int_0^{2\pi} \sin \theta (-\sin \theta \, d\theta) + (2\cos \theta + \cos \theta \sin \theta)(\cos \theta \, d\theta) \\ &= \int_0^{2\pi} -\sin^2 \theta + 2\cos^2 \theta + \cos^2 \theta \sin \theta \, d\theta \\ &= \int_0^{2\pi} (\cos^2 \theta - \sin^2 \theta) + \cos^2 \theta + \cos^2 \theta \sin \theta \, d\theta \\ &= \int_0^{2\pi} (\cos^2 \theta + \cos^2 \theta \sin \theta) + \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta\right) d\theta \\ &\quad \text{No contribution from these terms as } \theta \\ &= \int_0^{2\pi} \frac{1}{2} \, d\theta \\ &= \frac{1}{2} \times 2\pi = \pi \end{aligned}$$

Question 8

Evaluate the line integral

$$\oint_C \frac{x^2 y}{x^2 + y^2} dx,$$

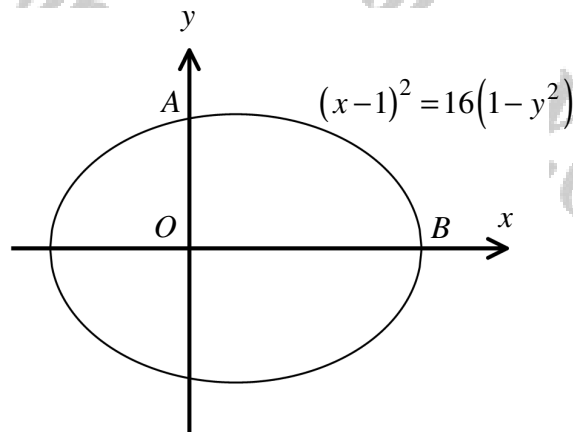
where C is a circle of radius 1, centre at the origin O , traced anticlockwise.

$$\frac{\pi}{4}$$

Handwritten solution for the line integral problem:

$$\begin{aligned} \int_C \frac{x^2 y}{x^2 + y^2} dz \dots & \text{PARAMETRIZE THE CIRCLE } x^2 + y^2 = 1 \\ & \left\{ \begin{array}{l} x = \cos \theta \\ y = \sin \theta \\ 0 \leq \theta \leq 2\pi \\ dz = -\sin \theta d\theta \end{array} \right. \\ & = \int_{\theta=0}^{2\pi} \frac{\cos^2 \theta \sin \theta}{1} (-\sin \theta d\theta) = \int_{\theta=0}^{2\pi} -\cos^2 \theta \sin^2 \theta d\theta \\ & = \int_{\theta=0}^{2\pi} -\left(\frac{1}{2} + \frac{1}{2} \cos 2\theta\right) \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) d\theta = \int_{\theta=0}^{2\pi} -\left(\frac{1}{4} - \frac{1}{4} \cos^2 2\theta\right) d\theta \\ & = \int_{\theta=0}^{2\pi} -\frac{1}{4} \cos^2 2\theta + \frac{1}{4} d\theta = \int_{\theta=0}^{2\pi} \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 4\theta\right) - \frac{1}{4} d\theta \\ & = \int_{\theta=0}^{2\pi} \frac{1}{8} + \frac{1}{8} \cos 4\theta - \frac{1}{4} d\theta = \int_{\theta=0}^{2\pi} -\frac{1}{8} + \frac{1}{8} \cos 4\theta d\theta \\ & = \left[-\frac{1}{8} \theta + \frac{1}{32} \sin 4\theta \right]_{\theta=0}^{2\pi} = -\frac{\pi}{4} \end{aligned}$$

Question 9



The figure above shows the ellipse with equation

$$(x-1)^2 = 16(1-y^2).$$

The ellipse meets the positive y and x axes at the points A and B , respectively, as shown in the figure.

The elliptic path C is the clockwise section from A to B .

Determine the value of each of the following line integrals.

a) $\int_C \left[(x^2 + xy) dx + \left(y^2 + \frac{1}{2}x^2 \right) dy \right].$

b) $\int_C \left[y^3 dx + \frac{1}{16}(x-1)^3 dy \right].$

$$\boxed{}, \boxed{\frac{125}{3} - \frac{5}{64}\sqrt{15}}, \boxed{\frac{1}{4}\sqrt{15}}$$

[solution overleaf]

$$\begin{aligned}
 &= \left[\frac{1}{3}x^3 + \frac{1}{5}y^5 + \frac{1}{2}xy \right] \left(\frac{1}{\sqrt{10}} \right) \\
 &= \frac{1}{3} \times 5^3 - \left(\frac{1}{5} \times \frac{1}{2} \right)^{1/5} \\
 &= \frac{125}{3} - \frac{1}{3} \frac{15}{64} \sqrt{10} = \frac{125}{3} - \frac{5}{64} \sqrt{10}
 \end{aligned}$$

4. CHOOSING FOR PATH INDEPENDENCE FOR (b)
 $\frac{\partial}{\partial y}(y^4) = 3y^2$ $\frac{\partial}{\partial x} \left(\frac{1}{10}(x-y)^4 \right) = \frac{2}{5}(x-y)^2$
 DIFFERENTIAL IS NOT
 SO IT DEPENDS ON THE PATH

PARAMETERIZE THE CURVE
 $\Rightarrow (x-y)^2 = 16(1-y^2)$
 $\Rightarrow (x-1)^2 = 16 - 16y^2$
 $\Rightarrow (x-1)^2 + 16y^2 = 16$
 $\Rightarrow \frac{(x-1)^2}{16} + y^2 = 1$
 $\left[\cos^2 t + \sin^2 t = 1 \right]$
 $\therefore x(t) = \frac{x-1}{4} \quad \& \quad \sin t = y$
 $\frac{x-1}{4} + 4 \cos^2 t \quad \& \quad y = \sin t$
 $dx = -4 \sin t \quad \& \quad dy = \cos t dt$

$$= \left[\begin{array}{c} 25 \sin 30^\circ \\ 4 \sin 60^\circ \cos 30^\circ \end{array} \right] \begin{array}{l} 0 \\ 0 \end{array}$$

$$\begin{array}{l} \theta = \arcsin \frac{\sqrt{3}}{2} \\ \theta = \arcsin \left(\frac{\sqrt{3}}{2} \right) \end{array}$$

$$= \left[\begin{array}{c} 4 \sin 60^\circ \cos 30^\circ \\ 0 \end{array} \right] \begin{array}{l} 0 \\ 0 \end{array}$$

$$\begin{array}{l} \theta = \arcsin \frac{\sqrt{3}}{2} \\ \theta = \arcsin \left(\frac{\sqrt{3}}{2} \right) \end{array}$$

$$= 0 = \frac{4 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}}{2}$$

$$= \frac{1}{2} \sqrt{3}$$

Question 10

The closed curve C bounds the finite region R in the x - y plane defined as

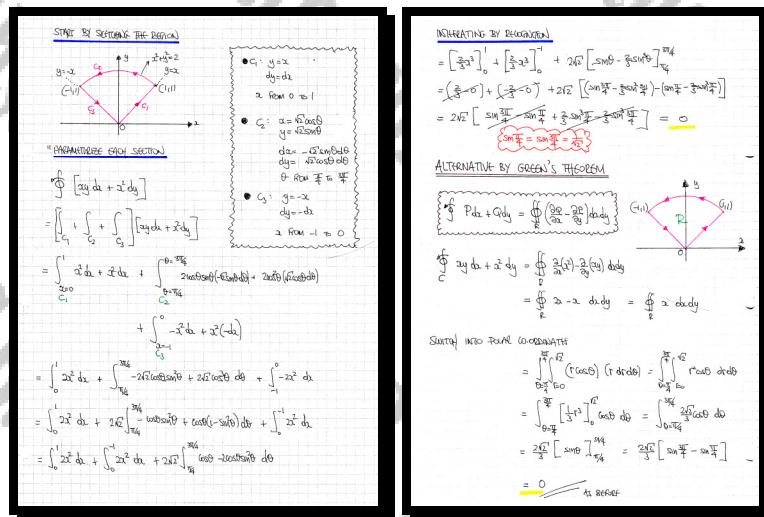
$$R(x, y) = \{x + y \geq 0 \cap x - y \leq 0 \cap x^2 + y^2 \leq 2\}.$$

Evaluate the line integral

$$\oint_C (xy \, dx + x^2 \, dy),$$

where C is traced anticlockwise.

