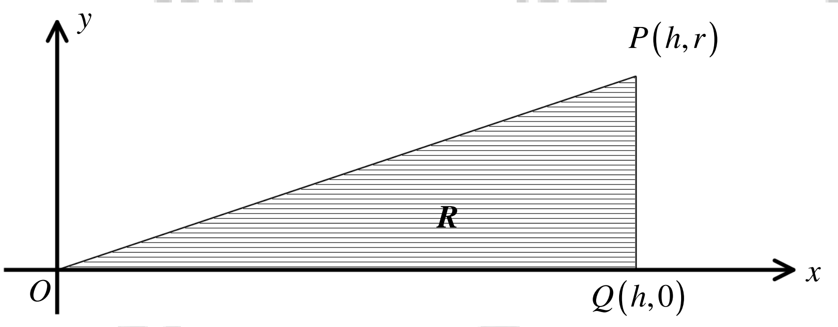
FURTHER INTEGRATION

Question 1



The figure above shows the straight line segment OP, joining the origin to the point $P(h,r)$, where h and r are positive coordinates.

The point $Q(h,0)$ lies on the x axis.

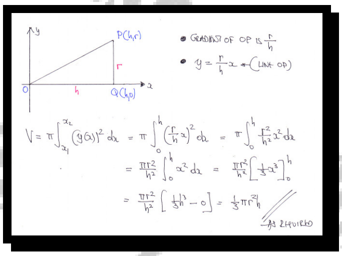
The shaded region R is bounded by the line segments OP, PQ and OQ.

The region R is rotated by 2π radians about the x axis to form a solid cone of height h and radius r.

Show by integration that the volume of the cone V is given by

$$V = \frac{1}{3}\pi r^2 h.$$

$\boxed{\text{proof}}$



Question 2

A curve C is defined parametrically

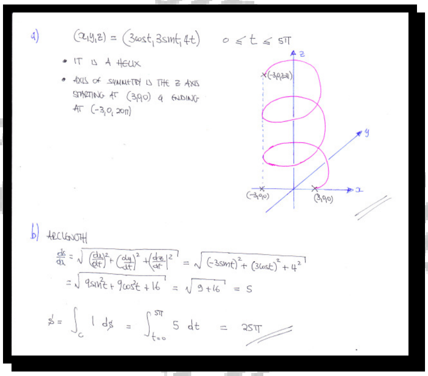
$$(x, y, z) = (3\cos t, 3\sin t, 4t), \quad 0 \le t \le 5\pi .$$

where t is a parameter.

a) Sketch the graph of C.

b) Find the length of C.

$\boxed{25\pi}$



Question 3

A finite region R is defined by the inequalities

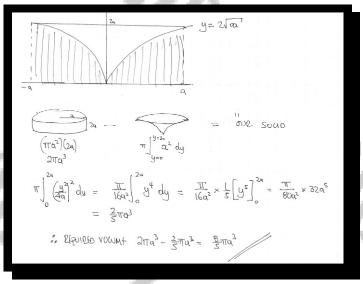
$$y^2 \le 4ax,\ 0 \le x \le a,\ y \ge 0,$$

where a is a positive constant.

The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine, in terms of π and a, the exact volume of this solid.

$\boxed{\frac{8}{5}\pi a^3}$



Question 4

A curve C is defined parametrically

$$(x, y, z) = \left(\mathrm{e}^{t}, \mathrm{e}^{t}\cos t, \mathrm{e}^{t}\sin t\right), \quad 0 \le t \le 2\pi\,.$$

where t is a parameter.

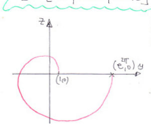
Describe the graph of C and find its length.

$$\boxed{\text{arclength} = \sqrt{3}\left[\mathrm{e}^{2\pi} - 1\right]}$$

$\boxed{\underline{r} = \left[e^{t}, e^{t}\cos t, e^{t}\sin t\right]}$

A "HELICAL SPIRAL"

PROFILE AS VIEWED FROM "ABOVE", I.E THE POSITIVE x AXIS



$\dot{\underline{r}} = \left[e^{t}, e^{t}\cos t - e^{t}\sin t, e^{t}\sin t + e^{t}\cos t\right]$

$\dot{\underline{r}} = e^{t}\left[1, \cos t - \sin t, \sin t + \cos t\right]$

$|\dot{\underline{r}}| = e^{t}\sqrt{1 + (\cos t - \sin t)^2 + (\sin t + \cos t)^2}$

$|\dot{\underline{r}}| = e^{t}\sqrt{1 + \cos^2 t - \cancel{2\cos t\sin t} + \sin^2 t + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t}$

$|\dot{\underline{r}}| = e^{t}\sqrt{3}$

THUS $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$s = \int_{0}^{2\pi} \sqrt{3}e^{t}\, dt$

$s = \left[\sqrt{3}e^{t}\right]_{0}^{2\pi}$

$s = \sqrt{3}\left[e^{2\pi} - 1\right]$

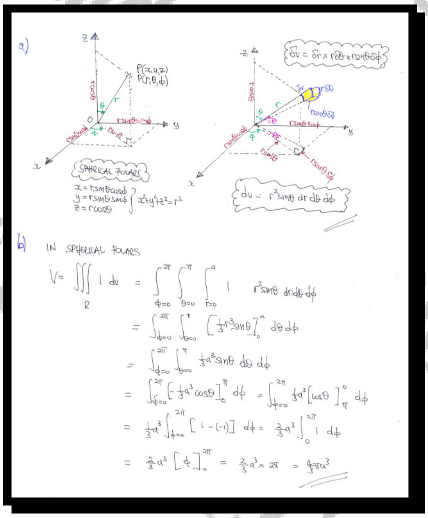
Question 5

a) Determine with the aid of a diagram an expression for the volume element in spherical polar coordinates, (r,θ,φ).

[*You may not use Jacobians in this part*]

b) Use spherical polar coordinates to obtain the standard formula for the volume of a sphere of radius a.

$$\boxed{dv = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$



Question 6

A family of curves C_n , $n = 1, 2, 3, 4, \ldots$ is defined parametrically by

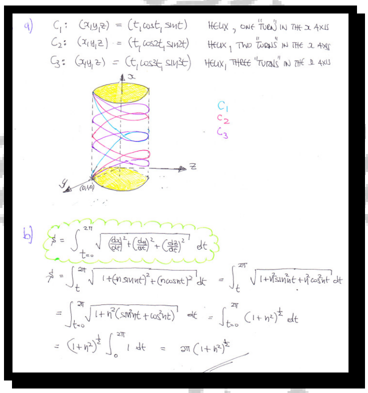
$$C_n : (x, y, z) = (t, \cos nt, \sin nt), \quad 0 \le t \le 2\pi .$$

where t is a parameter.

a) Sketch the graph of C_1, C_2 and C_3.

b) Find an expression for the length of C_n.

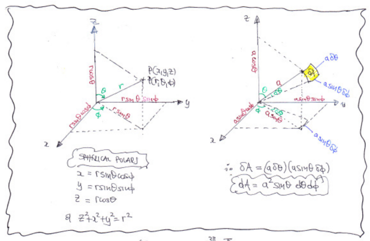
$$\boxed{2\pi\sqrt{1+n^2}}$$



Question 7

Use spherical polar coordinates, (r,θ,φ), to obtain the standard formula for the surface area of a sphere of radius a.

$$\boxed{dA = a^2 \sin\theta \, d\theta \, d\varphi \Rightarrow A = 4\pi a^2}$$



THUS SURFACE AREA $= \iint_S 1 \, dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 1 \quad a^2 \sin\theta \, d\theta \, d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} a^2 \sin\theta \quad d\theta \, d\phi$

$= \int_{\phi=0}^{2\pi} \left[-a^2\cos\theta\right]_0^{\pi} \, d\phi$

$= \int_{\phi=0}^{2\pi} \left[a^2\cos\theta\right]_{\pi}^{0} \, d\phi$

$= a^2 \int_{\phi=0}^{2\pi} \left[1-(-1)\right] \, d\phi$

$= 2a^2 \int_0^{2\pi} 1 \, d\phi$

$= 2a^2 \left[\phi\right]_0^{2\pi}$

$= 2a^2 \left[2\pi - 0\right]$

$= 4\pi a^2$

Question 8

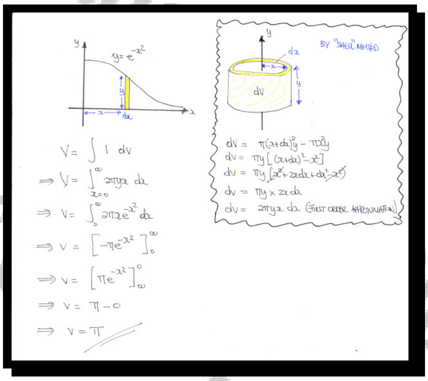
The infinite region R is defined by the inequalities.

$$y \le e^{-x^2}, \ x \ge 0, \ y \ge 0.$$

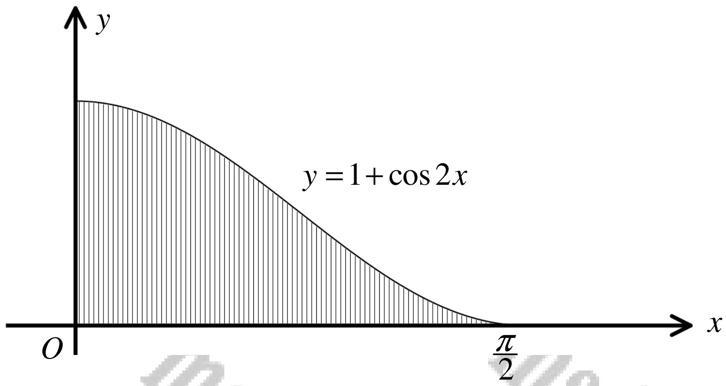
R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine the exact volume of this solid.

$\boxed{V = \pi}$



Question 9



The figure above shows the graph of the curve with equation

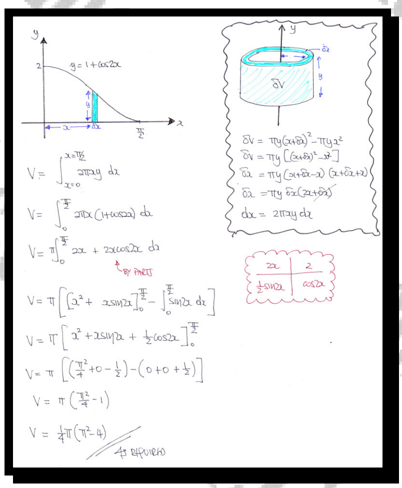
$$y = 1 + \cos 2x \, , \; 0 \le x \le \frac{\pi}{2} .$$

The shaded region bounded by the curve and the coordinate axes is rotated by 2π radians about the y axis to form a solid of revolution.

