INTRINSIC COORDINATES
Question 1 (**)

A curve $C$ has Cartesian equation

$$y = \arctan 2x, \quad x \in \mathbb{R}.$$ 

Find the magnitude of the radius of curvature at the point on $C$ where $x = \frac{1}{2}$.

\[ \sqrt{2} \]

Question 2 (**)

A curve $C$ has Cartesian equation

$$y = \cosh x, \quad x \in \mathbb{R}.$$ 

Find an intrinsic equation of $C$ in the form $s = f(\psi)$, where $s$ is measured from the point with coordinates $(0,1)$, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

$s = \tan \psi$
Question 3  (**)
A curve \( C \) has Cartesian equation

\[ y = \cosh x, \quad x \in \mathbb{R}. \]

Find a simplified expression, in terms of \( x \), for the radius of curvature at a general point on \( C \).

\[ \rho = \cosh^2 x \]

Question 4  (**)
A curve \( C \) has Cartesian equation

\[ y = \ln(\sec x), \quad 0 \leq x < \frac{\pi}{2}. \]

Find an intrinsic equation of \( C \) in the form \( s = f(\psi) \), where \( s \) is measured from the point with coordinates \((0,0)\), and \( \psi \) is the angle the tangent to \( C \) makes with the positive \( x \) axis.

\[ s = \ln|\tan \psi + \sec \psi| \]
Question 5 (**)  

A curve $C$ has intrinsic equation 

$$s = a \cos \psi, \; \psi \in [0, \pi],$$

where $s$ denotes the arc length measured from some fixed point and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

Show that the tangent to $C$ at the point where $s=0$ is parallel to the $y$ axis and determine the radius of curvature at that point.

$$\rho = a$$
Question 6  (**)
A curve \( C \) has intrinsic equation
\[
s = 2 \sin \psi, \quad \psi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],
\]
where \( s \) denotes the arc length measured from some fixed point and \( \psi \) is the angle the tangent to \( C \) makes with the positive \( x \) axis.

Show clearly that

\[
\rho = \sqrt{4 - s^2},
\]

where \( \rho \) is the radius of curvature at a general point on \( C \).

**proof**

Question 7  (**)
A curve \( C \) has Cartesian equation
\[
y = \text{arsinh} \, x, \quad x \in \mathbb{R}.
\]

Find the magnitude of the radius of curvature at the point on \( C \) where \( x = \sqrt{2} \).

\[
\frac{4\sqrt{2}}{}
\]
Question 8 (**)
A curve $C$ has Cartesian equation

$$y = \arcsin x, \; -1 \leq x \leq 1.$$ 

Find an expression, in terms of $x$, for the radius of curvature on $C$, giving the answer as a single simplified fraction.

$$\rho = \frac{(2-x^2)^{3/2}}{x}$$

Question 9 (**)
A curve $C$ has Cartesian equation

$$y = \arctan x^2, \; 0 \leq y < \frac{\pi}{2}.$$ 

Calculate the radius of curvature at the point on $C$ where $x = -1$.

$$-2\sqrt{2}$$
Question 10 (**+)

A curve $C$ has parametric equation $s$

$$x = \cosh t - t, \quad y = \cosh t + t, \quad t \in \mathbb{R}.$$ 

Find the exact value of the radius of curvature at the point on $C$ where $t = \ln 2$.

$$\rho = \frac{25}{16} \sqrt{2}$$

Question 11 (**+)

A curve $C$ has Cartesian equation

$$y = a \cosh \left(\frac{x}{a}\right), \quad \text{where} \quad a \quad \text{is a constant.}$$

Show that the radius of curvature on $C$ is given by $\frac{1}{a}y^2$
Question 12  (**+)**

A curve $C$ has Cartesian equation

\[ y = \frac{1}{2}(x-1)^3, \; x \in \mathbb{R}, \; x \geq 1. \]

Find an intrinsic equation of $C$ in the form $s = f(\psi)$, where $s$ is measured from the point with coordinates $(1,0)$, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

\[ s = \frac{2}{3}(\sec^3 \psi - 1) \]

Question 13  (**+)**

The radius of curvature at a general point on a curve $C$ is given by

\[ e^{\sin \psi} \cos \psi, \]

where $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

It is further given that when $\psi = 0$, $s = 1$, where $s$ is the arc length measured from some fixed point.