,



Question 11

Evaluate the line integral

$$\oint_C \left[\arctan\left(\frac{y}{x}\right) dx + \ln(x^2 + y^2) dy \right],$$

where C is the polar rectangle such that $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$, traced anticlockwise. $-\pi$

$$\oint_C \left[\arctan\left(\frac{y}{x}\right) dx + \ln(x^2 + y^2) dy \right]$$

$$= \int_C \arctan\left(\frac{y}{x}\right) dx + \ln(x^2 + y^2) dy$$

SPLIT into 4 sections C_1 to C_4 of length C_1 & C_3 have zero contribution

$C_1: y=0, dy=0$
 $C_2: x=2\cos\theta, y=2\sin\theta$
 θ from 0 to π
 $dx = -2\sin\theta d\theta$
 $dy = 2\cos\theta d\theta$
 $C_3: y=0, dy=0$
 $C_4: x=\cos\theta, y=\sin\theta$
 θ from π to 0
 $dx = -\sin\theta d\theta$
 $dy = \cos\theta d\theta$

$$= \int_0^\pi \theta(-2\sin\theta d\theta) + (\ln(4\cos^2\theta)) 2\cos\theta d\theta$$

$$+ \int_\pi^0 \theta(-\sin\theta d\theta) + (\ln(1))\cos\theta d\theta$$

$$= \int_0^\pi [-2\theta\sin\theta + (3\ln 2)\cos\theta + \theta\sin\theta] d\theta$$

$$= \int_0^\pi [(-\theta)\sin\theta + (3\ln 2)\cos\theta] d\theta$$

BY PARTS

$$= \left[(\theta\cos\theta) \right]_0^\pi - \left[-\theta\cos\theta + \int \cos\theta d\theta \right]_0^\pi$$

$$= \left[\theta\cos\theta \right]_0^\pi$$

$$= -\pi$$

Question 12

Evaluate the line integral

$$\oint_C [(2x-y)dx + (2y-x)dy],$$

where C is an ellipse with Cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

traced anticlockwise.

You may not use Green's theorem in this question.

0

Handwritten solution for Question 12:

Method 1 (Direct Integration):

$$\oint_C F \cdot dr = \oint_C (2x-y)dx + (2y-x)dy$$

$$= \int_0^{2\pi} [(2\cos t - 2\sin t)(-\sin t) + (2\sin t - \cos t)(2\cos t)] dt$$

$$= \int_0^{2\pi} [-2\cos t \sin t + 2\sin^2 t + 4\sin t \cos t - 2\cos^2 t] dt$$

$$= \int_0^{2\pi} [2\sin^2 t - 2\cos^2 t + 2\sin 2t] dt$$

$$= \int_0^{2\pi} [-2\cos 2t + 2\sin 2t] dt$$

$$= \int_0^{2\pi} -2\cos 2t dt + \int_0^{2\pi} 2\sin 2t dt$$

$$= \left[-\sin 2t \right]_0^{2\pi} + \left[-\cos 2t \right]_0^{2\pi} = 0 + 0 = 0$$

Method 2 (Green's Theorem):

$$\oint_C (2x-y)dx + (2y-x)dy = \iint_R \left(\frac{\partial}{\partial x}(2y-x) - \frac{\partial}{\partial y}(2x-y) \right) dx dy$$

$$= \iint_R (-1 - (-1)) dx dy = \iint_R 0 dx dy = 0$$

Question 13

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x - 3y)\mathbf{i} + (y - 2x)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the ellipse with cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

You may not use Green's theorem in this question.

6π

Handwritten solution for the line integral problem:

$E = (x-3y, y-2x)$ $C: \frac{x^2}{9} + \frac{y^2}{4} = 1$

PARAMETERISE THE ELLIPSE $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$
 $\cos t + \sin^2 t = 1$

$\therefore \begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases} \quad 0 \leq t < 2\pi$

$\frac{dx}{dt} = -3\sin t$
 $\frac{dy}{dt} = 2\cos t$

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (x-3y, y-2x) \cdot (dx, dy)$

$= \int_0^{2\pi} (3\cos t - 6\sin t)(-3\sin t) + (2\sin t - 6\cos t)(2\cos t) dt$

$= \int_0^{2\pi} -9\cos t \sin t + 18\sin^2 t + 4\sin t \cos t - 12\cos^2 t dt$

$= \int_0^{2\pi} -5\cos t \sin t + 18\sin^2 t - 8\cos^2 t dt$

$= \int_0^{2\pi} -\frac{5}{2}\sin(2t) + 9(1 - \cos(2t)) - 4(1 + \cos(2t)) dt$

$= \int_0^{2\pi} -\frac{5}{2}\sin(2t) + 5 - 4\cos(2t) dt$

$= \int_0^{2\pi} 5 dt$

$= 2\pi \times 5$

$= 10\pi$

NO CONSTANT AND TRIG UNIT

Question 14

$$\mathbf{F}(x, y) \equiv \left(-\frac{y}{x^2 + y^2} \right) \mathbf{i} + \left(\frac{x}{x^2 + y^2} \right) \mathbf{j}.$$

By considering the line integral of \mathbf{F} over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \frac{2\pi}{ab},$$

where a and b are real constants.

You may assume without proof that the line integral of \mathbf{F} yields the same value over any simple closed curve which contains the origin.

 , proof

SETTING UP THE LINE INTEGRAL OVER A CLOSED PATH C WITH COORDINATES (x, y)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \left(-\frac{y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy$$

PARAMETERISE OVER θ (A UNIT CIRCLE TRAVEL COUNTERCLOCKWISE)

- $x = \cos \theta$
- $y = \sin \theta$
- $dx = -\sin \theta d\theta$
- $dy = \cos \theta d\theta$
- θ runs from 0 to 2π

$$= \int_0^{2\pi} \frac{-\sin \theta}{\cos^2 \theta + \sin^2 \theta} (-\sin \theta d\theta) + \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} (\cos \theta d\theta)$$

$$= \int_0^{2\pi} \frac{\sin^2 \theta}{1} d\theta + \int_0^{2\pi} \frac{\cos^2 \theta}{1} d\theta = \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta$$

$$= \int_0^{2\pi} 1 d\theta = 2\pi$$

NOTE: INSTEAD OF USING A UNIT CIRCLE

- $x = a \cos \theta$
- $y = b \sin \theta$
- $dx = -a \sin \theta d\theta$
- $dy = b \cos \theta d\theta$
- θ runs from 0 to 2π

LET'S TRY ANOTHER WAY

$$\Rightarrow \oint_C \left(-\frac{y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{-b \sin \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} (-a \sin \theta d\theta) + \frac{a \cos \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} (b \cos \theta d\theta) = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{ab \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta + \frac{ab \cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{ab \sin^2 \theta + ab \cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{ab (\sin^2 \theta + \cos^2 \theta)}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{ab}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \frac{2\pi}{ab}$$

LET'S TRY ANOTHER WAY

$$\mathbf{F} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} \left(-\frac{y}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \left[0 - 0 \right] + \left[0 - 0 \right] = 0$$

LOOKING AT THE \mathbf{F} COMPONENTS

$$\frac{\partial}{\partial x} \left(-\frac{y}{x^2 + y^2} \right) = -\frac{y}{(x^2 + y^2)^2} \cdot 2x = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \frac{x}{(x^2 + y^2)^2} \cdot 2y = \frac{2xy}{(x^2 + y^2)^2}$$

$$= -\frac{2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2} = 0$$

YET THE INTEGRATION OVER A CLOSED PATH DOES NOT YIELD ZERO!

LOOK FURTHER

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y} \left(-\frac{y}{x^2 + y^2} \right) = \frac{-x}{(x^2 + y^2)^2}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{1 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = \frac{-x}{(x^2 + y^2)^2} - \frac{1 - x^2}{(x^2 + y^2)^2} = \frac{-x - 1 + x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = \frac{-x}{(x^2 + y^2)^2} - \frac{1 - x^2}{(x^2 + y^2)^2} = \frac{-x - 1 + x^2}{(x^2 + y^2)^2}$$

... (C) SATISFIES LAPLACE'S EQUATION, SO HARMONIC ...

... EQUALS ZERO, REVERSED OR NOT TO HAND ...

$f(x) = \frac{1}{x}$ AND THE INTEGRATION OVER A UNIT CIRCLE AT THE ORIGIN

$$\oint_C \frac{1}{z} dz = 2\pi i \quad (\text{STANDARD})$$

$$\oint_C \frac{1}{z+i} dz = 2\pi i$$

$$\oint_C \frac{z-i}{z^2 + 1} (dz + i dx) = 2\pi i$$

$$\oint_C \frac{z-i}{z^2 + 1} [z dz + i dx + y dy] = 2\pi i$$

$$\oint_C \left[\frac{z}{z^2 + 1} dz + \frac{i}{z^2 + 1} dz \right] = 2\pi i$$

THIS ANSWER MATCHES OUR FIRST ANSWER, BUT ONLY BECAUSE WE'VE CHOSEN THE CIRCLE

Created by T. Madas

LINE INTEGRALS

IN 3 DIMENSIONS

Created by T. Madas

Question 1

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x^2y)\mathbf{i} + (4xy^2)\mathbf{j} + (-6xz)\mathbf{k}.$$