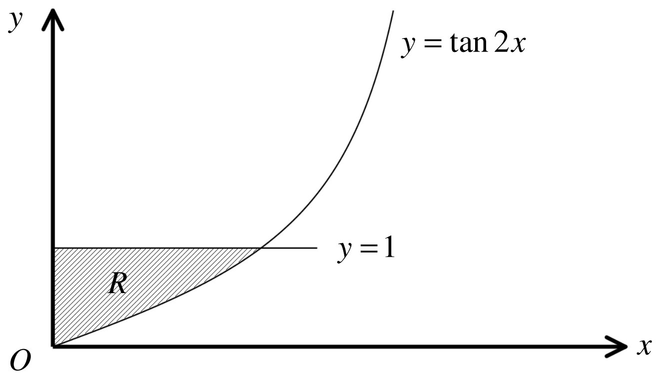
Show that the volume of the solid is

$$\tfrac{1}{4}\pi\left(\pi^2 - 4\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 10



The figure above shows the graph of the curve with equation

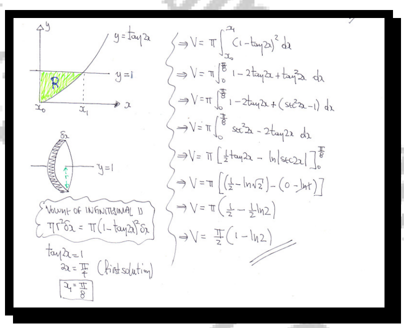
$$y = \tan 2x \,, \; 0 \le x \le \frac{\pi}{4} .$$

The finite region R is bounded by the curve, the y axis and the horizontal line with equation $y = 1$.

The region R is rotated by 2π radians about the line with equation $y = 1$ forming a solid of revolution.

Determine an exact volume for this solid.

$$\boxed{\frac{\pi}{2}(1 - \ln 2)}$$



Question 11

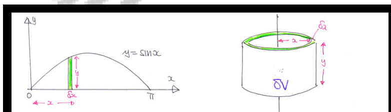
The finite region bounded the curve with equation

$$y = \sin x\,,\ 0 \le x \le \pi$$

and the x axis, is rotated by $360°$ about the y axis to form a solid of revolution.

Find, in exact form, the volume of the solid.

$$\boxed{V = 2\pi^2}$$



CONSIDER AN INFINITESIMAL STRIP OF THICKNESS δx AS SHOWN ABOVE ROTATED ABOUT THE y-AXIS

$\delta V = \pi(x+\delta x)^2 y - \pi x^2 y = \pi y\left[(x+\delta x)^2 - x^2\right] = \pi y\left[\cancel{x^2} + 2x\delta x + \delta x^2 - \cancel{x^2}\right]$

$\delta V \approx 2\pi x y\,\delta x$ (IGNORING δx^2)

SUMMING UP ALL THESE SHELLS FROM $x=0$ TO $x=\pi$

$V = \sum \delta V = \sum(2\pi x y\,\delta x)$

TAKING LIMITS

$V = \int_0^\pi 2\pi x(\sin x)\,dx = 2\pi\int_0^\pi x\sin x\,dx$

BY PARTS

x	1
$-\cos x$	$\sin x$

$V = 2\pi\left\{\left[-x\cos x\right]_0^\pi - \int_0^\pi -\cos x\,dx\right\}$

$V = 2\pi\left\{-\pi(-1) - 0 + \cancel{\int_0^\pi \cos x\,dx}\right\}$

$V = 2\pi\left[\pi\right]$

$V = 2\pi^2$

Question 12

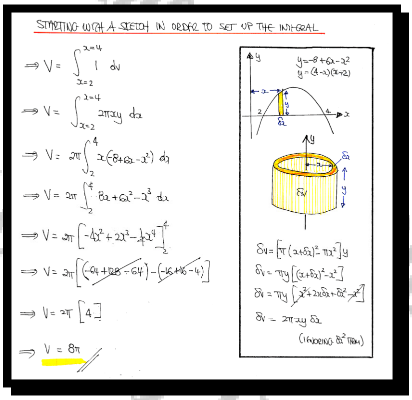
A quadratic curve C has equation

$$y = (4-x)(x-2), \quad x \in \mathbb{R}.$$

The finite region bounded by C and the x axis is fully revolved about the y axis, forming a solid of revolution S.

Determine in exact form the volume of S.

$\boxed{}$, $\boxed{V = 8\pi}$



STARTING WITH A SKETCH IN ORDER TO SET UP THE INTEGRAL

$$\Rightarrow V = \int_{x=2}^{x=4} 1 \, dV$$

$$\Rightarrow V = \int_{x=2}^{x=4} 2\pi x y \, dx$$

$$\Rightarrow V = 2\pi \int_2^4 x(-8+6x-x^2) \, dx$$

$$\Rightarrow V = 2\pi \int_2^4 -8x+6x^2-x^3 \, dx$$

$$\Rightarrow V = 2\pi \left[-4x^2+2x^3-\tfrac{1}{4}x^4\right]_2^4$$

$$\Rightarrow V = 2\pi\left[(\cancel{-64+128-64}) - (\cancel{-16+16}-4)\right]$$

$$\Rightarrow V = 2\pi[4]$$

$$\Rightarrow V = 8\pi$$

$$\delta V = \left[\pi(x+\delta x)^2 - \pi x^2\right] y$$

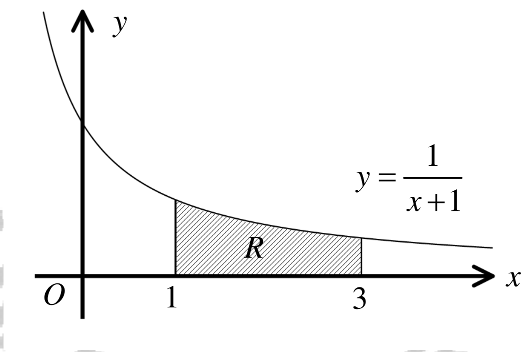
$$\delta V = \pi y\left[(x+\delta x)^2 - x^2\right]$$

$$\delta V = \pi y\left[\cancel{x^2} + 2x\delta x + \delta x^2 - \cancel{x^2}\right]$$

$$\delta V = 2\pi x y \, \delta x$$

(IGNORING δx^2 TERM)

Question 13



The figure above shows the graph of the curve with equation

$$y = \frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x = -1.$$

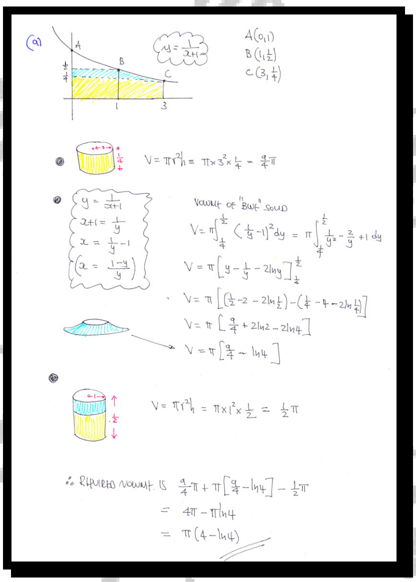
The finite region R is bounded by the curve, the x axis and the lines with equations $x = 1$ and $x = 3$.

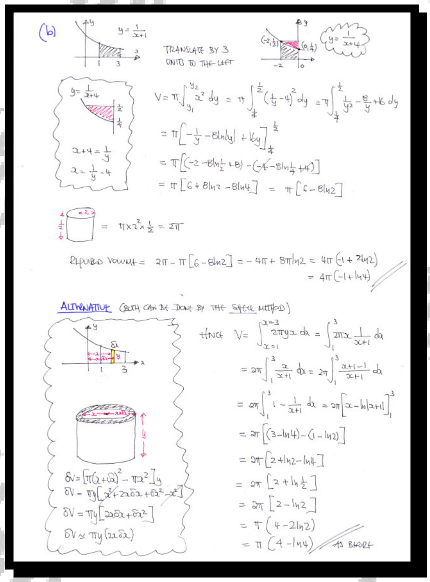
Determine the exact volume of the solid formed when the region R is revolved by 2π radians about …

a) … the y axis.

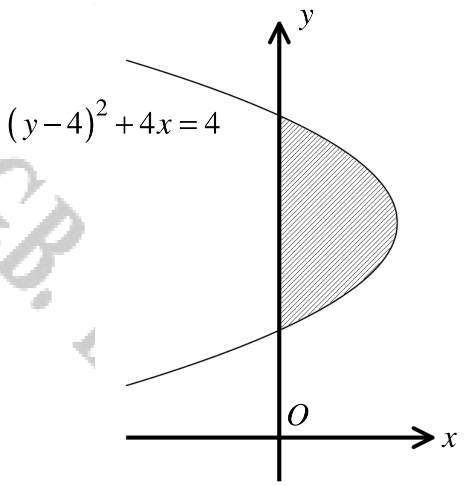
b) … the straight line with equation $x = 3$.

$\boxed{\pi(4 - \ln 4)}$, $\boxed{4\pi(-1 + \ln 4)}$





Question 14



The figure above shows the curve with equation

$$(y-4)^2+4x=4.$$

The finite region bounded the curve and the y axis, shown shaded in the figure, is rotated by a full turn about the x axis to form a solid of revolution.

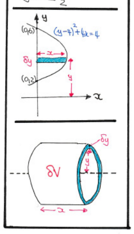
Find, in exact form, the volume of the solid.

$\boxed{}$, $\boxed{V=\frac{64\pi}{3}}$

FIND THE VOLUME BY THE "SHELL METHOD"

- CONSIDER AN INFINITESIMAL STRIP OF THICKNESS δy, ROTATED ABOUT THE x AXIS, FORMING A THIN INFINITESIMAL TUBE, AS IN THE DIAGRAMS OPPOSITE

$x=0$
$(y-4)^2=4$
$y-4=<^{2}_{-2}$
$y=<^{6}_{2}$



- THE VOLUME OF THE INFINITESIMAL TUBE IS

$\Longrightarrow \delta V = \pi x(y+\delta y)^2 - \pi x y^2$

$\Longrightarrow \delta V = \pi x\left[(y+\delta)^2 - y^2\right]$

$\Longrightarrow \delta V = \pi x\left[\cancel{y^2}+2y\delta y+\delta y^2-\cancel{y^2}\right]$

$\Longrightarrow \delta V = \pi x\left[2y\delta y+\delta y^2\right]$

$\Longrightarrow \delta V = 2\pi xy\,\delta y + \cancel{\pi x\delta y^2}$

$\Longrightarrow \delta V \approx 2\pi xy\,\delta y$ IGNORED COMPARED TO δy

- SUMMING UP ALL THESE TUBES FROM $y=2$ TO $y=6$

$V = \sum \delta V = \sum(2\pi xy\,\delta y)$

TAKING LIMITS AND CARRY OUT THE RESULTING INTEGRATIONS

$V = \int_{y=2}^{6} 2\pi xy\,dy = \int_{2}^{6} 2\pi y\left[1-\frac{1}{4}(y-4)^2\right]dy$

$V = \int_{2}^{6} 2\pi y\left[1-\frac{1}{4}(y^2-8y+16)\right]dy$

$\Longrightarrow V = \pi\int_{2}^{6} 2y\left[1-\frac{1}{4}y^2+2y-4\right]dy$

$\Longrightarrow V = \pi\int_{2}^{6} 2y\left[-\frac{1}{4}y^2+2y-3\right]dy$

$\Longrightarrow V = \pi\int_{2}^{6} -\frac{1}{2}y^3+4y^2-6y\,dy$

$\Longrightarrow V = \pi\left[-\frac{1}{8}y^4+\frac{4}{3}y^3-3y^2\right]_{2}^{6}$

$\Longrightarrow V = \pi\left[(-162+288-108)-(-2+\frac{32}{3}-12)\right]$

$\Longrightarrow V = \pi\left[18-(-\frac{10}{3})\right]$

$\Longrightarrow V = \frac{64\pi}{3}$

Question 15

A tube in the shape of a right circular cylinder of radius 4 m and height 0.5 m, emits heat from its curved surface only.