Find an intrinsic equation for $C$, in the form $s = f(\psi)$.

\[ s = e^{\sin \psi} \]
Question 14  (***)
A curve $C$ has Cartesian equation

$$y = \sinh^2 x, \ x \in \mathbb{R}.$$ 

Express the curvature at a general point on $C$ in terms of $\cosh 4x$.

$$\kappa = \frac{4}{1 + \cosh 4x}$$
Question 15  (***)

A curve $C$ has intrinsic equation

$$s = 2\psi,$$

where $s$ is measured from some arbitrary point, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

a) Describe $C$ geometrically, with reference to $\frac{ds}{d\psi}$.

b) Use a calculus method to obtain a Cartesian equation for $C$, in terms of suitable constants.

$$(x-a)^2 + (y-b)^2 = 4$$
Question 16 (***)

A curve $C$ has parametric equation $s$

\[ x = t - \sin 2t, \quad y = \cos 2t, \quad 0 \leq t < \frac{\pi}{2}. \]

The point $P$ lies on $C$ where $\cos t = \frac{1}{4}\sqrt{10}$.

Calculate the radius of curvature at $P$.

\[ \rho_p = \frac{8}{7} \]
Question 17 (***)

A curve $C$ has intrinsic equation

$$s = 12 \sin^2 \psi,$$

where $s$ is measured from a Cartesian origin, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

Show that a Cartesian equation of $C$ is

$$y^2 + (8 - x)^2 = 4.$$
Question 18 (****+)

A curve $C$ has Cartesian equation

$$y = \cosh^2 x, \quad x \in \mathbb{R}.$$ 

Show that the radius of curvature at the point on $C$ where $y = 4$ is $24\frac{1}{2}$.

proof
Question 19 (***)

The figure above shows the curve $C$ with Cartesian equation

$$y = x^2 - \frac{1}{3}x^3, \quad x \in \mathbb{R}, \quad x \geq 0.$$  

Show that the centre of curvature at the point $P\left(4, -\frac{2}{3}\right)$ on $C$, is $\left(-\frac{7}{2}, -\frac{32}{3}\right)$.

**proof**
Question 20 (***+)

A curve $C$ has Cartesian equation

$$y = \frac{1}{2} \left( 2x^2 - \ln x \right), \quad x \in \mathbb{R}, \quad x > 0.$$ 

The point $P$ lies on $C$ where $x = 1$.

a) Determine the radius of curvature at $P$.

b) Find the exact coordinates of the centre of curvature at $P$.

$$\rho = \frac{25}{16}, \quad \left( \frac{1}{16}, \frac{7}{4} \right)$$
Question 21 (***+)

A curve \( C \) has Cartesian equation

\[
y = \frac{2}{3}(x-1)^\frac{3}{2}, \quad x \in \mathbb{R}, \quad x \geq 1.
\]

The point \( P \) lies on \( C \) where \( x = 10 \).

a) Determine the radius of curvature at \( P \).

b) Find the exact coordinates of the centre of curvature at \( P \).

\[
\rho = 60\sqrt{10}, \quad (-170, 78)
\]
Question 22  (***)

A curve $C$ has Cartesian equation

$$y = 2\sin x, \ x \in \mathbb{R}.$$  

The point $P$ lies on $C$ where $x = \frac{\pi}{6}$.

a) Determine the radius of curvature at $P$.

b) Find the exact coordinates of the centre of curvature at $P$.

