LEGENDRE’S EQUATION

including Legendre’s functions and Legendre’s polynomials
Summary on Legendre Functions/Polynomials

Legendre’s Differential Equation

\[
(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n-1)y = 0, \quad n \in \mathbb{R}.
\]

General Solution of Legendre’s Differential Equation

\[
y = A \left[ \frac{(n+1)n}{2!} x^2 + \frac{(n+3)(n+1)n(n-2)}{4!} x^4 - \frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!} x^6 + \ldots \right] + \\
B \left[ \frac{(n+2)(n-1)}{3!} x^3 - \frac{(n+4)(n+2)(n-1)(n-3)}{3!} x^5 - \frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!} x^7 + \ldots \right]
\]

- If \( n \) is an even integer, the first solution terminates after a finite number of terms, while the second one produces an infinite series.
- If \( n \) is an odd integer, the second solution terminates after a finite number of terms, while the first solution produces an infinite series.
- The finite solutions are the Legendre Polynomials, also known as solutions of the first kind, denoted by \( P_n(x) \).
- The infinite series solutions are known as solutions of the second kind, denoted by \( Q_n(x) \).

The second solution \( Q_n(x) \) can be written in terms of \( P_n(x) \) by

\[
Q_n(x) = P_n(x) \int \frac{1}{(1-x^2)(P_n(x))^2} \, dx
\]
The infinite series form for the Legendre’s polynomial $P_n(x)$ is given by

$$P_n(x) = \sum_{k=0}^{N} \left[ \frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!} (-1)^k x^{n-2k} \right],$$

where $N$ is the floor function

$$N = \begin{cases} 
\frac{1}{2}n & \text{if } n \text{ is even} \\
\frac{1}{2}(n-1) & \text{if } n \text{ is odd}
\end{cases}$$

The generating function for the Legendre’s polynomial $P_n(x)$ is given by

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$
Question 1

Find the two independent solutions of Legendre’s equation

\[
\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R}.
\]

\[
y = A \left[ 1 - \frac{(n+1)n}{2!}x^2 + \frac{(n+3)(n+1)n(n-2)}{4!}x^4 - \frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6 + \ldots \right] + \\
B \left[ x^\frac{(n+2)(n-1)}{3!}x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{5!}x^5 - \frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7 + \ldots \right]
\]
Question 2
Legendre’s equation is given below

\[ (1-t^2) \frac{d^2w}{dt^2} - 2t \frac{dw}{dt} + n(n+1)w = 0, \quad n \in \mathbb{N}. \]

a) By assuming a series solution of the form

\[ w(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \]

show by a detailed method that

\[ a_{r+2} = \frac{(n-r)(n+r+1)}{(r+2)(r+1)} a_r. \]

b) By rewriting the recurrence relation of part (a) backwards, and taking the value of \( a_n \) as

\[ a_n = \prod_{m=1}^{n} \frac{(2n-2m+1)}{n!}, \]

show further that the Legendre’s polynomials \( P_n(t) \) can be written as

\[ P_n(t) = \sum_{k=0}^{N} \left[ \frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!} (-1)^k t^{n-2k} \right], \]

where \( N \) is the floor function

\[ N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases} \]

[ proof ]

[ solution overleaf ]
Question 3

It can be shown that the Legendre’s polynomials $P_n(x)$ can be written as

$$P_n(x) = \sum_{k=0}^{N} \left( \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} (-1)^k x^{n-2k} \right),$$

where $N$ is the floor function

$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

Show that the generating function for $P_n(x)$ satisfies

$$\left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left[t^n P_n(x)\right].$$

proof
**Question 4**

The generating function for the Legendre’s polynomials $P_n(x)$, satisfies

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left[ t^n P_n(x) \right].$$

Use this relationship to prove that

$$P_n(-x) = (-1)^n P_n(x).$$
Question 5

The generating function $g(x,t)$ for the Legendre’s polynomials $P_n(x)$, satisfies

$$g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]^2$$

Use this relationship to prove that

$$\frac{\partial}{\partial x} [g(x,t)] + \frac{\partial}{\partial t} [g(x,t)] = x \left[ g(x,t) \right]^3.$$
Question 6

\[ f(x) \equiv 10x^3 - 3x^2 + x - 1. \]

Express \( f(x) \) as a linear combination of Legendre’s polynomials, \( P_n(x) \).

You may assume

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{1}{2}(3x^2 - 1) \)
- \( P_3(x) = \frac{1}{2}(5x^3 - 3x) \)
- \( P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \)

\[ f(x) = 4P_3(x) - 2P_2(x) + 7P_1(x) - 2P_0(x) \]
Question 7

The generating function for the Legendre’s polynomials $P_n(x)$, satisfies

\[
(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} x^n P_n(x).
\]

By differentiating the above relationship with respect to $t$, prove that

\[
(2n+1)xP_n(x) - (n+1)P_{n+1}(x) + nP_{n-1}(x) = 0.
\]
Question 8

The generating function for the Legendre’s polynomials $P_n(x)$, satisfies

$$
(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].
$$

By separately differentiating the above relationship once with respect to $t$ and once with respect to $x$, prove that

$$
n P_n(x) = x P'_n(x) - P'_{n-1}(x).
$$
Question 9

The generating function for the Legendre’s polynomials \( P_n(x) \), satisfies

\[
(1 - 2xt + t^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x).
\]

a) Use this result to show that

\( P_n(1) = 1. \)

b) By using the result of part (a) and Legendre’s equation, deduce that

\( P_n'(1) = \frac{1}{2}n(n+1). \)
Question 10

Use trigonometric identities to show that

\[ \sin^2 \theta = \frac{8}{35} P_4(\cos \theta) - \frac{16}{21} P_2(\cos \theta) + \frac{8}{15} P_0(\cos \theta) \]