Evaluate the line integral

$$\int_{(0,0,0)}^{(10,4,8)} \mathbf{F} \cdot d\mathbf{r}, \quad \text{where } d\mathbf{r} = (dx, dy, dz)^T,$$

along a path given by the parametric equations

$$x = 5t, \quad y = t^2, \quad z = t^3.$$

$$\frac{800}{7}$$

Handwritten solution for the line integral problem:

$$\begin{aligned} \mathbf{F}(x,y,z) &= (x^2y, 4xy^2, -6xz) & \begin{aligned} x &= 5t & \Rightarrow dx &= 5dt \\ y &= t^2 & \Rightarrow dy &= 2t dt \\ z &= t^3 & \Rightarrow dz &= 3t^2 dt \end{aligned} \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=2} (25t^3, 40t^4, -30t^4) \cdot (5dt, 2t dt, 3t^2 dt) \\ &= \int_0^2 (125t^3 + 80t^5 - 90t^6) dt = \left[\frac{125}{4}t^4 + \frac{80}{6}t^6 - \frac{90}{7}t^7 \right]_0^2 \\ &= \frac{125}{4}(16) + \frac{80}{6}(64) - \frac{90}{7}(128) \\ &= 500 - \frac{6400}{7} = -\frac{800}{7} \end{aligned}$$

Question 2

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x^2 y)\mathbf{i} + (xy^2)\mathbf{j} + (yz)\mathbf{k}.$$

Evaluate the line integral

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r}, \quad \text{where } d\mathbf{r} = (dx, dy, dz)^T,$$

along a path of three straight line segments joining $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,2,0)$ and $(1,2,0)$ to $(1,2,3)$.

$$\boxed{}, \boxed{\frac{35}{3}}$$

LOOKING AT THE PATH IN SEPARATE SEGMENTS

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2 y) dx + (xy^2) dy + (yz) dz = \int_C x^2 y dx + xy^2 dy + yz dz$$

- FROM $(0,0,0)$ TO $(1,0,0)$ $y=0$ $dy=0$ $dz=0$ x RUNS FROM 0 TO 1
- FROM $(1,0,0)$ TO $(1,2,0)$ $x=1$ $dx=0$ $dz=0$ y RUNS FROM 0 TO 2
- FROM $(1,2,0)$ TO $(1,2,3)$ $x=1$ $dx=0$ $dy=0$ z RUNS FROM 0 TO 3

DETERMINING THE INTEGRAL

$$\dots = \int_{x=0}^{x=1} x^2 \cdot 0 dx + \int_{y=0}^{y=2} 1 \cdot y^2 dy + \int_{z=0}^{z=3} 1 \cdot 2z dz$$

$$= \left[\frac{1}{3} x^3 \right]_0^1 + \left[\frac{1}{3} y^3 \right]_0^2 + \left[z^2 \right]_0^3 = \frac{1}{3} + \frac{8}{3} + 9 = \frac{35}{3}$$

Question 3

It is given that

$$\mathbf{F}(x, y, z) = \mathbf{j} \wedge \mathbf{r},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t) = (t - t^2)\mathbf{i} + (2t - 2t^2)\mathbf{j} + (t^2 - t^3)\mathbf{k}, \quad 0 \leq t \leq 1.$$

1
30

$$\begin{aligned} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \end{vmatrix} = (z - 0, 0 - z, 0 - x) = (z, -z, -x) \\ C: \quad \mathbf{r}(t) &= [t - t^2, 2t - 2t^2, t^2 - t^3] \\ \frac{d\mathbf{r}}{dt} &= [1 - 2t, 2 - 4t, 2t - 3t^2] \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C (z, -z, -x) \cdot (dt, -2tdt, 2tdt - 3t^2dt) \\ &= \oint_C z dt - 2z dt - x(2t - 3t^2) dt \\ &= \int_0^1 (t^2 - t^3)(1 - 2t) dt - \int_0^1 (t - t^2)(2t - 3t^2) dt \\ &= \int_0^1 (t^2 - 2t^3 - t^3 + 2t^4 - 2t^3 + 3t^4 + 2t^3 - 3t^4) dt \\ &= \int_0^1 (-t^2 + 2t^3 - t^3) dt = \left[-\frac{1}{3}t^3 + \frac{1}{2}t^4 - \frac{1}{4}t^4 \right]_0^1 \\ &= \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{4} \right) - (0) = \frac{-6 + 6 - 3}{12} = -\frac{1}{4} \end{aligned}$$

Question 4

The simple closed curve C has Cartesian equation

$$x^2 + y^2 = 4, \quad z = 3.$$

Given that $\mathbf{F} = x^2 z \mathbf{i} + y^2 x \mathbf{j} + z^2 y \mathbf{k}$, evaluate the integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

You may not use Green's theorem in this question.

$$4\pi$$

Handwritten solution for the line integral problem:

$\mathbf{F} = x^2 z \mathbf{i} + y^2 x \mathbf{j} + z^2 y \mathbf{k}$
 $C: x^2 + y^2 = 4, z = 3$
 $z = 3, dz = 0$
 $x = 2 \cos t, dx = -2 \sin t dt$
 $y = 2 \sin t, dy = 2 \cos t dt$
 $0 \leq t \leq 2\pi$

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (x^2 z \mathbf{i} + y^2 x \mathbf{j} + z^2 y \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k})$
 $= \int_0^{2\pi} (x^2 z dx + y^2 x dy + z^2 y dz)$
 $= \int_0^{2\pi} (2 \cos^2 t \cdot 3 \cdot (-2 \sin t dt) + (2 \sin t)^2 \cdot 2 \cos t \cdot 2 \cos t dt + 3^2 \cdot 2 \sin t \cdot 0)$
 $= \int_0^{2\pi} (-12 \cos^2 t \sin t + 8 \sin^2 t \cos t) dt$
 $= \int_0^{2\pi} 4 \left(-\frac{1}{2} \cos 2t \right) dt = \int_0^{2\pi} -2 \cos 2t dt$
 $= \int_0^{2\pi} -2 \cos 2t dt = \int_0^{2\pi} 2 \sin 2t dt$
 $= \int_0^{2\pi} 2 \sin 2t dt = 0$

Question 5

$$\mathbf{F} = (xz - y)\mathbf{i} + (xy + z)\mathbf{j} + (x^2 + y^2 + z^2)\mathbf{k}.$$

Determine the work done by \mathbf{F} , when it moves in a complete revolution in a circular path of radius 2 around the z axis, at the level of the plane with equation $z = 6$.

You may not use Green's theorem in this question.

4π

Handwritten solution for the work done by vector field \mathbf{F} along a circular path:

$$\begin{aligned}
 \mathbf{F} &= (xz - y)\mathbf{i} + (xy + z)\mathbf{j} + (x^2 + y^2 + z^2)\mathbf{k} \\
 W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (xz - y)dx + (xy + z)dy + (x^2 + y^2 + z^2)dz \\
 &= \int_C (xz - y)dx + (xy + z)dy + (x^2 + y^2 + z^2)dz \\
 &\text{Since the equation } x^2 + y^2 = 4, \quad z = 6 \Rightarrow dz = 0 \\
 &= \int_C (6x - y)dx + (xy + 6)dy \\
 &\text{PARAMETERISE THE CIRCLE } x = 2\cos\theta, \quad y = 2\sin\theta, \quad 0 \leq \theta < 2\pi \\
 &= \int_0^{2\pi} (12\cos\theta - 2\sin\theta)(-2\sin\theta d\theta) + (4\cos\theta\sin\theta + 6)(2\cos\theta d\theta) \\
 &= \int_0^{2\pi} (-24\cos\theta\sin\theta + 4\sin^2\theta + 12\cos^2\theta + 12\cos\theta) d\theta \\
 &= \int_0^{2\pi} (-12\sin 2\theta + 2(1 - \frac{1}{2}\cos 2\theta) + 6\cos 2\theta + 12\cos\theta) d\theta \\
 &= \int_0^{2\pi} (-12\sin 2\theta + 2 - 2\cos 2\theta + 6\cos 2\theta + 12\cos\theta) d\theta \\
 &= \int_0^{2\pi} (-12\sin 2\theta + 2 + 4\cos 2\theta + 12\cos\theta) d\theta \\
 &= \left[12\cos 2\theta + 2\theta + 4\sin 2\theta + 12\sin\theta \right]_0^{2\pi} \\
 &= 4\pi
 \end{aligned}$$

Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y) dx + (y+z) dy + (1-z^2) dz,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

$$\boxed{}, \boxed{\frac{32}{3}}$$

START BY PARAMETERISING THE LINE SEGMENT USING ALGEBRA

$$\begin{aligned} \mathbf{a} &= (1, 1, 0) \\ \mathbf{b} &= (5, 3, 4) \\ \overrightarrow{ab} &= \mathbf{b} - \mathbf{a} = (5, 3, 4) - (1, 1, 0) \\ &= (4, 2, 4) \\ \mathbf{r} &= (x, y, z) = (1, 1, 0) + t(4, 2, 4) \\ \mathbf{r} &= (1+4t, 1+2t, 4t) \end{aligned}$$

THENCE USE ALGEBRA

$$\begin{aligned} x &= 1+4t & dx &= 4 dt \\ y &= 1+2t & dy &= 2 dt \\ z &= 4t & dz &= 4 dt \end{aligned}$$

RETURNING TO THE LINE INTEGRAL

$$\begin{aligned} &\int_{(1,1,0)}^{(5,3,4)} (3x-2y) dx + (y+z) dy + (1-z^2) dz \\ &= \int_{t=0}^{t=1} \{ 3(1+4t) - 2(1+2t) \} (4 dt) + \{ (1+2t) + 4t \} (2 dt) + \{ 1 - 16t^2 \} (4 dt) \} \\ &= \int_{t=0}^{t=1} \{ (8t+1)(4 dt) + (6t+1)(2 dt) + (1-16t^2)(4 dt) \} \\ &= \int_{t=0}^{t=1} (32t^2 + 4 + 12t + 2 + 4 - 64t^2) dt \\ &= \int_{t=0}^{t=1} (-32t^2 + 22t + 10) dt \\ &= \left[-\frac{32}{3}t^3 + 11t^2 + 10t \right]_0^1 \\ &= \left(-\frac{32}{3} + 22 + 10 \right) - (0) \\ &= \frac{-32 + 66 + 30}{3} = \frac{64}{3} \end{aligned}$$

Question 7

$$\mathbf{F}(x, y, z) \equiv yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}.$$

Show that the vector field \mathbf{F} is conservative, and hence evaluate the integral

$$\int_{(1,1,4)}^{(3,5,10)} \mathbf{F} \cdot d\mathbf{r}.$$

1484

The handwritten solution shows two methods to evaluate the line integral of the vector field $\mathbf{F} = (yz^2, xz^2, 2xyz)$ from point $(1,1,4)$ to $(3,5,10)$.