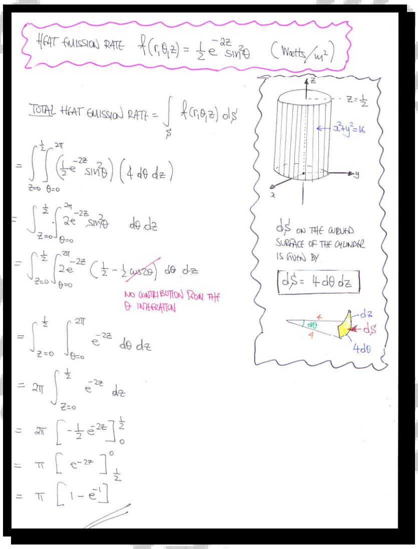
The heat emission rate, in Wm^{-2}, is given by

$$\frac{1}{2}\mathrm{e}^{-2z}\sin^2\theta\,,$$

where θ and z are standard cylindrical polar coordinates, whose origin is at the centre of one of the flat faces of the cylinder.

Given that the cylinder is contained in the part of space for which $z \geq 0$, determine the total heat emission rate from the tube.

$$\boxed{\pi\left(1-\mathrm{e}^{-1}\right)}$$



Question 16

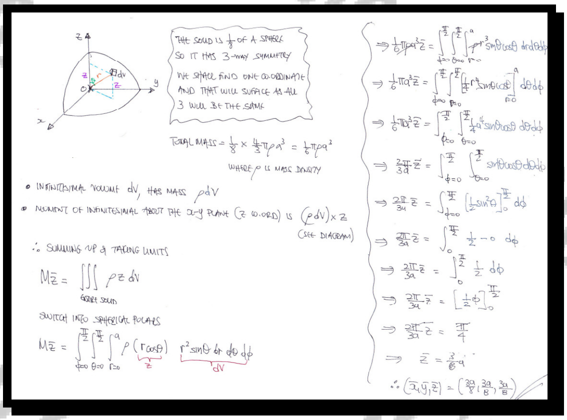
A uniform solid has equation

$$x^2 + y^2 + z^2 = a^2,$$

with $x > 0$, $y > 0$, $z > 0$, $a > 0$.

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the solid.

$$\boxed{\left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a\right)}$$



THE SOLID IS $\frac{1}{8}$ OF A SPHERE SO IT HAS 3-WAY SYMMETRY

WE SHALL FIND ONE CO-ORDINATE AND THAT WILL SUFFICE AS ALL 3 WILL BE THE SAME

TOTAL MASS $= \frac{1}{8} \times \frac{4}{3}\pi\rho a^3 = \frac{1}{6}\pi\rho a^3$

WHERE ρ IS MASS DENSITY

- INFINITESIMAL VOLUME dV, HAS MASS $\rho\, dV$
- MOMENT OF INFINITESIMAL ABOUT THE x-y PLANE (z CO-ORD) IS $(\rho\, dV) \times z$ (SEE DIAGRAM)

$\therefore$ SUMMING UP & TAKING LIMITS

$$M\bar{z} = \iiint_{\text{ENTIRE SOLID}} \rho z \, dV$$

SWITCH INTO SPHERICAL POLARS

$$M\bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} \rho \underbrace{(r\cos\theta)}_{z} \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\phi}_{dV}$$

$$\Rightarrow \frac{1}{6}\pi\cancel{\rho} a^3 \bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} \cancel{\rho} r^3 \sin\theta\cos\theta \, dr \, d\theta \, d\phi$$

$$\Rightarrow \frac{1}{6}\pi a^3 \bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{4}r^4 \sin\theta\cos\theta\right]_{r=0}^{a} d\theta \, d\phi$$

$$\Rightarrow \frac{1}{6}\pi a^3 \bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{4}a^4 \sin\theta\cos\theta \, d\theta \, d\phi$$

$$\Rightarrow \frac{2\pi}{3a}\bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta \, d\phi$$

$$\Rightarrow \frac{2\pi}{3a}\bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} d\phi$$

$$\Rightarrow \frac{2\pi}{3a}\bar{z} = \int_0^{\frac{\pi}{2}} \frac{1}{2} - 0 \, d\phi$$

$$\Rightarrow \frac{2\pi}{3a}\bar{z} = \int_0^{\frac{\pi}{2}} \frac{1}{2} \, d\phi$$

$$\Rightarrow \frac{2\pi}{3a}\bar{z} = \left[\frac{1}{2}\phi\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{2\pi}{3a}\bar{z} = \frac{\pi}{4}$$

$$\Rightarrow \bar{z} = \frac{3}{8}a$$

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8}\right)$$

Question 17

A hemispherical surface, of radius a m, is electrically charged.

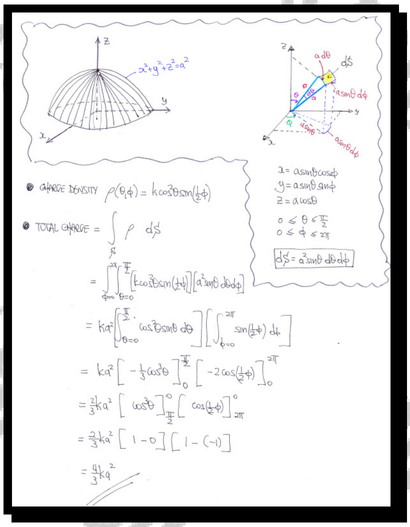
The electric charge density $\rho(\theta,\varphi)$, in $\text{C}\,\text{m}^{-2}$, is given by

$$\rho(\theta,\varphi) = k\cos^2(\theta)\sin\left(\tfrac{1}{2}\varphi\right),$$

where k is a positive constant, and θ and φ are standard spherical polar coordinates, whose origin is at the centre of the flat open face of the hemisphere.

Given that the hemisphere is contained in the part of space for which $z \geq 0$, determine the total charge on its surface.

$$\boxed{\frac{3}{4}ka^2}$$



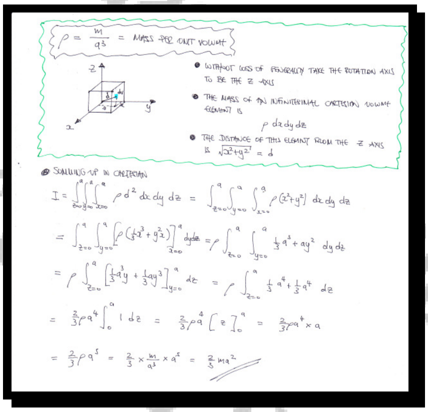
Question 18

A uniform solid cube, of mass m and side length a, is free to rotate about one of its edges, L.

Use multiple integration in Cartesian coordinates, to find the moment of inertia of this cube about L, giving the answer in terms of m and a.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

$$\boxed{\frac{2}{3}ma^2}$$



$\rho = \dfrac{m}{a^3}$ = MASS PER UNIT VOLUME

- WITHOUT LOSS OF GENERALITY TAKE THE ROTATION AXIS TO BE THE z AXIS
- THE MASS OF AN INFINITESIMAL CARTESIAN VOLUME ELEMENT IS

$$\rho \, dx \, dy \, dz$$

- THE DISTANCE OF THIS ELEMENT FROM THE z AXIS IS $\sqrt{x^2+y^2} = d$

SUMMING UP IN CARTESIAN

$$I = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \rho d^2 \, dx \, dy \, dz = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \rho(x^2+y^2) \, dx \, dy \, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\left[\rho\left(\tfrac{1}{3}x^3 + y^2x\right)\right]_{x=0}^{a} dy\,dz = \rho \int_{z=0}^{a}\int_{y=0}^{a} \tfrac{1}{3}a^3 + ay^2 \, dy \, dz$$

$$= \rho \int_{z=0}^{a}\left[\tfrac{1}{3}a^3y + \tfrac{1}{3}ay^3\right]_{y=0}^{a} dz = \rho \int_{z=0}^{a} \tfrac{1}{3}a^4 + \tfrac{1}{3}a^4 \, dz$$

$$= \tfrac{2}{3}\rho a^4 \int_0^a 1 \, dz = \tfrac{2}{3}\rho a^4 \left[z\right]_0^a = \tfrac{2}{3}\rho a^4 \times a$$

$$= \tfrac{2}{3}\rho a^5 = \tfrac{2}{3} \times \frac{m}{a^3} \times a^5 = \tfrac{2}{3}ma^2$$

Question 19

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

The air bubble density $\rho(z)$, in m^{-3}, is given by

$$\rho(z) = kz,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geq 0$, determine the total number of air bubbles in the solid.

M2B , $\boxed{\frac{1}{4}\pi k a^4}$

IF THE "BUBBLE DENSITY" IS GIVEN BY

$\rho(z) = kz$, THEN

$$\text{TOTAL BUBBLES} = \int_V \rho(z)\,dV$$

$$= \int_{\text{HEMISPHERE}} kz\,dV$$

WORKING IN SPHERICAL POLARS

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} k\underbrace{(r\cos\theta)}_{z}\underbrace{(r^2\sin\theta\,dr\,d\theta\,d\phi)}_{dV}$$

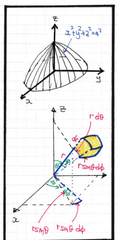
$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} kr^3\cos\theta\sin\theta\,dr\,d\theta\,d\phi$$

$$= \left[\int_{\phi=0}^{2\pi} k\,d\phi\right]\left[\int_{\theta=0}^{\frac{\pi}{2}} \cos\theta\sin\theta\,d\theta\right]\left[\int_{r=0}^{a} r^3\,dr\right]$$

$$= 2\pi k \times \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} \times \left[\tfrac{1}{4}r^4\right]_0^{a}$$

$$= 2\pi k \times \tfrac{1}{2} \times \tfrac{1}{4}a^4$$

$$= \tfrac{1}{4}\pi k a^4$$



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$x^2+y^2+z^2=a^2$

FOR THE HEMISPHERE

$0 \leq r \leq a$
$0 \leq \theta \leq \frac{\pi}{2}$
$0 \leq \phi \leq 2\pi$

$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

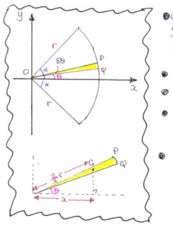
Question 20

A circular sector of radius r subtends an angle of 2α at its centre O. The position of the centre of mass of this sector lies at the point G, along its axis of symmetry.