$$\rho = \frac{\pi}{6} + 4\sqrt{3}, -3$$
Question 23 (***+)**

A cycloid $C$ has parametric equations

$$x = 2t + 2\sin t, \; y = 2 - 2\cos t, \; 0 \leq t < \frac{\pi}{2}.$$ 

a) Show clearly that

$$\frac{dy}{dx} = \tan\left(\frac{1}{2}t\right).$$

b) Find an intrinsic equation for $C$, in the form $s = f(\psi)$, where $s$ is measured from a Cartesian origin, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

c) Calculate the curvature at the point on $C$ where $s = 4$.

$$s = 8\sin \psi, \quad \kappa = \frac{1}{12}\sqrt{3}$$
Question 24 (***)

A curve \( C \) has parametric equations

\[
\begin{align*}
2x &= t^2, \\
3y &= 4t^3, \\
t &\in \mathbb{R}
\end{align*}
\]

The point \( P \) lies on \( C \) where \( t = 2 \).

a) Determine the radius of curvature at \( P \).

b) Find the exact coordinates of the centre of curvature at \( P \).

\[
\rho = \frac{125}{6}, \quad \left( -\frac{17}{2}, \frac{56}{3} \right)
\]
Question 25  (***)

A curve \( C \) has intrinsic equation

\[ s = \frac{1}{2} \psi^2, \]

where \( s \) is measured from the point with Cartesian coordinates \((1,0)\), and \( \psi \) is the angle the tangent to \( C \) makes with the positive \( x \) axis.

Use a calculus method to obtain two parametric equations for \( C \), in terms of a suitable parameter.

\[
\begin{align*}
x &= t \sin t + \cos t \\
y &= -t \cos t + \sin t
\end{align*}
\]
Question 26 (***)+

The radius of curvature at a general point on a curve $C$ is given by

$$2 \sin \psi,$$

where $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

It is further given that the arc length $s$ is measured from the point with Cartesian coordinates $(0, 1)$, where the value of $\psi$ at that point is $\frac{\pi}{3}$.

a) Find an intrinsic equation for $C$, in the form $s = f(\psi)$.

b) Show clearly that

$$x = \frac{1}{4} s (2 - s).$$

$$s = 1 - 2 \cos \psi.$$
Question 27 (***)

The radius of curvature at a general point on a curve $C$ is given by

$$2s + 1,$$

where $s$ is the arc length measured from the Cartesian origin.

It is further given when $s = 0$, $\psi = 0$, where $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

a) Find an intrinsic equation for $C$, in the form $s = f(\psi)$.

b) Determine a set of parametric equations for $C$.

You may assume without proof

$$\int e^{2u} \cos u \ du = \frac{1}{3} e^{2u} (2 \cos u + \sin u) + \text{constant}$$

$$\int e^{2u} \sin u \ du = \frac{1}{3} e^{2u} (2 \sin u - \cos u) + \text{constant}.$$
Question 28  (****)

A curve $C$ has parametric equations

$$x = t^3, \quad y = 4t^2 - t^4, \quad t \in \mathbb{R}.$$  

Find the exact coordinates of the centre of curvature at the point $P$ on $C$ where $t = 1$.

$$\left( \frac{34}{9}, \frac{11}{12} \right)$$
Question 29  

A cycloid \( C \) has parametric equations

\[
x = \theta + \sin \theta, \quad y = 1 - \cos \theta, \quad 0 \leq \theta \leq 2\pi.
\]

Find the exact coordinates of the centre of curvature at the point on \( C \) where \( \theta = \frac{2\pi}{3} \).

\[
\left( \frac{2\pi}{3}, \frac{\sqrt{3}}{2} \right)
\]
Question 30 (***)
The gradient at every point on a curve \( C \) is given by
\[
\frac{dy}{dx} = \frac{1}{2} s,
\]
where \( s \) is the arc length along \( C \) measured from the point \( P \) whose Cartesian coordinates are \((0,2)\). It is further given that \( \psi = 0 \) at \( P \), where \( \psi \) is the angle the tangent to \( C \) makes with the positive \( x \) axis.