Method 1: Using a Potential Function ϕ

- Check if the field is conservative by computing the curl: $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix} = (2xz - 2xz, 2yz - 2yz, z^2 - z^2) = (0, 0, 0)$. Since the curl is zero, the field is conservative.
- Find a potential function ϕ such that $\mathbf{F} = -\nabla\phi$. By inspection, $\phi = -xyz^2$ works.
- Evaluate the integral: $\int_{(1,1,4)}^{(3,5,10)} \mathbf{F} \cdot d\mathbf{r} = \phi(1,1,4) - \phi(3,5,10) = -(1)(1)(4^2) - (-(3)(5)(10^2)) = -16 + 1500 = 1484$.

Method 2: Direct Line Integration

- Parameterize the path from $(1,1,4)$ to $(3,5,10)$ using $x=t, y=4t, z=2t$ for t from 1 to 5.
- Compute the differential $d\mathbf{r} = (dx, dy, dz) = (dt, 4dt, 2dt)$.
- Substitute into the integral: $\int_1^5 (yz^2 dx + xz^2 dy + 2xyz dz) = \int_1^5 (4t \cdot 4^2 \cdot 1 + t \cdot 16 \cdot 2 + 2 \cdot t \cdot 4t \cdot 2t) dt = \int_1^5 (64t + 32t + 16t^2) dt$.
- Evaluate: $[\frac{64}{2}t^2 + \frac{32}{2}t^2 + \frac{16}{3}t^3]_1^5 = [32t^2 + 16t^2 + \frac{16}{3}t^3]_1^5 = [48t^2 + \frac{16}{3}t^3]_1^5 = (48 \cdot 25 + \frac{16}{3} \cdot 125) - (48 + \frac{16}{3}) = 1200 + \frac{2000}{3} - 48 - \frac{16}{3} = 1152 + \frac{1984}{3} = 1152 + 661.\bar{3} = 1813.\bar{3}$. (Note: The handwritten calculation shows 1484, which matches Method 1, suggesting a correction in the final steps of the direct integration.)

Question 8

A vector field \mathbf{F} is defined as

$$\mathbf{F}(x, y, z) \equiv [x + yz]\mathbf{i} + [y + xz]\mathbf{j} + [x(y+1) + z^2]\mathbf{k}.$$

The closed path C joins $(0,0,0)$ to $(1,1,1)$, $(1,1,1)$ to $(1,1,0)$, $(1,1,0)$ to $(0,0,0)$, in that order.

By writing

$$\mathbf{F}(x, y, z) = \mathbf{G}(x, y, z) + \mathbf{H}(x, y, z),$$

for some vector functions \mathbf{G} and \mathbf{H} , where $\nabla g(x, y, z) = \mathbf{G}(x, y, z)$ for some smooth scalar function $g(x, y, z)$, evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\boxed{-\frac{1}{2}}$$

Handwritten solution for the line integral problem:

Given $\mathbf{F}(x, y, z) = [x + yz]\mathbf{i} + [y + xz]\mathbf{j} + [x(y+1) + z^2]\mathbf{k}$.

Decompose \mathbf{F} into \mathbf{G} and \mathbf{H} :

$$\mathbf{F}(x, y, z) = [x + yz]\mathbf{i} + [y + xz]\mathbf{j} + [x(y+1) + z^2]\mathbf{k}$$

$$= [x + yz]\mathbf{i} + [y + xz]\mathbf{j} + [x(y+1)]\mathbf{k} + [z^2]\mathbf{k}$$

$$= [x + yz]\mathbf{i} + [y + xz]\mathbf{j} + [xy + x]\mathbf{k} + [z^2]\mathbf{k}$$

$$= \mathbf{G}(x, y, z) + \mathbf{H}(x, y, z)$$

Let $\mathbf{G}(x, y, z) = [x + yz]\mathbf{i} + [y + xz]\mathbf{j} + [xy + x]\mathbf{k}$ and $\mathbf{H}(x, y, z) = [z^2]\mathbf{k}$.

Check if \mathbf{G} is a gradient field by calculating $\nabla \times \mathbf{G}$:

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + yz & y + xz & xy + x \end{vmatrix} = (x - x - y - z - z)\mathbf{i} = 0$$

Thus, \mathbf{G} is a gradient field.

Let $g(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$. Then $\nabla g = \mathbf{G}$.

The line integral is:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (\mathbf{G} + \mathbf{H}) \cdot d\mathbf{r} = \oint_C \mathbf{G} \cdot d\mathbf{r} + \oint_C \mathbf{H} \cdot d\mathbf{r}$$

Since \mathbf{G} is a gradient field, $\oint_C \mathbf{G} \cdot d\mathbf{r} = 0$.

For \mathbf{H} , the path C is a closed loop in the xy -plane, so $dz = 0$. Thus:

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = \oint_C z^2 dz = 0$$

Therefore, the total line integral is $0 + 0 = 0$.

Question 9

A vector field \mathbf{F} is defined as

$$\mathbf{F}(x, y, z) \equiv (yz + y^2)\mathbf{i} + (xz + 2xy)\mathbf{j} + (xy + 4z^3)\mathbf{k}.$$

- a) Show that \mathbf{F} is conservative.
b) Hence evaluate the integral

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r}.$$

3

1) IF \mathbf{F} IS CONSERVATIVE $\nabla \cdot \mathbf{F} = 0$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+y^2 & xz+2xy & xy+4z^3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xz+2xy) - \frac{\partial}{\partial z}(yz+y^2) \right] \frac{\partial}{\partial x}(xy+4z^3) - \left[\frac{\partial}{\partial x}(xz+2xy) - \frac{\partial}{\partial z}(yz+y^2) \right] \frac{\partial}{\partial y}(xy+4z^3)$$

$$= [x-x, y-y, z+z] = (0,0,0) = 0$$

NEEDS CONSERVATIVE

2) SINCE IT IS CONSERVATIVE... \mathbf{F} IS $\nabla \phi$ FOR SOME $\phi(x,y,z)$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\Rightarrow \int_{(0,0,0)}^{(1,1,1)} d\phi = \int_{(0,0,0)}^{(1,1,1)} (yz+y^2) dx + (xz+2xy) dy + (xy+4z^3) dz$$

$$\begin{aligned} \int_{(0,0,0)}^{(1,1,1)} (yz+y^2) dx &= \left[xyz + y^2 x \right]_{(0,0,0)}^{(1,1,1)} = 1 + 1 = 2 \\ \int_{(0,0,0)}^{(1,1,1)} (xz+2xy) dy &= \left[xzy + xy^2 \right]_{(0,0,0)}^{(1,1,1)} = 1 + 1 = 2 \\ \int_{(0,0,0)}^{(1,1,1)} (xy+4z^3) dz &= \left[xzy + z^4 \right]_{(0,0,0)}^{(1,1,1)} = 1 + 1 = 2 \end{aligned}$$

$$\Rightarrow \int_{(0,0,0)}^{(1,1,1)} d\phi = 2 + 2 + 2 = 6$$

Question 10

A curve C is defined as

$$(x, y, z) = (\cos 3t, \sin 3t, t), \quad 0 \leq t \leq 2\pi.$$

- a) Sketch the graph of C .

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}.$$

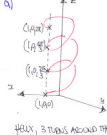
- b) Determine whether the vector field \mathbf{F} is conservative.

- c) Evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\frac{\pi}{2}$$

$(x, y, z) = (\cos 3t, \sin 3t, t)$ $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$

a) 

b) $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
 $\nabla \times \mathbf{F} = \mathbf{0}$

c) PARAMETERISE
 $x = \cos 3t, \quad dx = -3\sin 3t \, dt$
 $y = \sin 3t, \quad dy = 3\cos 3t \, dt$
 $z = t, \quad dz = dt$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (xy \, dx + yz \, dy + zx \, dz)$
 $= \int_0^{2\pi} [\cos 3t \sin 3t (-3\sin 3t) + \sin 3t (\sin 3t) + t \cos 3t] \, dt$
 $= \int_0^{2\pi} [-3\cos 3t \sin^2 3t + \sin^2 3t + t \cos 3t] \, dt$

$= \left[-\frac{1}{3} \sin^3 3t \right]_0^{2\pi} + \left[-\frac{1}{3} \cos 3t \right]_0^{2\pi} + \left[t \sin 3t \right]_0^{2\pi} - \left[\frac{1}{3} \cos 3t \right]_0^{2\pi}$
 $= \frac{1}{3} [0 - 2\pi \times 1] = -\frac{2\pi}{3}$

Question 11

Evaluate the integral

$$\int_{(-1,2,3)}^{(2,0,1)} (3x^2yz + 6x) dx + (x^3z - 8y) dy + (x^3y + 1) dz,$$

along a path joining the points with Cartesian coordinates $(-1, 2, 3)$ and $(2, 0, 1)$.