You may assume

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{1}{2}(3x^2 - 1) \)
- \( P_3(x) = \frac{1}{2}(5x^3 - 3x) \)
- \( P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \)

proof
Question 11

The generating function \( g(x,t) \) for the Legendre’s polynomials \( P_n(x) \), satisfies

\[
g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left[ t^n P_n(x) \right]^2.
\]

Verify that \( g = g(x,t) \) is a solution of the differential equation

\[	r \frac{\partial^2}{\partial t^2} [t,g] + \frac{\partial}{\partial x} \left[ (1 - x^2) \frac{\partial g}{\partial x} \right] = 0.
\]

\textbf{proof}
Question 12

The generating function for the Legendre’s polynomials $P_n(x)$, satisfies

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

a) Use this result to show that

$$P_n(-1) = (-1)^n.$$

b) By using the result of part (a) and Legendre’s equation, deduce that

$$P_n'(-1) = \frac{1}{2} n(n+1)(-1)^{n+1}.$$
Question 13

The Legendre’s polynomial \( P_n(x) \) is a solution of the differential equation

\[
(1-x^2)\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R}.
\]

Show that

\[
P_n'(x) = \frac{n(n+1)}{1-x^2} \int_1^x P_n(x) \, dx.
\]

**proof**
Question 14

The generating function for the Legendre’s polynomials \( P_n(x) \), satisfies

\[
(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].
\]

a) By differentiating the above relationship with respect to \( t \), prove that

\[
(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x).
\]

b) By separately differentiating the generating function for the Legendre’s polynomials once with respect to \( t \) and once with respect to \( x \), prove that

\[
n P_n(x) = x P'_n(x) - P'_{n-1}(x).
\]

c) Use parts (a) and (b) to show that

\[
(2n+1) P_n(x) = P'_{n+1}(x) + P'_{n-1}(x).
\]

d) Use parts (b) and (c) to deduce that

\[
(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x).
\]
Question 15

The generating function for the Legendre’s polynomials \( P_n(x) \), satisfies

\[
(1 - 2xt + t^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].
\]

Use this result to show that …

… if \( n \) is even, \( P_n(x) \) is an even polynomial in \( x \).

… if \( n \) is odd, \( P_n(x) \) is an odd polynomial in \( x \).
Question 16

The generating function for the Legendre’s Polynomials $P_n(x)$ satisfies

\[
(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].
\]

a) Use this result to show that

\[
P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} n!}.
\]

b) Deduce the value of $P_{2n+1}(0)$.

\[
P_{2n+1}(0) = 0
\]
Question 17
Legendre’s equation is given below
\[
(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R}.
\]

Use the substitution \( x = \cos \theta \) to show that
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \frac{dy}{d\theta} \sin \theta \right) + n(n+1)y = 0.
\]

\[\text{proof}\]
Question 18
Find the two independent solutions of Legendre’s equation
\[
(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0.
\]

\[
y = Ax + B \left[ \ln \left( \frac{1+x}{1-x} \right) - 1 \right]
\]
Question 19

The generating function for the Legendre’s Polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left[t^n P_n(x)\right].$$

Using this result, and integrating both sides with respect to $t$, from 0 to 1, show that

$$\sum_{n=0}^{\infty} \left[\frac{P_n(\cos \theta)}{n+1}\right] = \ln \left[1 + \csc \left(\frac{1}{2} \theta\right)\right].$$

**proof**
Question 20

The generating function \( g \) for the Legendre’s polynomials \( P_n(x) \), satisfies

\[
g(x, t) = \left(1 - 2xt + t^2 \right)^{-1/2} = \sum_{n=0}^{\infty} \left[ t^n P_n(x) \right].
\]

a) By differentiating \( g \) with respect to \( t \), prove that

\[
(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).
\]

b) By differentiating \( g \) once with respect to \( t \) and once with respect to \( x \), prove that

\[
nP_n(x) = xP'_n(x) - P_{n-1}(x).
\]

c) Use parts (a) and (b) to show that

\[
(2n+1)P_n(x) = P'_{n+1}(x) + P'_{n-1}(x).
\]

d) Use parts (b) and (c) to deduce that

\[
(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x).
\]

e) Use parts (b) and (d) to show that

\[
\left(1 - x^2\right)P'_n(x) = n\left[P_{n-1}(x) - xP_n(x)\right].
\]

f) Use parts (a) and (e) to show that

\[
\left(1 - x^2\right)P'_n(x) = (n+1)\left[XP_n(x) - P_{n+1}(x)\right].
\]

[ solution overleaf ]
Question 21
Find one series solution for the Legendre’s equation

\[(1-x^2)\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \ n \in \mathbb{R},\]

about \(x = 1\).

\[y = A \sum_{r=0}^{\infty} \left[ \frac{(n+r)!}{(n-r)!} \times \frac{1}{(r!)^2} \times \left(\frac{x-1}{2}\right)^r \right] \]