a) Show clearly that
\[
\frac{ds}{dx} = \frac{1}{2} \sqrt{s^2 + 4}.
\]

b) Express \( s \) as a function of \( x \).

c) Deduce that
\[
y = 2 \cosh \left( \frac{1}{2} x \right).
\]
Question 31  (****)

An ellipse has equation

$$3x^2 + y^2 = 18.$$ 

The point $P\left(\sqrt{3}, 3\right)$ lies on $C$.

a) Determine the radius of curvature at $P$.

b) Find the exact coordinates of the centre of curvature at $P$.

$$\rho = -4, \left(-\sqrt{3}, 1\right)$$
Question 32 (****)

A curve $C$ has intrinsic equation

$$s = \ln \left( \tan \frac{\psi}{2} \right), \quad 0 < \psi < \pi,$$

where $s$ is measured from a fixed point, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$-axis.

It is given that the tangent to $C$ at a Cartesian origin has infinite gradient.

Show that a Cartesian equation of $C$ is

$$e^x = \cos y.$$

**proof**
A curve \( C \) has Cartesian equation
\[
y = \ln \left( x + 1 + \sqrt{x^2 + 2x} \right), \quad x \in \mathbb{R}.
\]

Determine the radius of curvature at the point on \( C \) where \( x = 2 \).

\[
\rho = 9
\]
Question 34  (***)

A curve $C$ has Cartesian equation

$$\sin y = e^x, \quad x \leq 0.$$ 

Find an intrinsic equation for $C$, in the form $s = f(\psi)$, where $s$ is measured from the point with Cartesian coordinates $(0, \pi/2)$, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

$$s = \ln \tan \left( \frac{\psi}{2} \right) \quad \text{or} \quad e^s = \tan \left( \frac{\psi}{2} \right)$$
Question 35  (***)

A curve \( C \) has intrinsic equation

\[
s = 8 \left( \sec^3 \psi - 1 \right), \quad 0 \leq \psi < \frac{\pi}{2},
\]

where \( s \) is the arc length measured from a Cartesian origin \( O \), and \( \psi \) is the angle the tangent to \( C \) makes with the positive \( x \) axis. It is further given that \( \psi = 0 \) at the origin \( O \).

Show that a Cartesian equation of \( C \) is

\[
y^2 = \frac{x^3}{27}.
\]

\[
\boxed{\text{proof}}
\]
A curve $C$ has intrinsic equation

$$s = 4 \sin \psi, \quad 0 \leq \psi \leq \pi,$$

where $s$ is the arc length is measured from the Cartesian origin $O$, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

It is given that the tangent to $C$ at $O$ has zero gradient.

Show that the parametric equations of $C$ are

$$x = t + \sin t, \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi.$$
Question 37  (***)

A curve has Cartesian equation

\[ y = \ln(\sin x), \quad 0 < x < \pi. \]

Show that an intrinsic equation of the curve is

\[ s = \ln \left| \frac{2}{\tan \psi + \sec \psi} \right|, \]

where \( s \) is the arc length measured from the point where \( \psi = \arctan \frac{3}{4} \), where \( \psi \) is the angle the tangent to the curve makes with the positive \( x \) axis.
Question 38  (****)

A curve $C$ has parametric equations

$$x = 6 \tan^2 t, \quad y = 4 \tan^3 t, \quad 0 \leq t < \frac{\pi}{2}.$$ 

It is further given that when $t = 0$, $s = 4$, where $s$ is the arc length measured from some fixed point.

