LEIBNIZ' S RULE OF DIFFERENTIATION

Leibniz Theorem

If $y = u(x)v(x)$ then

$$y_n = \sum_{r=1}^{n} \binom{n}{r} u_r v_{n-r} = u_n + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \dots \quad ,$$

where $u_m = \dfrac{d^m u}{dx^m}$ and $v_m = \dfrac{d^m v}{dx^m}$.

n^th order differential coefficients

$$\frac{d^n}{dx^n}\left(x^a\right) = y_n = \frac{a!}{(a-n)!} a^{a-n}$$

$$\frac{d^n}{dx^n}\left(\mathrm{e}^{ax}\right) = y_n = a^n \, \mathrm{e}^{ax}$$

$$\frac{d^n}{dx^n}(\sin ax) = y_n = a^n \sin\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\cos ax) = y_n = a^n \cos\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\sinh ax) = y_n = \frac{1}{2} a^n \left[\left[1 - (-1)^n\right] \sinh ax + \left[1 + (-1)^n\right] \cosh ax\right]$$

$$\frac{d^n}{dx^n}(\cosh ax) = y_n = \frac{1}{2} a^n \left[\left[1 + (-1)^n\right] \sinh ax + \left[1 - (-1)^n\right] \cosh ax\right]$$

Question 1 (***)

$$y = x^3 \mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

Use the Leibniz rule to show that

$$\frac{d^k y}{dx^k} = \mathrm{e}^{2x} 2^{k-3} f(x,k), \ k \in \mathbb{N},$$

where $f(x,k)$ is a function to be found.

$\boxed{}$, $\boxed{\dfrac{d^k y}{dx^k} = \mathrm{e}^{2x} 2^{k-3}\left[8x^3 + 12kx^2 + 6k(k-1)x + k(k-1)(k-2)\right]}$

$y = x^3 e^{2x}$ TO BE DIFFERENTIATED k TIMES

PICK SENSIBLE CHOICES FOR u & v

$u = e^{2x}$ (DIFFERENTIATING ANY NUMBER OF TIMES IS EASY)

$v = x^3$ (AFTER 4 DIFFERENTIATIONS IT VANISHES

LEIBNIZ RULE STATES

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + n \frac{d^{n-1} u}{dx^{n-1}} \frac{d^1 v}{dx^1} + \frac{n(n-1)}{2!} \frac{d^{n-2} u}{dx^{n-2}} \frac{d^2 v}{dx^2} + \dots$$

IN THIS QUESTION

$$\frac{d^k}{dx^k}\left(x^3 e^{2x}\right) = 2^k e^{2x} \cdot x^3 + k \cdot 2^{k-1} e^{2x}(3x^2) + \frac{k(k-1)}{2!} 2^{k-2} e^{2x}(6x)$$

$$+ \frac{k(k-1)(k-2)}{3!} 2^{k-3} e^{2x}(6) + \text{"REST IS ZERO"}$$

$$= e^{2x}\left[2^k x^3 + 3kx^2 2^{k-1} + \tfrac{1}{2}k(k-1) \times 6x 2^{k-2} + \tfrac{1}{6}k(k-1)(k-2) \times 6 \times 2^{k-3}\right]$$

$$= e^{2x}\left[x^3 2^k + 3kx^2 2^{k-1} + 3k(k-1)x 2^{k-2} + k(k-1)(k-2) 2^{k-3}\right]$$

$$= e^{2x} 2^{k-3}\left[x^3 2^3 + 3kx^2 2^2 + 3k(k-1)x \times 2^1 + k(k-1)(k-2)\right]$$

$$= e^{2x} 2^{k-3}\left[8x^3 + 12kx^2 + 6k(k-1)x + k(k-1)(k-2)\right]$$

Question 2 (*+)**

$$y = x^4 \cos x, \ x \in \mathbb{R}.$$

Use the Leibniz rule to find a simplified expression for $\dfrac{d^6 y}{dx^6}$.

$$\boxed{}, \quad \boxed{\frac{d^6 y}{dx^6} = 24x\left(20 - x^2\right)\sin x - \left(x^4 - 180x^2 + 360\right)\cos x}$$

USING LEIBNIZ RULE FOR PRODUCTS

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{d^1 v}{dx^1} + \frac{n(n-1)}{2!}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + \frac{n(n-1)(n-2)}{3!}\frac{d^{n-3}u}{dx^{n-3}}\frac{d^3 v}{dx^3} + \dots$$