Use calculus to show that

$$|OG| = \frac{2r\sin\alpha}{3\alpha}.$$

$\boxed{\text{proof}}$



- CONSIDER AN INFINITESIMAL SECTOR $\widehat{OPQ}$. AS $\delta\theta \ll 1$ THE SECTOR IS AN APPROXIMATE TRIANGLE WHOSE CENTRE OF MASS LIES $\frac{2}{3}$ ALONG ITS MEDIAN FROM O, I.E $\frac{2}{3}r$
- LET ρ = MASS PER UNIT AREA
- THE MASS OF THE INFINITESIMAL IS $m_1 = \rho\left(\frac{1}{2}r^2\delta\theta\right)$
- THE MASS OF THE ENTIRE LAMINA IS $\frac{1}{2}r^2(2\alpha)\rho = \rho\alpha r^2$
- $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$ (OR "TAKE MOMENTS ABOUT y")

$$\bar{x} = \frac{\sum \frac{1}{2}\rho r^2 \delta\theta \times x}{\rho\alpha r^2} = \frac{\frac{1}{2}\cancel{\rho}\cancel{r^2}\sum\left(\frac{2}{3}r\cos\theta\right)}{\cancel{\rho}\alpha\cancel{r^2}}$$

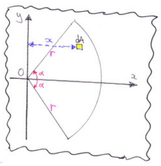
TAKING LIMITS

$$\bar{x} = \frac{1}{2\alpha}\int_{\theta=-\alpha}^{\theta=\alpha} \frac{2}{3}r\cos\theta \, d\theta = \frac{r}{3\alpha}\int_{-\alpha}^{\alpha}\cos\theta \, d\theta = \dots \text{ EVEN INTEGRAND}$$

$$= \frac{2r}{3\alpha}\int_0^{\alpha}\cos\theta \, d\theta = \frac{2r}{3\alpha}\left[\sin\theta\right]_0^{\alpha} = \frac{2r\sin\alpha}{3\alpha}.$$

AS THE CENTRE OF MASS LIES ON THE CENTRE OF SYMMETRY OF THE SECTOR THEN $\bar{x}$ IS ALSO THE CENTRE OF MASS OF THE SECTOR

I.E $\bar{r} = \frac{2r\sin\alpha}{3\alpha}$



- LET ρ = MASS PER UNIT AREA
- MASS OF INFINITESIMAL AREA IS $\rho \, dA$
- MOMENT OF INFINITESIMAL ABOUT THE x-AXIS IS $(\rho \, dA)x$
- TOTAL MASS OF THE SECTOR IS $\frac{1}{2}\rho r^2(2\alpha) = \alpha r^2\rho$
- SUMMING MOMENTS ABOUT THE y-AXIS

$$M\bar{x} = \int_{\text{SECTOR}} \rho x \, dA$$

- $\bar{x}$ WILL ALSO BE THE POSITION OF THE CENTRE OF MASS BY SYMMETRY

- TAKING LIMITS

$$\alpha r^2\rho\bar{x} = \int_{\text{SECTOR}} \rho x \, dA$$

- SWITCH INTO PLANE POLARS

$$\Rightarrow \underbrace{\alpha r^2\rho}_{\text{CONSTANT}}\bar{x} = \int_{\theta=-\alpha}^{\alpha}\int_{r=0}^{\overset{\text{CONSTANT}}{r}} \rho(r\cos\theta)\, r \, dr \, d\theta$$

$$\Rightarrow \underbrace{\alpha r^2\rho}_{\text{CONSTANT}}\bar{x} = \rho\int_{-\alpha}^{\alpha}\int_0^r r^2\cos\theta \, dr \, d\theta$$

$$\Rightarrow \alpha r^2\bar{x} = \int_{-\alpha}^{\alpha}\left[\frac{1}{3}r^3\right]_0^r\cos\theta \, d\theta$$

$$\Rightarrow \alpha r^2\bar{x} = \frac{1}{3}r^3\int_{-\alpha}^{\alpha}\cos\theta \, d\theta$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \int_{-\alpha}^{\alpha}\cos\theta \, d\theta$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \left[\sin\theta\right]_{-\alpha}^{\alpha}$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \left[\sin\alpha - \sin(-\alpha)\right]$$

$$\Rightarrow \frac{3\alpha}{r}\bar{x} = \sin\alpha + \sin\alpha$$

$$\Rightarrow \frac{3\alpha\bar{x}}{r} = 2\sin\alpha$$

$$\Rightarrow \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Question 21

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

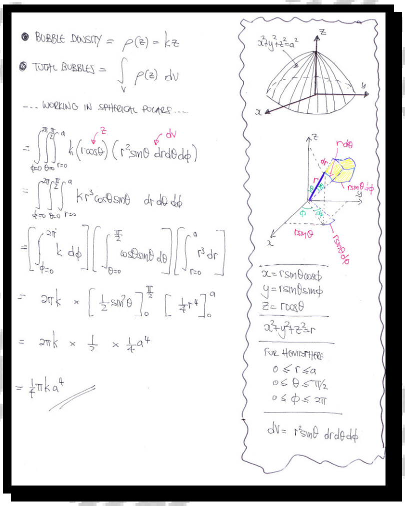
The air bubble density $\rho(z)$, in m^{-3}, is given by

$$\rho(z) = kz\,,$$

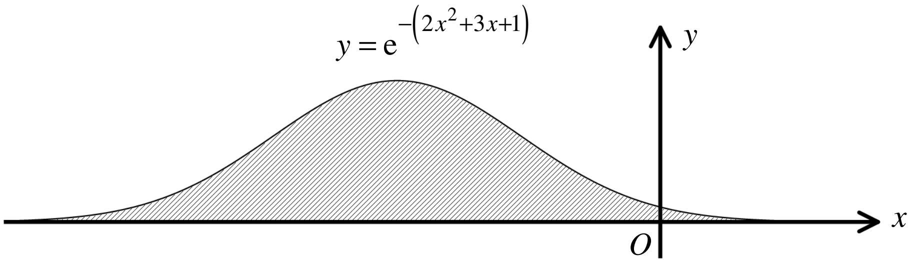
where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geq 0$, determine the total number of air bubbles in the solid.

$\boxed{}$, $\boxed{\frac{1}{4}\pi k a^4}$



Question 22

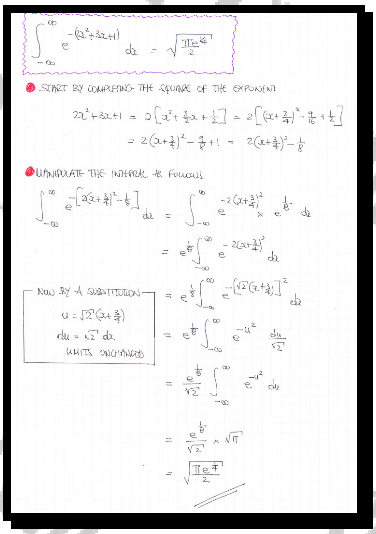


The figure above shows the curve with equation

$$y = \mathrm{e}^{-(2x^2+3x+1)}, \quad x \in \mathbb{R}.$$

Show that the area between the curve and the x axis is $\sqrt{\frac{1}{2}\pi \mathrm{e}^{\frac{1}{4}}}$

$\boxed{\text{proof}}$



$$\int_{-\infty}^{\infty} e^{-(2x^2+3x+1)}\,dx = \sqrt{\frac{\pi e^{\frac{1}{4}}}{2}}$$

- START BY COMPLETING THE SQUARE OF THE EXPONENT

$$2x^2+3x+1 = 2\left[x^2+\tfrac{3}{2}x+\tfrac{1}{2}\right] = 2\left[\left(x+\tfrac{3}{4}\right)^2-\tfrac{9}{16}+\tfrac{1}{2}\right]$$
$$= 2\left(x+\tfrac{3}{4}\right)^2-\tfrac{9}{8}+1 = 2\left(x+\tfrac{3}{4}\right)^2-\tfrac{1}{8}$$

- MANIPULATE THE INTEGRAL AS FOLLOWS

$$\int_{-\infty}^{\infty} e^{-\left[2\left(x+\frac{3}{4}\right)^2-\frac{1}{8}\right]}\,dx = \int_{-\infty}^{\infty} e^{-2\left(x+\frac{3}{4}\right)^2} \times e^{\frac{1}{8}}\,dx$$
$$= e^{\frac{1}{8}}\int_{-\infty}^{\infty} e^{-2\left(x+\frac{3}{4}\right)^2}\,dx$$
$$= e^{\frac{1}{8}}\int_{-\infty}^{\infty} e^{-\left[\sqrt{2}\left(x+\frac{3}{4}\right)\right]^2}\,dx$$

NOW BY A SUBSTITUTION
$u = \sqrt{2}\left(x+\frac{3}{4}\right)$
$du = \sqrt{2}\,dx$
LIMITS UNCHANGED

$$= e^{\frac{1}{8}}\int_{-\infty}^{\infty} e^{-u^2}\,\frac{du}{\sqrt{2}}$$
$$= \frac{e^{\frac{1}{8}}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-u^2}\,du$$
$$= \frac{e^{\frac{1}{8}}}{\sqrt{2}} \times \sqrt{\pi}$$
$$= \sqrt{\frac{\pi e^{\frac{1}{4}}}{2}}$$

Question 23

The position vector of a curve C is given by

$$\mathbf{r}(t) = \cos(\cosh t)\mathbf{i} + \sin(\cosh t)\mathbf{j} + t\mathbf{k},$$

where t is a scalar parameter with $0 \le t \le a$, $a \in \mathbb{R}$.

Determine the length of C.