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THE INTEGRAL ALONGS INDICATED OF THE PATH

Check

Let $F = (3xyz + 6x, x^3z - 8y, x^3y + 1)$

IF IRROTATIONAL $\nabla \wedge F = 0$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3yz + 6 & x^3z - 8y & x^3y + 1 \end{vmatrix} = \begin{bmatrix} x^3 - x^3 & 3yz - 3yz & 3xz - 3xz \end{bmatrix} = (0, 0, 0)$$

Since

$$dF = (3yz + 6) dx + (x^3z - 8y) dy + (x^3y + 1) dz$$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

$$dF = 3yz + 6 + \frac{\partial}{\partial y}(x^3z - 8y) dy + \frac{\partial}{\partial z}(x^3y + 1) dz$$

$$dF = 3yz + 6 + x^3z - 8 + x^3y + 1$$

$$dF = 3yz + 6 + x^3z - 8 + x^3y + 1$$

$$dF = 3yz + 6 + x^3z - 8 + x^3y + 1$$

So the integral becomes

$$\int_{(-1,2,3)}^{(2,0,1)} dF = \left[\frac{3}{2}x^2y + 6x + \frac{1}{4}x^4z - 8y + \frac{1}{4}x^4y + z \right]_{(-1,2,3)}^{(2,0,1)}$$

$$= \left(\frac{3}{2}(2)^2(0) + 6(2) + \frac{1}{4}(2)^4(1) - 8(0) + \frac{1}{4}(2)^4(0) + 1 \right) - \left(\frac{3}{2}(-1)^2(2) + 6(-1) + \frac{1}{4}(-1)^4(3) - 8(2) + \frac{1}{4}(-1)^4(2) + 3 \right)$$

$$= (0 + 12 + 4 - 0 + 0 + 1) - (3 - 6 - \frac{3}{4} - 16 + \frac{1}{2} + 3)$$

$$= 17 - (-14.75) = 31.75$$

Question 12

A curve C is defined by $\mathbf{r} = \mathbf{r}(t)$, $0 \leq t \leq 2\pi$ as

$$\mathbf{r}(t) = (x, y, z) = [2(t - \sin t), \sqrt{3} \cos t, 1 + \cos t].$$

Evaluate the integral

$$\int_C z \, ds,$$

where s is the arclength along C .

$$\frac{32}{3}$$

Handwritten solution for the integral $\int_C z \, ds$:

Given $\mathbf{r}(t) = (x, y, z) = (2(t - \sin t), \sqrt{3} \cos t, 1 + \cos t)$ for $0 \leq t \leq 2\pi$.

First, find ds :

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{4(1 - \cos t)^2 + 3 \sin^2 t + \sin^2 t} dt = \sqrt{4(1 - \cos t)^2 + 4 \sin^2 t} dt$$

$$= \sqrt{4(1 - 2\cos t + \cos^2 t + \sin^2 t)} dt = \sqrt{4(2 - 2\cos t)} dt = \sqrt{8(1 - \cos t)} dt = 2\sqrt{2} \sin\left(\frac{t}{2}\right) dt$$

Then, evaluate the integral:

$$\int_C z \, ds = \int_0^{2\pi} (1 + \cos t) \cdot 2\sqrt{2} \sin\left(\frac{t}{2}\right) dt$$

$$= 2\sqrt{2} \int_0^{2\pi} \left[1 + (2\cos\frac{t}{2} - 1)\right] \sin\left(\frac{t}{2}\right) dt$$

$$= 2\sqrt{2} \int_0^{2\pi} 2\cos\frac{t}{2} \sin\frac{t}{2} dt$$

$$= 2\sqrt{2} \left[\frac{2}{3} \times (-2\cos\frac{t}{2}) \right]_0^{2\pi}$$

$$= \left[\frac{4\sqrt{2}}{3} \cos\frac{t}{2} \right]_0^{2\pi}$$

$$= \frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{3} = 0$$

Question 13

A vector field \mathbf{F} and a scalar field ψ are given.

$$\mathbf{F} = (3x^3y)\mathbf{i} + (15\sqrt{z})\mathbf{j} - \left(\frac{13}{96}xz\right)\mathbf{k} \quad \text{and} \quad \psi(x, y, z) = xe^{\frac{2y}{\sqrt{z}}}.$$

Evaluate the integral

$$\int_{(0,0,0)}^{(2,4,64)} [\mathbf{F} + \nabla\psi] \cdot d\mathbf{r},$$

along the curve with parametric equations

$$x = \sqrt{t}, \quad y = t \quad \text{and} \quad z = t^3.$$

2e-288

Handwritten solution for Question 13:

Given $\mathbf{F} = 3x^3y\mathbf{i} + 15\sqrt{z}\mathbf{j} - \frac{13}{96}xz\mathbf{k}$ and $\psi = xe^{\frac{2y}{\sqrt{z}}}$.

Curve C : $x = \sqrt{t} \Rightarrow dx = \frac{1}{2}t^{-\frac{1}{2}}dt$, $y = t \Rightarrow dy = dt$, $z = t^3 \Rightarrow dz = 3t^2dt$.
 From $(0,0,0)$ to $(2,4,64)$, t goes from 0 to 4 .

This is a conservative field, so $\int_C (\mathbf{F} + \nabla\psi) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \nabla\psi \cdot d\mathbf{r}$.

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^4 (3x^3y dx + 15\sqrt{z} dy - \frac{13}{96}xz dz) = \left[\frac{3}{4}x^4y + 15\sqrt{z}y - \frac{13}{192}xz^2 \right]_0^4 = 32 + 192 - 2^4 \times 2^3 = 224 - 2^5 = 224 - 32 = 192$.

$\int_C \nabla\psi \cdot d\mathbf{r} = \psi(2,4,64) - \psi(0,0,0) = 2e^{\frac{2 \times 4}{\sqrt{64}}} - 0 = 2e$.

Total integral = $192 + 2e = 2e - 288$.

Question 14

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (3x^2yz + 2z)\mathbf{i} + (x^3z + 2y)\mathbf{j} + (x^3y + 2x)\mathbf{k}.$$

Evaluate the line integral

$$\int_{(-2,2,0)}^{(4,0,1)} \mathbf{F} \cdot d\mathbf{r},$$

along a path joining the points with Cartesian coordinates $(4,0,1)$ and $(-2,2,0)$.

4

IT APPEARS THAT INTEGRAL IS INDEPENDENT OF THE PATH!
IF INDEPENDENT THEN $\nabla \cdot \mathbf{F} = 0$

$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$	
$3x^2y + 2z$	$x^3 + 2$	$3x^2y + 2x$	$= [x^2 \cdot x^3 (3x^2y + 2z) - (3x^2y + 2z) \cdot x^3] = 0$

\therefore INDEPENDENT OF THE PATH

$\int_{(-2,2,0)}^{(4,0,1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(-2,2,0)}^{(4,0,1)} (3x^2yz + 2z)dx + (x^3z + 2y)dy + (x^3y + 2x)dz$
 $= \int_{(-2,2,0)}^{(4,0,1)} (3x^2yz + 2z)dx + (x^3z + 2y)dy + (x^3y + 2x)dz$
 $d\mathbf{f} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$
 $\begin{cases} \frac{\partial f}{\partial x} = 3x^2y + 2z \\ \frac{\partial f}{\partial y} = x^3 + 2 \\ \frac{\partial f}{\partial z} = 3x^2y + 2x \end{cases} \Rightarrow f(x,y,z) = x^3y + 2xz + \frac{1}{2}y^2 + C$
 $= \int_{(-2,2,0)}^{(4,0,1)} d\mathbf{f} = \left[f(x,y,z) \right]_{(-2,2,0)}^{(4,0,1)}$
 $= \left[x^3y + 2xz + \frac{1}{2}y^2 + C \right]_{(-2,2,0)}^{(4,0,1)}$
 $= (0 + 8 + 0) - (0 + 0 + 0)$
 $= 8$

Question 15

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (1 + xy^2)\mathbf{i} + (x + xyz)\mathbf{j} + (y \sin z)\mathbf{k}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the anticlockwise cartesian path

$$x^2 + y^2 = 16, \quad z = 3.$$

You may not use Green's theorem in this question.