HERE $y = x^4 \cos x$

v VANISHES AFTER A FEW DIFFERENTIATIONS

u (ITS DERIVATIVE HAS A PATTERN)

$$\boxed{\frac{d^k}{dx^k}\left[\cos(ax)\right] = a^k \cos\left(ax + \frac{k\pi}{2}\right)}$$

USING THE RULE WE OBTAIN

$$\frac{d^6}{dx^6}\left(\overset{v}{x^4}\overset{u}{\cos x}\right) = 1\frac{d^6}{dx^6}(\cos x)\,x^4 + 6\frac{d^5}{dx^5}(\cos x)\frac{d^1}{dx^1}(x^4) + \frac{6\times5}{2!}\frac{d^4}{dx^4}(\cos x)\frac{d^2}{dx^2}(x^4) + \frac{6\times5\times4}{3!}\frac{d^3}{dx^3}(\cos x)\frac{d^3}{dx^3}(x^4)$$

$$+ \frac{6\times5\times4\times3}{4!}\frac{d^2}{dx^2}(\cos x)\frac{d^4}{dx^4}(x^4) + \cancel{\frac{6\times5\times4\times3\times2}{5!}\frac{d^1}{dx^1}(\cos x)\frac{d^5}{dx^5}(x^4)} + \cancel{1\frac{d^0}{dx^0}(\cos x)\frac{d^6}{dx^6}(x^4)}$$

$$= x^4\cos(x+3\pi) + 6\cos\left(x+\tfrac{5}{2}\pi\right)\cdot 4x^3 + 15\cos(x+2\pi)\cdot 12x^2 + 20\cos\left(x+\tfrac{3}{2}\pi\right)\cdot 24x + 15\cos(x+\pi)\cdot 24$$

$$= x^4\cos(x+\pi) + 24x^3\cos\left(x+\tfrac{\pi}{2}\right) + 180x^2\cos x + 480x\cos\left(x-\tfrac{\pi}{2}\right) + 360\cos(x+\pi)$$

$$= x^4(-\cos x) + 24x^3(-\sin x) + 180x^2\cos x + 480x\sin x + 360(-\cos x)$$

$$= 24x(20-x^2)\sin x - (x^4 - 180x^2 + 360)\cos x$$

Question 3 (***+)

The function with equation $y = f(x)$ is differentiable n times, $n \in \mathbb{N}$, and satisfies the following relationship.

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0.$$

Use the Leibniz rule to show that at $x = 0$

$$\frac{d^{n+2}y}{dx^{n+2}} = \left(4-n^2\right)\frac{d^n y}{dx^n}.$$

$\boxed{}$, $\boxed{\text{proof}}$

WRITE THE O.D.E AS FOLLOWS

$$y = y_0, \quad \frac{dy}{dx} = y_1, \quad \frac{d^2y}{dx^2} = y_2, \quad \frac{d^3y}{dx^3} = y_3 \cdots \quad \frac{d^ny}{dx^n} = y_n$$

$$\Rightarrow y_2(x^2+1) + y_1 x - 4y_0 = 0$$

DIFFERENTIATE n TIMES BY LEIBNIZ RULE

$$\Rightarrow \frac{d^n}{dx^n}\left[\overset{u}{y_2}\overset{v}{(x^2+1)}\right] + \frac{d^n}{dx^n}\left[\overset{u}{y_1}\overset{v}{x}\right] - 4\frac{d^n}{dx^n}\left[y_0\right] = \cancel{\frac{d^n}{dx^n}[0]}$$

$$\Rightarrow y_{n+2}(x^2+1) + ny_{n+1}(2x) + \frac{n(n-1)}{2!}y_n(2) + \dots \text{ZERO TERMS}$$
$$+ y_{n+1}x + ny_n(1) + \dots \text{ZERO TERMS}$$
$$- 4y_n = 0$$

$$\Rightarrow y_{n+2}(x^2+1) + (2nx+x)y_{n+1} + (n(n-1)+n-4)y_n = 0$$

$$\Rightarrow y_{n+2}(x^2+1) + (2n+1)xy_{n+1} + (n^2-4)y_n = 0$$

FINALLY SET $x=0$

$$\Rightarrow y_{n+2} + (n^2-4)y_n = 0$$

$$\Rightarrow \frac{d^{n+2}y}{dx^{n+2}} - (4-n^2)\frac{d^ny}{dx^n} = 0$$

$$\Rightarrow \frac{d^{n+2}y}{dx^{n+2}} = (4-n^2)\frac{d^ny}{dx^n}$$

AS REQUIRED