$\boxed{\text{arclength} = \sinh a}$

$\mathbf{r}(t) = [\cos(\cosh t), \sin(\cosh t), t] \quad 0 \le t \le a$

$s = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$

$\dot{x} = -\sin(\cosh t) \times \sinh t \implies \dot{x}^2 = \sin^2(\cosh t)\sinh^2 t$

$\dot{y} = \cos(\cosh t) \times \sinh t \implies \dot{y}^2 = \cos^2(\cosh t)\sinh^2 t$

$\dot{z} = 1$

Thus

$s = \int_0^a \sqrt{\sin^2(\cosh t)\sinh^2 t + \cos^2(\cosh t)\sinh^2 t + 1}$

$s = \int_0^a \sqrt{\sinh^2 t\,[\cancel{\sin^2(\cosh t) + \cos^2(\cosh t)}] + 1}\, dt$

$s = \int_0^a \sqrt{\sinh^2 t + 1}\, dt = \int_0^a \cosh t\, dt$

$s = [\sinh t]_0^a = \sinh a - \sinh 0 = \sinh a$

Question 24

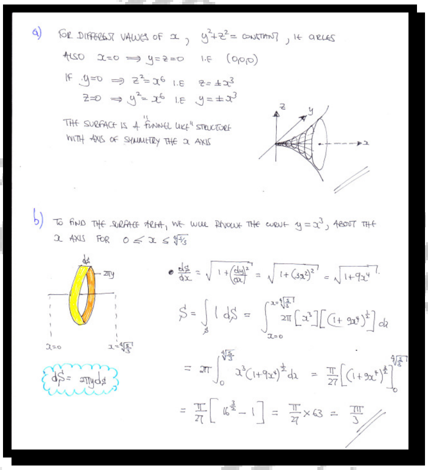
A surface S is has Cartesian equation

$$y^2 + z^2 = x^6, \quad 0 \le x \le \sqrt[4]{\frac{5}{3}}.$$

a) Sketch the graph of S.

b) Find the area of S.

$$\boxed{\frac{7}{3}\pi}$$



a) FOR DIFFERENT VALUES OF x, $y^2+z^2=$ CONSTANT, IE CIRCLES

ALSO $x=0 \Rightarrow y=z=0$ I.E $(0,0,0)$

IF $y=0 \Rightarrow z^2=x^6$ I.E $z=\pm x^3$

$z=0 \Rightarrow y^2=x^6$ I.E $y=\pm x^3$

THE SURFACE IS A "FUNNEL LIKE" STRUCTURE WITH AXIS OF SYMMETRY THE x AXIS

b) TO FIND THE SURFACE AREA, WE WILL REVOLVE THE CURVE $y=x^3$, ABOUT THE x AXIS FOR $0 \le x \le \sqrt[4]{5/3}$

- $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+(3x^2)^2} = \sqrt{1+9x^4}$

$$S = \int_S 1\,dS = \int_{x=0}^{x=\sqrt[4]{\frac{5}{3}}} 2\pi\left[x^3\right]\left[(1+9x^4)^{\frac{1}{2}}\right]dx$$

$$= 2\pi\int_0^{\sqrt[4]{\frac{5}{3}}} x^3(1+9x^4)^{\frac{1}{2}}\,dx = \frac{\pi}{27}\Big[(1+9x^4)^{\frac{3}{2}}\Big]_0^{\sqrt[4]{\frac{5}{3}}}$$

$$= \frac{\pi}{27}\left[16^{\frac{3}{2}}-1\right] = \frac{\pi}{27}\times 63 = \frac{7\pi}{3}$$

Question 25

A solid sphere has equation

$$x^2 + y^2 + z^2 = a^2 .$$

The density, ρ, at the point of the sphere with coordinates (x_1, y_1, z_1) is given by

$$\rho = \sqrt{x_1^2 + y_1^2} .$$

Determine the **average** density of the sphere.

$\boxed{\text{MM2}}$, $\boxed{\bar{\rho} = \frac{3}{16}\pi a}$

STARTING BY THE DEFINITION OF MASS FOR VARIABLE DENSITY

$$\text{MASS} = \int_V \rho(x,y,z)\, dV = \int_{\substack{\text{VOLUME OF} \\ x^2+y^2+z^2=a^2}} \sqrt{x^2+y^2}\, dx\, dy\, dz$$

SWITCH INTO SPHERICAL POLARS

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{\underbrace{(r\sin\theta\cos\phi)^2}_{x^2} + \underbrace{(r\sin\theta\sin\phi)^2}_{y^2}}\ \underbrace{r^2\sin\theta\, dr\, d\theta\, d\phi}_{dV}$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} (r\sin\theta)\cancel{\sqrt{\cos^2\phi + \sin^2\phi}}\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\sin^2\theta\, dr\, d\theta\, d\phi$$

INTEGRATING WITH RESPECT TO r FIRST

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[\tfrac{1}{4}r^4\sin^2\theta\right]_0^a d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \tfrac{1}{4}a^4\sin^2\theta\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \tfrac{1}{4}a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^2\theta\, d\theta\, d\phi$$

INTEGRATING WITH RESPECT TO ϕ NEXT

$$\text{MASS} = \tfrac{1}{4}a^4 \times 2\pi \times \int_{\theta=0}^{\pi} \tfrac{1}{2} - \tfrac{1}{2}\cancel{\cos 2\theta}\, d\theta$$

$$\text{MASS} = \tfrac{1}{4}a^4 \times 2\pi \times \tfrac{1}{2}\pi$$

$$\text{MASS} = \tfrac{1}{4}a^4\pi^2$$

NOW THE VOLUME OF A SPHERE OF RADIUS a IS $V = \frac{4}{3}\pi a^3$

$$\therefore \text{AVERAGE DENSITY} = \frac{\text{MASS}}{\text{VOLUME}} = \frac{\frac{1}{4}\pi^2 a^4}{\frac{4}{3}\pi a^3} = \frac{3}{16}\pi a$$

Question 26

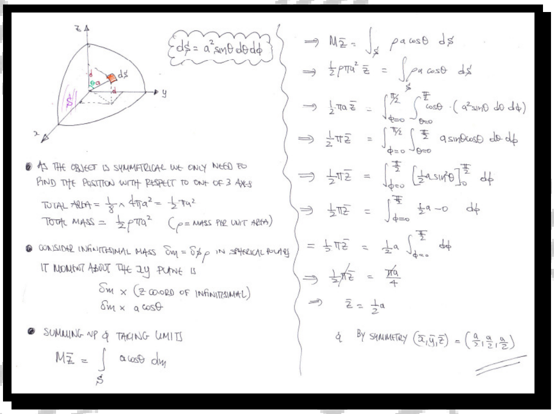
A thin uniform spherical shell with equation

$$x^2 + y^2 + z^2 = a^2, \ a > 0,$$

occupies the region in the first octant.

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the shell.

$$\boxed{\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)}$$



$dS = a^2 \sin\theta \, d\theta \, d\phi$

- AS THE OBJECT IS SYMMETRICAL WE ONLY NEED TO FIND THE POSITION WITH RESPECT TO ONE OF 3 AXES

  TOTAL AREA $= \frac{1}{8} \times 4\pi a^2 = \frac{1}{2}\pi a^2$

  TOTAL MASS $= \frac{1}{2}\rho\pi a^2$ (ρ = MASS PER UNIT AREA)

- CONSIDER INFINITESIMAL MASS $\delta m = \delta S \rho$ IN SPHERICAL POLARS

  IT MOMENT ABOUT THE xy PLANE IS

  $\delta m \times$ (z COORD OF INFINITESIMAL)

  $\delta m \times a\cos\theta$

- SUMMING UP & TAKING LIMITS

  $M\bar{z} = \int_S a\cos\theta \, dm$

$\Rightarrow M\bar{z} = \int_S \rho a \cos\theta \, dS$

$\Rightarrow \frac{1}{2}\rho\pi a^2 \bar{z} = \int_S \rho a \cos\theta \, dS$

$\Rightarrow \frac{1}{2}\pi a \bar{z} = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\frac{\pi}{2}} \cos\theta \cdot (a^2 \sin\theta \, d\theta \, d\phi)$

$\Rightarrow \frac{1}{2}\pi \bar{z} = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\frac{\pi}{2}} a \sin\theta\cos\theta \, d\theta \, d\phi$

$\Rightarrow \frac{1}{2}\pi \bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \left[\frac{1}{2} a \sin^2\theta\right]_0^{\frac{\pi}{2}} d\phi$

$\Rightarrow \frac{1}{2}\pi \bar{z} = \int_{\phi=0}^{\frac{\pi}{2}} \frac{1}{2}a - 0 \, d\phi$

$= \frac{1}{2}\pi \bar{z} = \frac{1}{2}a \int_{\phi=0}^{\frac{\pi}{2}} d\phi$

$\Rightarrow \frac{1}{2}\pi \bar{z} = \frac{\pi a}{4}$

$\Rightarrow \bar{z} = \frac{1}{2}a$

& BY SYMMETRY $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

Question 27

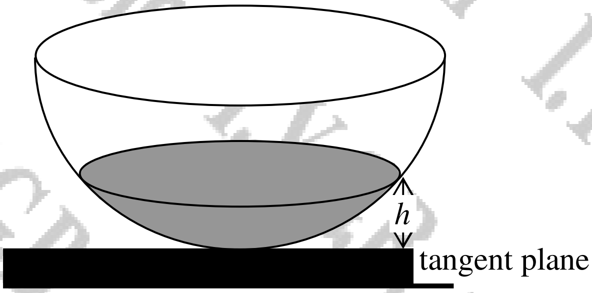


figure 1

Figure 1 shows a hemispherical bowl of radius r cm containing water up to a certain level h cm. The shape of the water in the bowl is called a spherical segment.

It is required to find a formula for the volume of a spherical segment as a function of the radius r cm and the distance of its plane face from the tangent plane, h cm.

The circle with equation

$$x^2 + y^2 = r^2, \; x \geq 0$$

is to be used to find a formula for the volume of a spherical segment.

The part of the circle in the first quadrant between $x = r - h$ and $x = r$ is shown shaded in figure 2, and is labelled as the region R.

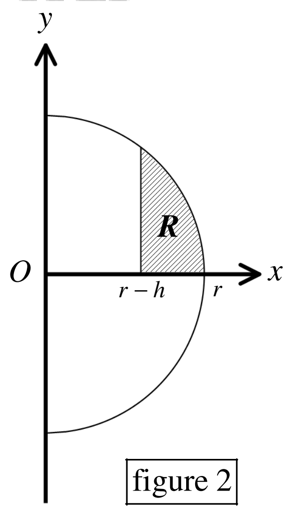


figure 2

[continues overleaf]

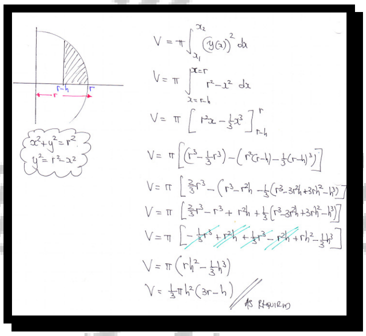
[continued from overleaf]

Show by integration that the volume of the spherical segment V is given by

$$V = \frac{1}{3}\pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

proof



Question 28

A thin plate occupies the region in the x-y plane with equation

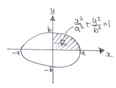
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The mass per unit area of the plate ρ, is given by

$$\rho(x,y)=x^2y^2.$$

Find a simplified expression for the mass of the plate.

$$\boxed{\frac{\pi a^3 b^3}{24}}$$



AS THE DENSITY FUNCTION IS EVEN IN x & y USE SYMMETRY

$$M = 4\int_R \rho(x,y)\,dx\,dy = 4\int_{y=0}^{b}\int_{x=0}^{x=a\sqrt{1-\frac{y^2}{b^2}}} x^2y^2\,dx\,dy$$

$$= 4\int_{y=0}^{b} \frac{1}{3}y^2\left[x^3\right]_0^{x=a\sqrt{1-\frac{y^2}{b^2}}}\,dy = \frac{4}{3}\int_{y=0}^{b} y^2a^3\left(1-\frac{y^2}{b^2}\right)^{\frac{3}{2}}dy$$

$$= \frac{4a^3}{3}\int_0^b y^2\left(\frac{b^2-y^2}{b^2}\right)^{\frac{3}{2}}dy = \frac{4a^3}{3b^3}\int_0^b y^2(b^2-y^2)^{\frac{3}{2}}dy$$