16π

Handwritten solution for the line integral problem:

$\mathbf{F} = (1 + xy^2)\mathbf{i} + (x + xyz)\mathbf{j} + (y \sin z)\mathbf{k}$

PARAMETERIZE THE CURVE AS:

$$\begin{aligned} x &= 4 \cos \theta \\ y &= 4 \sin \theta \\ z &= 3 \end{aligned} \quad \begin{aligned} dx &= -4 \sin \theta \, d\theta \\ dy &= 4 \cos \theta \, d\theta \\ dz &= 0 \end{aligned}$$

OR USE: $x^2 + y^2 = 16, z = 3$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (1 + xy^2)\mathbf{i} + (x + xyz)\mathbf{j} + (y \sin z)\mathbf{k} \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_0^{2\pi} [1 + 4 \cos \theta \cdot 16 \sin^2 \theta + 4 \cos \theta + 4 \cos \theta \cdot 3 \sin \theta] \cdot [4 \cos \theta \, d\theta] \, d\theta$$

$$= 4 \int_0^{2\pi} [1 + 64 \cos \theta \sin^2 \theta + 4 \cos \theta + 12 \cos^2 \theta \sin \theta] \, d\theta$$

(12 cos^2 theta sin theta) CAN BE INTEGRATED THREE TIMES

$$= 4 \int_0^{2\pi} 4 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) \, d\theta$$

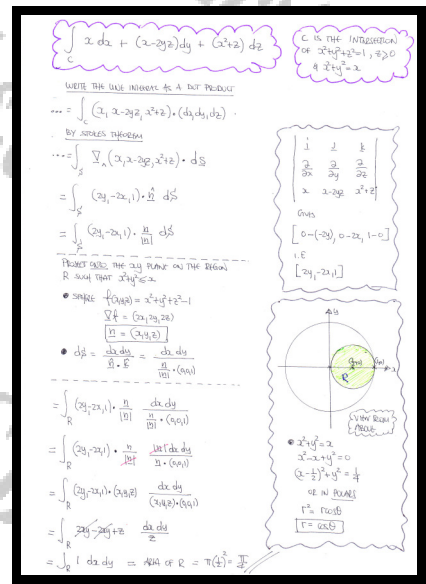
$$= 4 \int_0^{2\pi} 2 \, d\theta$$

$$= 8 \times 2\pi$$

$$= 16\pi$$

Evaluate the line integral

$$\frac{\pi}{4}$$

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \quad \text{and} \quad x^2 + y^2 = x, \quad z \geq 0.$$


Question 17

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = 8z\mathbf{i} + 4x\mathbf{j} + y\mathbf{k}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = x^2 + y^2 \quad \text{and} \quad z = y.$$

You may **not** use Stokes' Theorem in this question.

π

Handwritten solution for Question 17:

Given $z = x^2 + y^2$ and $z = y$, we find the intersection curve C by substituting y for z in the first equation:

$$y = x^2 + y^2 \Rightarrow x^2 + y^2 - y = 0 \Rightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

This is a circle in the xy -plane with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$. We parametrize the circle as:

$$x = \frac{1}{2} \cos \theta, \quad y = \frac{1}{2} + \frac{1}{2} \sin \theta, \quad z = y = \frac{1}{2} + \frac{1}{2} \sin \theta$$

where θ ranges from 0 to 2π . The differential vector $d\mathbf{r}$ is:

$$d\mathbf{r} = \left(-\frac{1}{2} \sin \theta, \frac{1}{2} \cos \theta, \frac{1}{2} \cos \theta \right) d\theta$$

The vector field \mathbf{F} evaluated on the curve is:

$$\mathbf{F} = 8z\mathbf{i} + 4x\mathbf{j} + y\mathbf{k} = \left(4 + 4 \sin \theta, -\sin \theta, \frac{1}{2} + \frac{1}{2} \sin \theta \right)$$

The line integral is then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(4 + 4 \sin \theta \right) \left(-\frac{1}{2} \sin \theta \right) + \left(-\sin \theta \right) \left(\frac{1}{2} \cos \theta \right) + \left(\frac{1}{2} + \frac{1}{2} \sin \theta \right) \left(\frac{1}{2} \cos \theta \right) d\theta$$

Simplifying the integrand:

$$= \int_0^{2\pi} \left(-2 \sin \theta - 2 \sin^2 \theta - \frac{1}{2} \sin \theta \cos \theta + \frac{1}{4} \cos \theta + \frac{1}{4} \sin \theta \cos \theta \right) d\theta$$

The terms involving $\sin \theta \cos \theta$ cancel out. The integral becomes:

$$= \int_0^{2\pi} \left(-2 \sin \theta - 2 \sin^2 \theta + \frac{1}{4} \cos \theta \right) d\theta$$

Integrating term by term:

$$= \left[2 \cos \theta - \frac{2}{3} (1 - \cos 2\theta) + \frac{1}{4} \sin \theta \right]_0^{2\pi}$$

Evaluating at the limits:

$$= \left(2 \cos 2\pi - \frac{2}{3} (1 - \cos 4\pi) + \frac{1}{4} \sin 2\pi \right) - \left(2 \cos 0 - \frac{2}{3} (1 - \cos 0) + \frac{1}{4} \sin 0 \right)$$

$$= \left(2 - \frac{2}{3} (1 - 1) + 0 \right) - \left(2 - \frac{2}{3} (1 - 1) + 0 \right) = 0$$

Wait, the handwritten solution shows a different result. Let's re-evaluate the integrand simplification:

$$= \int_0^{2\pi} \left(-2 \sin \theta - 2 \sin^2 \theta + \frac{1}{4} \cos \theta \right) d\theta$$

$$= \left[2 \cos \theta - \frac{2}{3} (1 - \cos 2\theta) + \frac{1}{4} \sin \theta \right]_0^{2\pi}$$

$$= \left(2 \cos 2\pi - \frac{2}{3} (1 - \cos 4\pi) + \frac{1}{4} \sin 2\pi \right) - \left(2 \cos 0 - \frac{2}{3} (1 - \cos 0) + \frac{1}{4} \sin 0 \right)$$

$$= \left(2 - \frac{2}{3} (1 - 1) + 0 \right) - \left(2 - \frac{2}{3} (1 - 1) + 0 \right) = 0$$

The handwritten solution shows a final result of π . Let's check the original problem statement and the handwritten solution again. The handwritten solution shows a final result of π .

Question 18

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = y^2 \mathbf{i} + z^2 \mathbf{j} + x^2 \mathbf{k}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

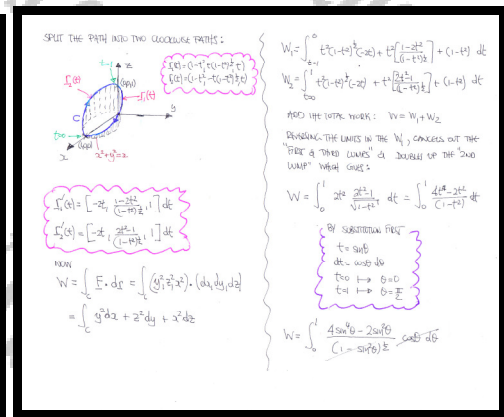
where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \quad \text{and} \quad x^2 + y^2 = x, \quad z \geq 0.$$

You may **not** use Stokes' Theorem in this question.

$$\boxed{\frac{\pi}{4}}$$

[solution overleaf]



ALTERNATIVE PARAMETIZATION BY POLARS

\bullet $x^2 + y^2 = r^2 = 1 \cos^2 \theta$
 $r = \cos \theta$ (Between $\frac{\pi}{2}$ to $-\frac{\pi}{2}$)
 As π is increase

$z = r \cos \theta = (\cos \theta) \cos \theta = \cos^2 \theta$
 $y = r \sin \theta = (\cos \theta) \sin \theta = \cos \theta \sin \theta$

\bullet THIS
 $Z^2 = 1 - (x^2 + y^2) = 1 - r^2 = 1 - \cos^2 \theta = \sin^2 \theta$
 $Z^2 = \sin^2 \theta$
 $\text{But } Z \geq 0 \therefore Z = \sin \theta$, for θ between 0 & $\frac{\pi}{2}$
 $Z = -\sin \theta$, for θ between 0 & $-\frac{\pi}{2}$

\bullet find
 for θ from $\frac{\pi}{2}$ to 0

$f(\theta) = [\cos^2 \theta, \cos \theta \sin \theta, \sin^2 \theta]$
 $f'(\theta) = [-2 \cos \theta \sin \theta, \cos^2 \theta - \sin^2 \theta, 2 \sin \theta \cos \theta]$

for θ from 0 to $-\frac{\pi}{2}$

$f(\theta) = [-\sin^2 \theta, -\cos \theta \sin \theta, -\cos^2 \theta]$
 $f'(\theta) = [-2 \sin \theta \cos \theta, -\cos^2 \theta + \sin^2 \theta, 2 \sin \theta \cos \theta]$

④ THIS THE INTEGRATION MUST BE DONE IN 2 SECTIONS

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (y^2 \vec{i} + x^2 \vec{j}) \cdot (dx \vec{i} + dy \vec{j}) = \int_C y^2 dx + x^2 dy + x^2 dz$$