Show clearly that

$$s = 4 \sec^3 \psi,$$

where $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

**proof**
Question 39 (****+)

An astroid is given parametrically by

\[ x = 4\cos^3\theta, \quad y = 4\sin^3\theta, \quad 0 \leq \theta < 2\pi. \]

a) Show that if the arc length \( s \) is measured from the point \((4,0)\), and \( \psi \) is the angle the tangent to the astroid makes with the positive \( x \) axis, then

\[ s = 6\sin^2\psi. \]

b) Determine the coordinates of the centre of curvature at the point \( P \) on the astroid where \( \theta = \frac{\pi}{6} \)

\[ C(3\sqrt{3},5) \]
Question 40  (***)

A curve $C$ has parametric equations

$$x = 2\sinh t, \quad y = \cosh^2 t, \quad t \in \mathbb{R}.$$ 

It is further given that the arc length $s$ is measured from the point where $t = 0$.

Show clearly that

$$s = \ln(\tan \psi + \sec \psi) + \tan \psi \sec \psi,$$

where $\psi$ is the angle the tangent to $C$ makes with the positive $x$-axis.

\[\text{proof}\]
Question 41 (****+)

The position vector of a curve $C$ is given by

$$\mathbf{r}(t)=\left(\frac{2}{1+t^2}-1\right)i+\left(\frac{2t}{1+t^2}\right)j,$$

where $t$ is a scalar parameter with $t \in \mathbb{R}$.

Find an expression for the position vector of $C$, giving the answer in the form

$$\mathbf{r}(s)=f(s)i+g(s)j,$$

where $s$ is the arc length of a general point on $C$, measure from the point $(1,0)$.

$$\mathbf{r}(s)=(\cos s)i+(\sin s)j$$
Question 42 (***)

A curve $C$ has intrinsic equation

$$s = \ln (\tan \psi + \sec \psi) + \tan \psi \sec \psi, \ 0 \leq \psi < \frac{\pi}{2},$$

where $s$ is the arc length is measured from the point with Cartesian coordinates $(0,1)$, and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

It is further given that the gradient at $(0,1)$ is zero.

Show that the Cartesian equation of $C$ is

$$y = \frac{1}{4}x^2 + 1.$$
Question 43  (*****)

The gradient at every point on a curve \( C \) is given by

\[
\frac{dy}{dx} = \frac{1}{2} s,
\]

where \( s \) is the arc length along \( C \) measured from the point \( P \) whose Cartesian coordinates are \((0,2)\).

It is further given that \( \psi = 0 \) at \( P \), where \( \psi \) is the angle the tangent to \( C \) makes with the positive \( x \) axis.

a) Show clearly that

\[
x = 2 \ln \left| \sec \psi + \tan \psi \right|, \quad y = 2 \sec \psi.
\]

b) Eliminate \( \psi \) to show further that

\[
y = 2 \cosh \left( \frac{1}{2} x \right).
\]
Question 44 (*****)

The radius of curvature $\rho$ at any point on a curve with Cartesian equation $y = f(x)$ is given by

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$ 

a) Given that the curve can be parameterized as $x = g(t), y = h(t)$, for some parameter $t$, show that

$$\rho = \left(\frac{x^2 + y^2}{\dot{x}\dot{y} - \ddot{y}\ddot{x}}\right)^{\frac{3}{2}},$$

where a dot above a variable denoted differentiation with respect to $t$.

A curve $C$ is given parametrically by

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad 0 \leq t < 2\pi.$$ 

b) Find an expression for $\rho$ on $C$, giving the answer in terms of $t$. 

$$\rho = t.$$
Question 45  (*****)

A curve $C$ has Cartesian equation $y = f(x)$.

The same curve has intrinsic equation $s = g(\psi)$, where $s$ is measured from an arbitrary point and $\psi$ is the angle the tangent to $C$ makes with the positive $x$ axis.

The radius of curvature $\rho$ at any point on $C$ is defined as $\frac{ds}{d\psi}$.

\[ a) \text{ Show clearly that } \rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \]

\[ b) \text{ Given that } C \text{ can be suitably parameterized as } x = h_1(t), y = h_2(t), \text{ for some parameter } t, \text{ show further that } \]

\[ \rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{\ddot{x}\dot{y} - \ddot{y}\dot{x}}, \]

where a dot above a variable denotes differentiation with respect to $t$.