... SUBSTITUTION $y=b\sin\theta$, $dy=b\cos\theta\,d\theta$, $b^2-y^2=b^2(1-\sin^2\theta)=b^2\cos^2\theta$; $y=b \mapsto \theta=\frac{\pi}{2}$, $y=0 \mapsto \theta=0$

$$= \frac{4a^3}{3b^3}\int_0^{\frac{\pi}{2}} b^2\sin^2\theta\, b^3\cos^3\theta\,(b\cos\theta\,d\theta) = \frac{4}{3}a^3b^3\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta\,d\theta$$

... BY TRIG IDENTITIES, COMPLEX NUMBERS OR BETA/GAMMA FUNCTIONS

$$= \frac{2}{3}a^3b^3\times\left[2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\times\frac{3}{2}-1}(\cos\theta)^{2\times\frac{5}{2}-1}d\theta\right] = \frac{2}{3}a^3b^3\,B\left(\tfrac{3}{2},\tfrac{5}{2}\right)$$

$$= \frac{2}{3}a^3b^3\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)} = \frac{2}{3}a^3b^3\times\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)}{3!}$$

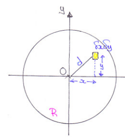
$$= \frac{2}{3}a^3b^3\times\frac{\frac{3}{8}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{6} = \frac{2}{3}\times\frac{3}{8}\times\frac{1}{6}\times(\sqrt{\pi})^2a^3b^3 = \frac{\pi}{24}a^3b^3$$

Question 29

A uniform circular lamina has mass M and radius a.

Use double integration in plane polar coordinates to find the moment of inertia of the lamina, when the axis of rotation is perpendicular to the plane of the lamina and passes through its centre.

$$\boxed{\frac{1}{2}Ma^2}$$



- LET THE AXIS OF ROTATION BE THROUGH O AND PERPENDICULAR TO THE PLANE OF THE LAMINA (HERE THE x-y PLANE)
- LET $\rho = \frac{M}{\pi a^2}$ = MASS PER UNIT AREA
- THE MASS OF INFINITESIMAL ELEMENT WILL BE
  $\rho\, \delta x \delta y$
- AND ITS MOMENT OF INERTIA ABOUT THE GIVEN AXIS WILL BE
  $(\rho\, \delta x \delta y) d^2$
  $\rho\, \delta x \delta y\, (x^2+y^2)$
- SUMMING UP & TAKING LIMITS

$$I = \int_R \rho (x^2+y^2)\, dx\, dy$$

- SWITCH INTO POLARS

$$I = \rho \int_R r^2 (r\, dr\, d\theta) = \frac{M}{\pi a^2} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r^3\, dr\, d\theta$$

$$= \frac{M}{\pi a^2} \times 2\pi \times \int_{r=0}^{a} r^3\, dr = \frac{2M}{a^2}\left[\frac{1}{4}r^4\right]_0^a$$

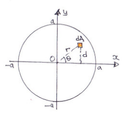
$$= \frac{2M}{a^2} \times \frac{1}{4}a^4 = \frac{1}{2}Ma^2$$

Question 30

A uniform circular lamina has mass M and radius a.

Use double integration to find the moment of inertia of the lamina, when the axis of rotation is a diameter.

$$\boxed{\frac{1}{4}Ma^2}$$



- $\rho = \frac{M}{\pi a^2}$ (MASS PER UNIT AREA)
- LET THE AXIS OF ROTATION BE THE x AXIS
- INFINITESIMAL MOMENT OF INERTIA IS $(\rho \, dA) \, d^2$
- IN POLARS $d = r\sin\theta$

SUMMING UP ALL INFINITESIMALS IN PLANE POLARS

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \rho d^2 \,(r \, dr \, d\theta) = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{M}{\pi a^2}(r^2\sin^2\theta) r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{M}{\pi a^2} r^3 \sin^2\theta \, dr \, d\theta = \int_{\theta=0}^{2\pi}\left[\frac{M}{4\pi a^2} r^4 \sin^2\theta\right]_{r=0}^{r=a} d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{Ma^2}{4\pi}\sin^2\theta \, d\theta = \frac{Ma^2}{4\pi}\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{Ma^2}{4\pi}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{Ma^2}{4\pi}\left[(\pi - 0) - (0 - 0)\right]$$

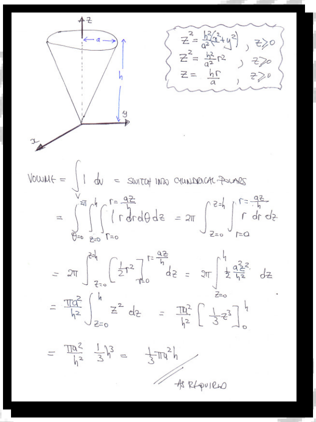
$$= \frac{1}{4}Ma^2$$

Question 31

Use cylindrical polar coordinates (r,θ,z) to show that the volume of a right circular cone of height h and base radius a is

$$\frac{1}{3}\pi a^2 h.$$

$\boxed{\text{proof}}$



$z^2 = \frac{h^2}{a^2}(x^2+y^2)$, $z \geqslant 0$

$z^2 = \frac{h^2}{a^2}r^2$, $z \geqslant 0$

$z = \frac{hr}{a}$, $z \geqslant 0$

VOLUME $= \int_V 1 \, dv$ = SWITCH INTO CYLINDRICAL POLARS

$= \int_{\theta=0}^{2\pi} \int_{z=0}^{h} \int_{r=0}^{r=\frac{az}{h}} 1 \, r \, dr \, d\theta \, dz = 2\pi \int_{z=0}^{z=h} \int_{r=0}^{r=\frac{az}{h}} r \, dr \, dz$

$= 2\pi \int_{z=0}^{z=h} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\frac{az}{h}} dz = 2\pi \int_{z=0}^{h} \frac{1}{2}\frac{a^2z^2}{h^2} \, dz$

$= \frac{\pi a^2}{h^2} \int_{z=0}^{h} z^2 \, dz = \frac{\pi a^2}{h^2}\left[\frac{1}{3}z^3\right]_0^h$

$= \frac{\pi a^2}{h^2} \frac{1}{3}h^3 = \frac{1}{3}\pi a^2 h$

AS REQUIRED

Question 32

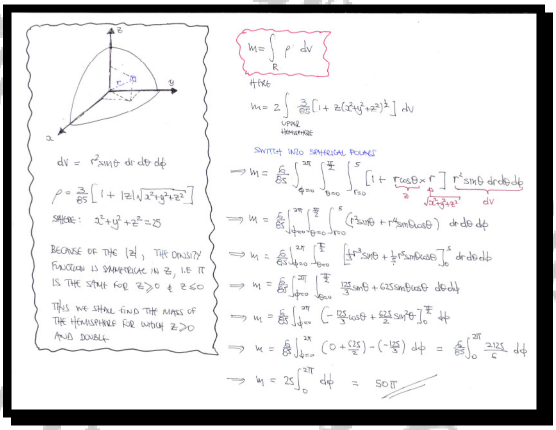
A solid sphere has radius 5 and is centred at the Cartesian origin O.

The density ρ at point $P(x_1, y_1, z_1)$ of the sphere satisfies

$$\rho = \frac{3}{85}\left[1 + |z_1|\sqrt{x_1^2 + y_1^2 + z_1^2}\right].$$

Use spherical polar coordinates, (r, θ, φ), to find the mass of the sphere.

$\boxed{m = 50\pi}$



$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$\rho = \frac{3}{85}\left[1 + |z|\sqrt{x^2+y^2+z^2}\right]$

SPHERE: $x^2+y^2+z^2=25$

BECAUSE OF THE $|z|$, THE DENSITY FUNCTION IS SYMMETRICAL IN z, I.E IT IS THE SAME FOR $z \geqslant 0$ & $z \leqslant 0$

THUS WE SHALL FIND THE MASS OF THE HEMISPHERE FOR WHICH $z \geqslant 0$ AND DOUBLE

$\boxed{m = \int_R \rho \, dV}$

HERE

$m = 2\int_{\text{UPPER HEMISPHERE}} \frac{3}{85}\left[1 + z(x^2+y^2+z^2)^{\frac{1}{2}}\right] dV$

SWITCH INTO SPHERICAL POLARS

$\Rightarrow m = \frac{6}{85}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{5}\left[1 + \underbrace{r\cos\theta}_{z} \times \underbrace{r}_{\sqrt{x^2+y^2+z^2}}\right]\underbrace{r^2\sin\theta \, dr \, d\theta \, d\phi}_{dV}$

$\Rightarrow m = \frac{6}{85}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{5}\left(r^2\sin\theta + r^4\sin\theta\cos\theta\right) dr \, d\theta \, d\phi$

$\Rightarrow m = \frac{6}{85}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{1}{3}r^3\sin\theta + \frac{1}{5}r^5\sin\theta\cos\theta\right]_0^5 \, d\theta \, d\phi$

$\Rightarrow m = \frac{6}{85}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \frac{125}{3}\sin\theta + 625\sin\theta\cos\theta \, d\theta \, d\phi$

$\Rightarrow m = \frac{6}{85}\int_{\phi=0}^{2\pi}\left[-\frac{125}{3}\cos\theta + \frac{625}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} d\phi$

$\Rightarrow m = \frac{6}{85}\int_{\phi=0}^{2\pi}\left(0 + \frac{625}{2}\right) - \left(-\frac{125}{3}\right) d\phi = \frac{6}{85}\int_0^{2\pi}\frac{2125}{6} \, d\phi$

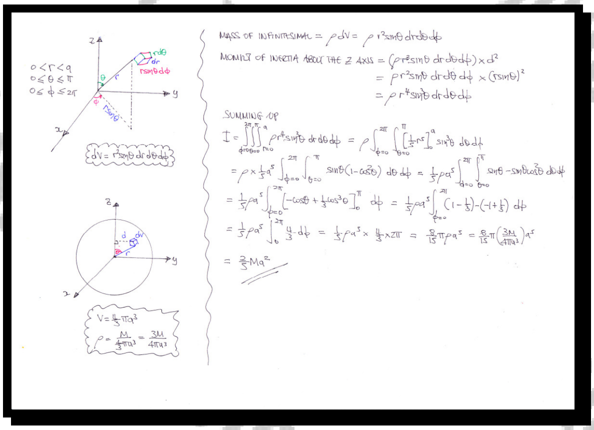
$\Rightarrow m = 25\int_0^{2\pi} d\phi = 50\pi$

Question 33

A solid uniform sphere has mass M and radius a.

Use spherical polar coordinates, (r,θ,φ), to show that the moment of inertial of this sphere about one of its diameters is $\frac{2}{5}Ma^2$.