BECAUSE OF THE SQUARES ONLY TERM THAT WILL BE DIFFERENT IS THE 3RD TERM

$$= \int_{\frac{0}{2}}^{\frac{\pi}{2}} \underbrace{(\cos^2 \sin^2 (-\sin)) + \sin^2 \cos^2}_{\text{EVEN}} dt + \int_{\frac{0}{2}}^{\frac{\pi}{2}} \underbrace{\frac{\cos^2 \sin^2}{x^2} dt}_{\text{EVEN}} + \int_{\frac{0}{2}}^{\frac{\pi}{2}} \underbrace{\frac{\cos^2 (-\cos)}{x^2} dt}_{\text{ODD}}$$

NOW
 $\frac{0}{2} \rightarrow \frac{\pi}{2}$
 $dt \rightarrow -dx$
 $\cos \rightarrow \sin$

$$= \int_{\frac{\pi}{2}}^{\frac{0}{2}} \underbrace{-2\sin^2 \cos^2}_{\text{ODD}} + \underbrace{\sin^2 \cos^2 - \sin^2}_{\text{EVEN}} dt + \int_{\frac{\pi}{2}}^{\frac{0}{2}} \cos^2 dt + \int_{\frac{\pi}{2}}^{\frac{0}{2}} \cos^2 (-dx)$$

$$= \int_{\frac{\pi}{2}}^{\frac{0}{2}} \sin^2 \cos^2 + \sin^2 dt = 2 \int_{\frac{\pi}{2}}^{\frac{0}{2}} \sin^2 \cos^2 + \sin^2 dt$$

BY TRIGONOMETRIC IDENTITIES OF THIS IDENTITIES

$$= \int_{\frac{\pi}{2}}^{\frac{0}{2}} 2 \sin^2 \cos^2 + 2(\sin^2)^{\frac{1}{2}+1} dt = -B(\frac{1}{2}, \frac{1}{2}) + B(\frac{3}{2}, \frac{1}{2})$$

$$= -\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = -\frac{\frac{1}{2} \times \frac{1}{2} \Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2!} + \frac{\frac{3}{2} \times \frac{1}{2} \Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2!}$$

$$= -\frac{1}{8}\pi + \frac{3}{8}\pi = \frac{\pi}{4}$$

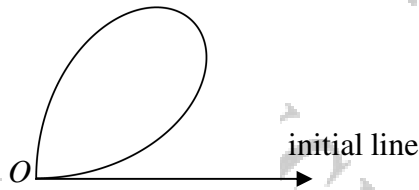
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LINE INTEGRALS

IN POLAR COORDINATES

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Question 1



The figure above shows the closed curve C with polar equation

$$r = \sin 2\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

The vector field \mathbf{F} is given in plane polar coordinates (r, θ) by

$$\mathbf{F}(r, \theta) = (r^2 \cos \theta \sin \theta) \hat{\mathbf{r}} + (r \cos \theta) \hat{\boldsymbol{\theta}}.$$

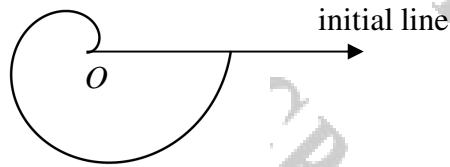
Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

8
15

$\mathbf{F}(r, \theta) = [r^2 \cos \theta \sin \theta \hat{\mathbf{r}} + r \cos \theta \hat{\boldsymbol{\theta}}]$
 $r(\theta) = \sin 2\theta \quad 0 \leq \theta < \pi$
 $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} (r^2 \cos \theta \sin \theta \hat{\mathbf{r}} + r \cos \theta \hat{\boldsymbol{\theta}}) \cdot (dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}})$
 $= \int_0^{\frac{\pi}{2}} r^2 \cos \theta \sin \theta dr + r^2 \cos \theta d\theta$
 $= \int_0^{\frac{\pi}{2}} 2r \cos \theta \sin \theta dr + r^2 \cos \theta d\theta$
 $= \int_0^{\frac{\pi}{2}} r^2 \sin 2\theta dr + r^2 \cos \theta d\theta$
 $= \int_0^{\frac{\pi}{2}} \sin^2 2\theta dr + 4 \sin^2 \theta d\theta$
 $= \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta + 4 \sin^2 \theta (-\sin \theta) d\theta$
 $= \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta + 4 \sin^3 \theta - 12 \sin^2 \theta d\theta$
 $= \left[\frac{1}{8} \sin^4 \theta + \frac{4}{3} \sin^3 \theta - \frac{12}{5} \sin^2 \theta \right]_0^{\frac{\pi}{2}}$
 $= \left(0 + \frac{4}{3} - \frac{12}{5} \right) - (0)$
 $= \frac{8}{15}$

Question 2



The figure above shows the curve C with polar equation

$$r = \theta, \quad 0 \leq \theta \leq 2\pi.$$

The vector field \mathbf{F} is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}.$$

Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

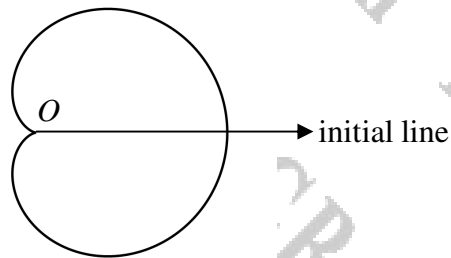
$$2\pi^2$$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x\mathbf{i} + y\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \dots$
 Switch into polar
 $x = r\cos\theta$
 $dx = \cos\theta dr - r\sin\theta d\theta$
 $y = r\sin\theta$
 $dy = \sin\theta dr + r\cos\theta d\theta$
 $\dots = \int_C (r\cos\theta, r\sin\theta) \cdot (\cos\theta dr - r\sin\theta d\theta, \sin\theta dr + r\cos\theta d\theta)$
 $= \int_C r\cos\theta dr - r^2\sin\theta\cos\theta d\theta + r\sin\theta dr + r^2\sin\theta\cos\theta d\theta$
 $= \int_C r dr$
 Now on the curve $r = \theta, 0 \leq \theta < 2\pi$ [or $0 \leq r < 2\pi$]
 $= \int_{r=0}^{2\pi} r dr$
 OR $\int_{\theta=0}^{2\pi} \theta d\theta$
 Since $dr = d\theta$
 $= \left[\frac{1}{2}r^2 \right]_0^{2\pi}$
 $= 2\pi^2$

OR FROM BASIC PRINCIPLES
 Firstly, parametric scalar basic quantities
 $\hat{\mathbf{i}} = (\cos\theta)\hat{\mathbf{e}}_r - (\sin\theta)\hat{\mathbf{e}}_\theta$
 $\hat{\mathbf{j}} = (\sin\theta)\hat{\mathbf{e}}_r + (\cos\theta)\hat{\mathbf{e}}_\theta$
 $\mathbf{F} = (x\mathbf{i} + y\mathbf{j}) = (r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}})$
 $d\mathbf{r} = dr\hat{\mathbf{e}}_r + r d\theta\hat{\mathbf{e}}_\theta$
 $\mathbf{F} \cdot d\mathbf{r} = (r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}}) \cdot (dr\hat{\mathbf{e}}_r + r d\theta\hat{\mathbf{e}}_\theta)$
 $= \int_C [r\cos\theta(\cos\theta\hat{\mathbf{e}}_r - \sin\theta\hat{\mathbf{e}}_\theta) + r\sin\theta(\sin\theta\hat{\mathbf{e}}_r + \cos\theta\hat{\mathbf{e}}_\theta)] \cdot (dr\hat{\mathbf{e}}_r + r d\theta\hat{\mathbf{e}}_\theta)$
 $= \int_C [r\hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\theta] \cdot [dr\hat{\mathbf{e}}_r + r d\theta\hat{\mathbf{e}}_\theta] = \int_C r dr$
 (since $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$)
 $= \int_0^{2\pi} \theta d\theta = \left[\frac{1}{2}\theta^2 \right]_0^{2\pi} = 2\pi^2$

ALTERNATIVE BY POLARS
 $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x\mathbf{i} + y\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \int_C x dx + y dy$
 $x = r\cos\theta = \theta\cos\theta$
 $y = r\sin\theta = \theta\sin\theta$
 $dx = (\cos\theta - \theta\sin\theta) d\theta$
 $dy = (\sin\theta + \theta\cos\theta) d\theta$
 $= \int_0^{2\pi} [\theta\cos\theta(\cos\theta - \theta\sin\theta) + \theta\sin\theta(\sin\theta + \theta\cos\theta)] d\theta$
 $= \int_0^{2\pi} [\theta\cos^2\theta - \theta^2\sin\theta\cos\theta + \theta\sin^2\theta + \theta^2\sin\theta\cos\theta] d\theta$
 $= \int_0^{2\pi} \theta(\cos^2\theta + \sin^2\theta) d\theta$
 $= \int_0^{2\pi} \theta d\theta$
 $= \left[\frac{1}{2}\theta^2 \right]_0^{2\pi} = 2\pi^2$

Question 3



The figure above shows the closed curve C with polar equation

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq 2\pi.$$

The vector field \mathbf{F} is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