$\boxed{\text{proof}}$



$0 < r < a$
$0 \le \theta \le \pi$
$0 \le \phi \le 2\pi$

$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$V = \frac{4}{3}\pi a^3$

$\rho = \frac{M}{\frac{4}{3}\pi a^3} = \frac{3M}{4\pi a^3}$

MASS OF INFINITESIMAL $= \rho \, dV = \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$

MOMENT OF INERTIA ABOUT THE z AXIS $= (\rho r^2 \sin\theta \, dr \, d\theta \, d\phi) \times d^2$
$= \rho r^2 \sin\theta \, dr \, d\theta \, d\phi \times (r\sin\theta)^2$
$= \rho r^4 \sin^3\theta \, dr \, d\theta \, d\phi$

SUMMING UP

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \rho r^4 \sin^3\theta \, dr \, d\theta \, d\phi = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[\tfrac{1}{5}r^5\right]_0^a \sin^3\theta \, d\theta \, d\phi$$

$$= \rho \times \tfrac{1}{5}a^5 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta(1-\cos^2\theta) \, d\theta \, d\phi = \tfrac{1}{5}\rho a^5 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos^2\theta \, d\theta \, d\phi$$

$$= \tfrac{1}{5}\rho a^5 \int_{\phi=0}^{2\pi} \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi} d\phi = \tfrac{1}{5}\rho a^5 \int_{\phi=0}^{2\pi} \left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right) d\phi$$

$$= \tfrac{1}{5}\rho a^5 \int_0^{2\pi} \tfrac{4}{3} \, d\phi = \tfrac{1}{5}\rho a^5 \times \tfrac{4}{3} \times 2\pi = \tfrac{8}{15}\pi\rho a^5 = \tfrac{8}{15}\pi\left(\tfrac{3M}{4\pi a^3}\right)a^5$$

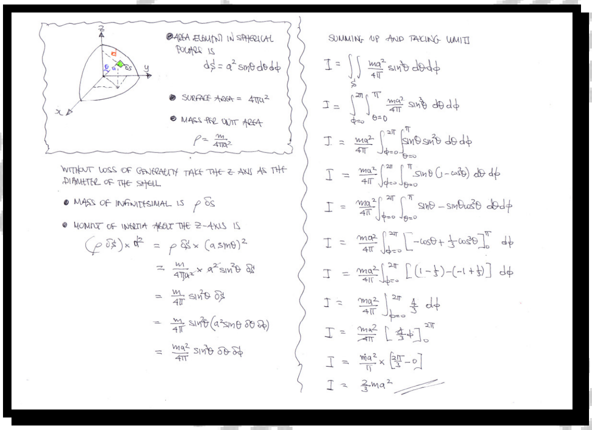
$$= \tfrac{2}{5}Ma^2$$

Question 34

A thin uniform spherical shell has mass m and radius a.

Use spherical polar coordinates, (r,θ,φ), to show that the moment of inertial of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof



- AREA ELEMENT IN SPHERICAL POLARS IS
$$dS = a^2 \sin\theta \, d\theta \, d\phi$$
- SURFACE AREA $= 4\pi a^2$
- MASS PER UNIT AREA
$$\rho = \frac{m}{4\pi a^2}$$

WITHOUT LOSS OF GENERALITY TAKE THE Z AXIS AS THE DIAMETER OF THE SHELL

- MASS OF INFINITESIMAL IS $\rho \, \delta S$
- MOMENT OF INERTIA ABOUT THE Z-AXIS IS

$$(\rho \, \delta S) \times d^2 = \rho \, \delta S \times (a\sin\theta)^2$$
$$= \frac{m}{4\pi a^2} \times a^2 \sin^2\theta \, \delta S$$
$$= \frac{m}{4\pi} \sin^2\theta \, \delta S$$
$$= \frac{m}{4\pi} \sin^2\theta \left(a^2 \sin\theta \, \delta\theta \, \delta\phi\right)$$
$$= \frac{ma^2}{4\pi} \sin^3\theta \, \delta\theta \, \delta\phi$$

SUMMING UP AND TAKING LIMITS

$$I = \iint_S \frac{ma^2}{4\pi} \sin^3\theta \, d\theta \, d\phi$$
$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{ma^2}{4\pi} \sin^3\theta \, d\theta \, d\phi$$
$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \sin^2\theta \, d\theta \, d\phi$$
$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta (1-\cos^2\theta) \, d\theta \, d\phi$$
$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos^2\theta \, d\theta \, d\phi$$
$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi} d\phi$$
$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \left[\left(1-\tfrac{1}{3}\right)-\left(-1+\tfrac{1}{3}\right)\right] d\phi$$
$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \frac{4}{3} \, d\phi$$
$$I = \frac{ma^2}{4\pi} \left[\tfrac{4}{3}\phi\right]_0^{2\pi}$$
$$I = \frac{ma^2}{\pi} \times \left[\frac{2\pi}{3} - 0\right]$$
$$I = \frac{2}{3}ma^2$$

Question 35

A building whose plan measures 10 m long by 10 m wide has vertical walls and a suspended fabric roof. The height, z m, of the roof above the ground is modelled in three dimensional Cartesian space by the equation

$$z = \frac{y\left(x^2+y\right)}{50} + 2, \quad -5 \le x \le 5, \quad 0 \le y \le 10.$$

a) Sketch the graph of the surface which models the roof of the building.

Give a brief description of its shape including its key features with relevant coordinates such as the maximum height and minimum height of the roof.

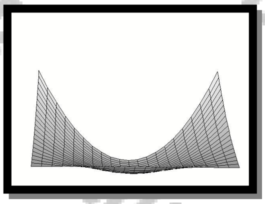
b) Determine the volume of the building enclosed by vertical walls and the suspended fabric roof.

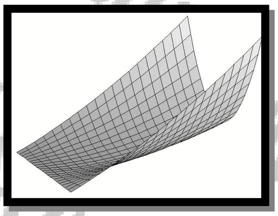
c) Show that the area of the fabric roof is given by

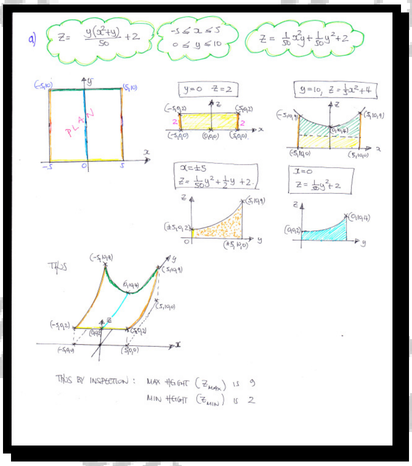
$$\frac{1}{25}\int_{y=0}^{10}\int_{x=0}^{5} \sqrt{G(x,y)}\,dx\,dy,$$

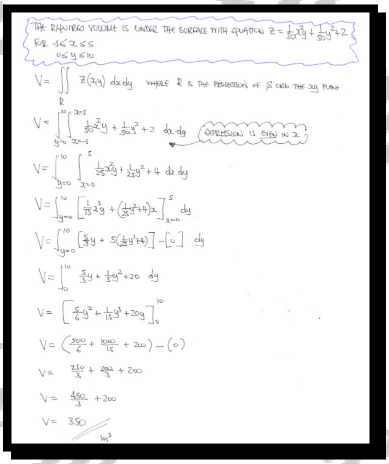
where $G(x,y)$ is a function to be found.

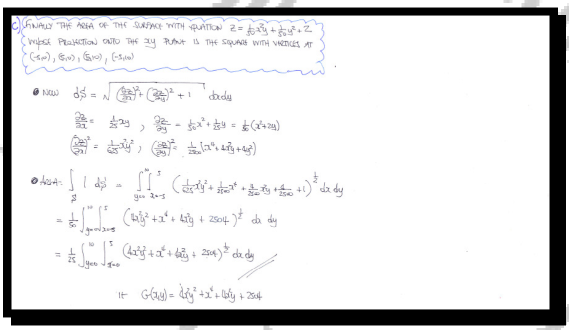
$\boxed{\text{Volume} = 350}$, $\boxed{G(x,y) = 4x^2y^2 + x^4 + 4x^2y + 2504}$











Question 36

A thin plate occupies the region in the x-y plane defined by the inequalities

$$0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 2x .$$

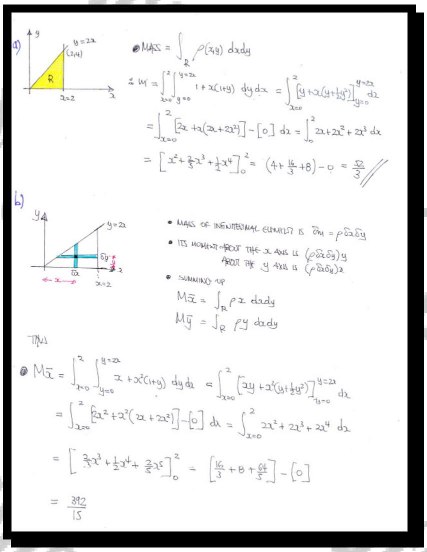
The mass per unit area of the plate ρ, is given by

$$\rho(x,y) = 1 + x(1+y) .$$

a) Find the mass of the plate.

b) Determine the coordinates of the centre of mass of the plate.

$\boxed{m = \frac{52}{3}}$, $\boxed{(\bar{x},\bar{y}) = \left(\frac{98}{65},\frac{114}{65}\right)}$



a)

• MASS $= \int_R \rho(x,y)\, dx\,dy$

$\therefore M = \int_{x=0}^{2}\int_{y=0}^{y=2x} 1 + x(1+y)\, dy\,dx = \int_{x=0}^{2} \left[y + x\left(y + \frac{1}{2}y^2\right)\right]_{y=0}^{y=2x} dx$

$= \int_{x=0}^{2} \left[2x + x(2x+2x^2)\right] - [0]\, dx = \int_0^2 2x + 2x^2 + 2x^3\, dx$

$= \left[x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4\right]_0^2 = \left(4 + \frac{16}{3} + 8\right) - 0 = \frac{52}{3}$

b)

• MASS OF INFINITESIMAL ELEMENT IS $\delta m = \rho\,\delta x\,\delta y$

• ITS MOMENT ABOUT THE x AXIS IS $(\rho\,\delta x\,\delta y)y$

ABOUT THE y AXIS IS $(\rho\,\delta x\,\delta y)x$

• SUMMING UP

$M\bar{x} = \int_R \rho x\, dx\,dy$

$M\bar{y} = \int_R \rho y\, dx\,dy$

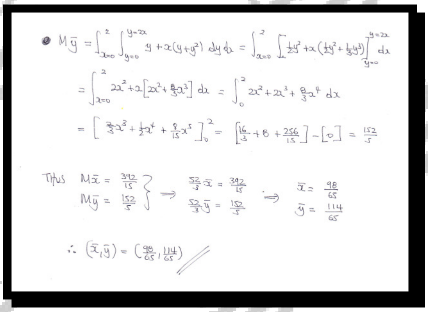
THUS

• $M\bar{x} = \int_{x=0}^{2}\int_{y=0}^{y=2x} x + x^2(1+y)\, dy\,dx = \int_{x=0}^{2}\left[xy + x^2\left(y + \frac{1}{2}y^2\right)\right]_{y=0}^{y=2x} dx$