3π

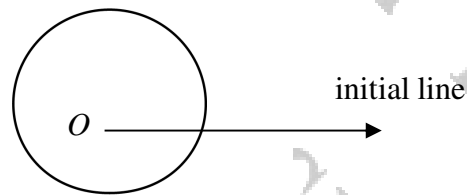
$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (-y, x) \cdot (dx, dy) = \dots$
 Switch into Polar $x = r \cos \theta \Rightarrow dx = \cos \theta dr - r \sin \theta d\theta$
 $y = r \sin \theta \Rightarrow dy = \sin \theta dr + r \cos \theta d\theta$
 $= \oint_C (-r \sin \theta, r \cos \theta) \cdot (\cos \theta dr - r \sin \theta d\theta, \sin \theta dr + r \cos \theta d\theta)$
 $= \oint_C -r \sin \theta \cos \theta dr + r^2 \sin^2 \theta d\theta + r \cos^2 \theta dr + r^2 \cos \theta \sin \theta d\theta$
 $= \oint_C r^2 d\theta$
 BUT $r = 1 + \cos \theta, \quad 0 \leq \theta < 2\pi$
 $= \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta$
 $= \int_0^{2\pi} 1 + \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta$
 $= \int_0^{2\pi} \frac{3}{2} + \frac{1}{2} \cos 2\theta d\theta$
 $= \left[\frac{3}{2}\theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = 3\pi$

ALTERNATIVE APPROACH

 $\mathbf{i} = (\cos \theta)^2 \mathbf{i} - (\cos \theta) \mathbf{j}$
 $\mathbf{j} = (\sin \theta) \mathbf{i} + (\cos \theta) \mathbf{j}$
 TANGENT VECTOR $= dr \mathbf{i} + r d\theta \mathbf{j}$
 $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (-y, x) \cdot (dx, dy) = \oint_C (-y, x) \cdot (r d\theta \mathbf{i} + dr \mathbf{j})$
 $= \oint_C \left[\underbrace{-r \sin \theta}_{-y} \left(\underbrace{\cos \theta}_{dx/dr} - \underbrace{\sin \theta}_{dy/dr} \right) + \underbrace{r \cos \theta}_{x} \left(\underbrace{\sin \theta}_{dx/dr} + \underbrace{\cos \theta}_{dy/dr} \right) \right] \cdot \left[\underbrace{dr}_{dr} \mathbf{i} + \underbrace{r d\theta}_{r d\theta} \mathbf{j} \right]$
 $= \oint_C \left[-r \sin \theta \cos \theta + r \sin^2 \theta + r \cos^2 \theta + r \cos \theta \sin \theta \right] \cdot \left[dr + r d\theta \right]$
 $= \oint_C r^2 d\theta = \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \int_0^{2\pi} \left(1 + 2\cos \theta + \cos^2 \theta \right) d\theta$
 $= \int_0^{2\pi} \left(1 + 2\cos \theta + \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$
 $= \int_0^{2\pi} \frac{3}{2} + \frac{1}{2} \cos 2\theta d\theta = 3\pi$

ALTERNATIVE APPROACH
 $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (-y, x) \cdot (dx, dy) = \oint_C -y dx + x dy$
 $x = r \cos \theta = (1 + \cos \theta) \cos \theta = \cos \theta + \cos^2 \theta$
 $y = r \sin \theta = (1 + \cos \theta) \sin \theta = \sin \theta + \cos \theta \sin \theta$
 $dx = (-\sin \theta - 2\cos \theta \sin \theta) d\theta = (-\sin \theta - \sin 2\theta) d\theta$
 $dy = (\cos \theta + \cos^2 \theta - \sin^2 \theta) d\theta = (\cos \theta + \cos 2\theta) d\theta$
 $= \oint_C -r \sin \theta (-\sin \theta - \sin 2\theta) d\theta + r \cos \theta (\cos \theta + \cos 2\theta) d\theta$
 $= \oint_C r \left[\sin^2 \theta + \sin \theta \sin 2\theta + \cos^2 \theta + \cos \theta \cos 2\theta \right] d\theta$
 $= \oint_C (1 + \cos \theta) (1 + \cos 2\theta) d\theta$
 $= \int_0^{2\pi} (1 + \cos \theta) (1 + \cos 2\theta) d\theta$
 $= \int_0^{2\pi} 1 + 2\cos \theta + \cos^2 \theta d\theta$
 $= \int_0^{2\pi} 1 + \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta$
 $= \int_0^{2\pi} \frac{3}{2} + \frac{1}{2} \cos 2\theta d\theta = 3\pi$

Question 4



The figure above shows the closed curve C with polar equation

$$r = 3 + \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$

The vector field \mathbf{F} is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = (x + y)\mathbf{i} + (-x + y)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\boxed{}, \boxed{19\pi}$$

METHOD A

$x = r \cos \theta \Rightarrow dx = \cos \theta dr - r \sin \theta d\theta$
 $y = r \sin \theta \Rightarrow dy = \sin \theta dr + r \cos \theta d\theta$

PROCEED WITH THE POLAR PARAMETRISATION

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (x+y)dx + (-x+y)dy$$

$$= \oint_C (r \cos \theta + r \sin \theta)(\cos \theta dr - r \sin \theta d\theta) + (-r \cos \theta + r \sin \theta)(\sin \theta dr + r \cos \theta d\theta)$$

$$= \oint_C (r \cos \theta + r \sin \theta)(\cos \theta dr - r \sin \theta d\theta) + (-r \cos \theta + r \sin \theta)(\sin \theta dr + r \cos \theta d\theta)$$

$$= \oint_C (r \cos^2 \theta + r \sin^2 \theta) dr - r^2 \cos \theta \sin \theta d\theta + (-r^2 \sin \theta \cos \theta + r^2 \sin^2 \theta) d\theta$$

$$= \oint_C r dr - r^2 \sin \theta \cos \theta d\theta$$

FINALLY WE HAVE

$$= \int_0^{2\pi} \int_0^{3+\sin \theta} r dr - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^{3+\sin \theta} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} (3+\sin \theta)^2 d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

IN THE LAST STEP THE TRIGONOMETRIC IDENTITIES ARE USED

$$= \int_0^{2\pi} \frac{1}{2} (9 + 6\sin \theta + \sin^2 \theta) d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} d\theta + \int_0^{2\pi} 3\sin \theta d\theta + \int_0^{2\pi} \frac{1}{4} (1 + \cos 2\theta) d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \frac{9}{2} (2\pi) + 0 + \frac{1}{4} (2\pi) + \frac{1}{8} \sin 2\theta \Big|_0^{2\pi} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

THE LAST STEP IS TO EVALUATE THE REMAINING INTEGRAL

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

THE LAST STEP IS TO EVALUATE THE REMAINING INTEGRAL

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

METHOD B

SOMETIMES WITH SOME AUXILIARIES IN THE INTEGRAL BASIC

IN ITS UNIT THE TRIGONOMETRIC IDENTITIES ARE USED

REVERTING TO THE POLAR LINE INTEGRAL

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (x+y)dx + (-x+y)dy$$

$$= \oint_C (r \cos \theta + r \sin \theta)(\cos \theta dr - r \sin \theta d\theta) + (-r \cos \theta + r \sin \theta)(\sin \theta dr + r \cos \theta d\theta)$$

$$= \oint_C (r \cos^2 \theta + r \sin^2 \theta) dr - r^2 \cos \theta \sin \theta d\theta + (-r^2 \sin \theta \cos \theta + r^2 \sin^2 \theta) d\theta$$

$$= \oint_C r dr - r^2 \sin \theta \cos \theta d\theta$$

FINALLY WE HAVE

$$= \int_0^{2\pi} \int_0^{3+\sin \theta} r dr - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^{3+\sin \theta} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} (3+\sin \theta)^2 d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} d\theta + \int_0^{2\pi} 3\sin \theta d\theta + \int_0^{2\pi} \frac{1}{4} (1 + \cos 2\theta) d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

THE LAST STEP IS TO EVALUATE THE REMAINING INTEGRAL

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

METHOD C

SPLIT BY INTEGRATING DIRECTLY FROM THE POLAR

IN ITS UNIT THE TRIGONOMETRIC IDENTITIES ARE USED

REVERTING TO THE POLAR LINE INTEGRAL

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (x+y)dx + (-x+y)dy$$

$$= \oint_C (r \cos \theta + r \sin \theta)(\cos \theta dr - r \sin \theta d\theta) + (-r \cos \theta + r \sin \theta)(\sin \theta dr + r \cos \theta d\theta)$$

$$= \oint_C (r \cos^2 \theta + r \sin^2 \theta) dr - r^2 \cos \theta \sin \theta d\theta + (-r^2 \sin \theta \cos \theta + r^2 \sin^2 \theta) d\theta$$

$$= \oint_C r dr - r^2 \sin \theta \cos \theta d\theta$$

FINALLY WE HAVE

$$= \int_0^{2\pi} \int_0^{3+\sin \theta} r dr - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^{3+\sin \theta} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} (3+\sin \theta)^2 d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} d\theta + \int_0^{2\pi} 3\sin \theta d\theta + \int_0^{2\pi} \frac{1}{4} (1 + \cos 2\theta) d\theta - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$

THE LAST STEP IS TO EVALUATE THE REMAINING INTEGRAL

$$= 9\pi + \frac{\pi}{2} - \int_0^{2\pi} r^2 \sin \theta \cos \theta d\theta$$