$= \int_{x=0}^{2} \left[2x^2 + x^2(2x+2x^2)\right] - [0]\, dx = \int_{x=0}^{2} 2x^2 + 2x^3 + 2x^4\, dx$

$= \left[\frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{2}{5}x^5\right]_0^2 = \left[\frac{16}{3} + 8 + \frac{64}{5}\right] - [0]$

$= \frac{392}{15}$



• $M\bar{y} = \int_{x=0}^{2}\int_{y=0}^{y=2x} y + x(y+y^2)\, dy\,dx = \int_{x=0}^{2}\left[\frac{1}{2}y^2 + x\left(\frac{1}{2}y^2 + \frac{1}{3}y^3\right)\right]_{y=0}^{y=2x} dx$

$= \int_{x=0}^{2} 2x^2 + x\left[2x^2 + \frac{8}{3}x^3\right] dx = \int_0^2 2x^2 + 2x^3 + \frac{8}{3}x^4\, dx$

$= \left[\frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{8}{15}x^5\right]_0^2 = \left[\frac{16}{3} + 8 + \frac{256}{15}\right] - [0] = \frac{152}{5}$

Thus $\left.\begin{aligned} M\bar{x} &= \frac{392}{15} \\ M\bar{y} &= \frac{152}{5}\end{aligned}\right\} \Rightarrow \begin{aligned} \frac{52}{3}\bar{x} &= \frac{392}{15} \\ \frac{52}{3}\bar{y} &= \frac{152}{5}\end{aligned} \Rightarrow \begin{aligned} \bar{x} &= \frac{98}{65} \\ \bar{y} &= \frac{114}{65}\end{aligned}$

$\therefore (\bar{x},\bar{y}) = \left(\frac{98}{65},\frac{114}{65}\right)$

Question 37

The position vector of a curve C is given by

$$\mathbf{r}(t) = \left(\frac{2}{1+t^2} - 1\right)\mathbf{i} + \left(\frac{2t}{1+t^2}\right)\mathbf{j},$$

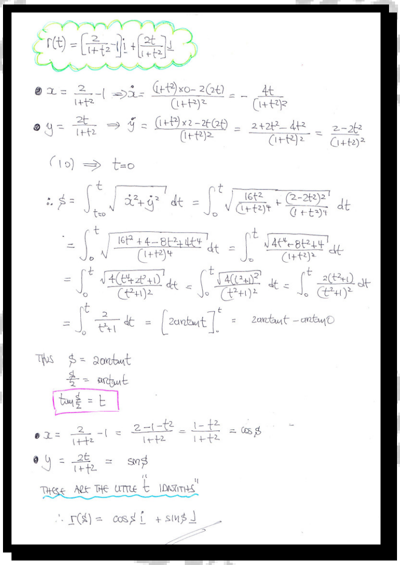
where t is a scalar parameter with $t \in \mathbb{R}$.

Find an expression for the position vector of C, giving the answer in the form

$$\mathbf{r}(s) = f(s)\mathbf{i} + g(s)\mathbf{j},$$

where s is the arc length of a general point on C, measured from the point $(1,0)$.

$$\boxed{\mathbf{r}(s) = (\cos s)\mathbf{i} + (\sin s)\mathbf{j}}$$



$$\boxed{\mathbf{r}(t) = \left[\frac{2}{1+t^2} - 1\right]\mathbf{i} + \left[\frac{2t}{1+t^2}\right]\mathbf{j}}$$

- $x = \frac{2}{1+t^2} - 1 \Rightarrow \dot{x} = \frac{(1+t^2)\times 0 - 2(2t)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$
- $y = \frac{2t}{1+t^2} \Rightarrow \dot{y} = \frac{(1+t^2)\times 2 - 2t(2t)}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$

$(1,0) \Rightarrow t=0$

$$\therefore s = \int_{t=0}^{t} \sqrt{\dot{x}^2+\dot{y}^2}\,dt = \int_0^t \sqrt{\frac{16t^2}{(1+t^2)^4} + \frac{(2-2t^2)^2}{(1+t^2)^4}}\,dt$$

$$= \int_0^t \sqrt{\frac{16t^2+4-8t^2+4t^4}{(1+t^2)^4}}\,dt = \int_0^t \frac{\sqrt{4t^4+8t^2+4}}{(1+t^2)^2}\,dt$$

$$= \int_0^t \frac{\sqrt{4(t^4+2t^2+1)}}{(t^2+1)^2}\,dt = \int_0^t \frac{\sqrt{4(t^2+1)^2}}{(t^2+1)^2}\,dt = \int_0^t \frac{2(t^2+1)}{(t^2+1)^2}\,dt$$

$$= \int_0^t \frac{2}{t^2+1}\,dt = \left[2\arctan t\right]_0^t = 2\arctan t - \arctan 0$$

Thus $s = 2\arctan t$

$\frac{s}{2} = \arctan t$

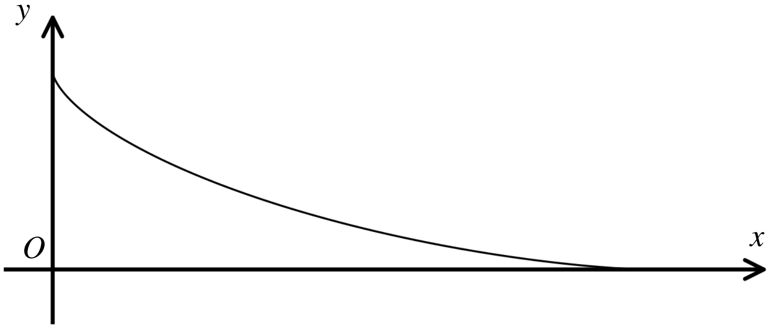
$\boxed{\tan\frac{s}{2} = t}$

- $x = \frac{2}{1+t^2} - 1 = \frac{2-1-t^2}{1+t^2} = \frac{1-t^2}{1+t^2} = \cos s$
- $y = \frac{2t}{1+t^2} = \sin s$

THESE ARE THE LITTLE "t IDENTITIES"

$\therefore \mathbf{r}(s) = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$

Question 38



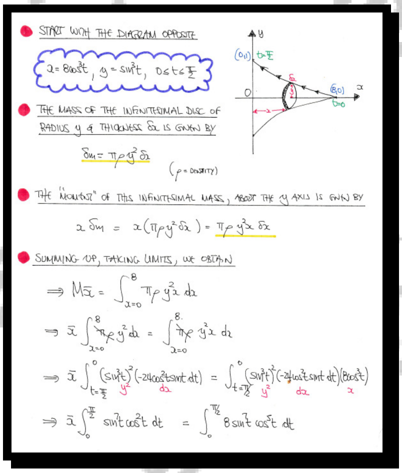
The figure above shows the curve with parametric equations

$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \tfrac{1}{2}\pi .$$

The finite region bounded by the curve and the coordinate axes is revolved fully about the x axis, forming a solid of revolution S.

Determine the x coordinate of the centre of mass of S.

$\boxed{\bar{x} = \dfrac{21}{16}}$



START WITH THE DIAGRAM OPPOSITE

$x = 8\cos^3 t, \ y = \sin^3 t, \ 0 \le t \le \frac{\pi}{2}$

THE MASS OF THE INFINITESIMAL DISC OF RADIUS y & THICKNESS δx IS GIVEN BY

$\delta m = \pi \rho y^2 \delta x$ $\quad (\rho = \text{DENSITY})$

THE "MOMENT" OF THIS INFINITESIMAL MASS, ABOUT THE y AXIS IS GIVEN BY

$x\,\delta m = x(\pi \rho y^2 \delta x) = \pi \rho y^2 x\, \delta x$

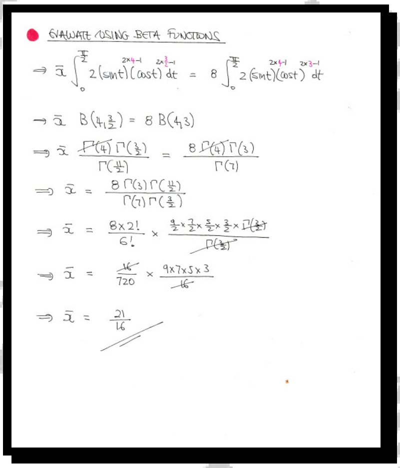
SUMMING UP, TAKING LIMITS, WE OBTAIN

$\Rightarrow M\bar{x} = \int_{x=0}^{8} \pi \rho y^2 x \, dx$

$\Rightarrow \bar{x} \int_{x=0}^{8} \cancel{\pi\rho}\, y^2 \, dx = \int_{x=0}^{8} \cancel{\pi\rho}\, y^2 x \, dx$

$\Rightarrow \bar{x} \int_{t=\frac{\pi}{2}}^{0} \underbrace{(\sin^3 t)^2}_{y^2} \underbrace{(-24\cos^2 t \sin t \, dt)}_{dx} = \int_{t=\pi/2}^{0} \underbrace{(\sin^3 t)^2}_{y^2} \underbrace{(-24\cos^2 t \sin t \, dt)}_{dx} \underbrace{(8\cos^3 t)}_{x}$

$\Rightarrow \bar{x} \int_0^{\frac{\pi}{2}} \sin^7 t \cos^2 t \, dt = \int_0^{\pi/2} 8 \sin^7 t \cos^5 t \, dt$



EVALUATE USING BETA FUNCTIONS

$\Rightarrow \bar{x} \int_0^{\frac{\pi}{2}} 2 (\sin t)^{2\times 4 - 1} (\cos t)^{2 \times \frac{3}{2} - 1} dt = 8 \int_0^{\frac{\pi}{2}} 2 (\sin t)^{2\times 4 - 1} (\cos t)^{2\times 3 - 1} dt$

$\Rightarrow \bar{x}\, B(4, \tfrac{3}{2}) = 8\, B(4,3)$

$\Rightarrow \bar{x} \dfrac{\cancel{\Gamma(4)}\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{2})} = \dfrac{8\cancel{\Gamma(4)}\Gamma(3)}{\Gamma(7)}$

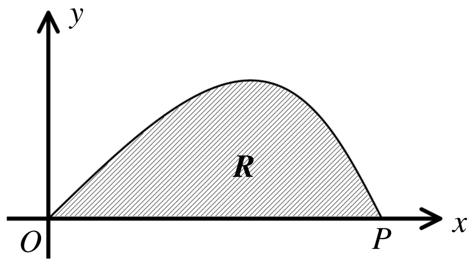
$\Rightarrow \bar{x} = \dfrac{8\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(7)\Gamma(\frac{3}{2})}$

$\Rightarrow \bar{x} = \dfrac{8 \times 2!}{6!} \times \dfrac{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times \cancel{\Gamma(\frac{3}{2})}}{\cancel{\Gamma(\frac{3}{2})}}$

$\Rightarrow \bar{x} = \dfrac{\cancel{16}}{720} \times \dfrac{9\times7\times5\times3}{\cancel{16}}$

$\Rightarrow \bar{x} = \dfrac{21}{16}$

Question 39



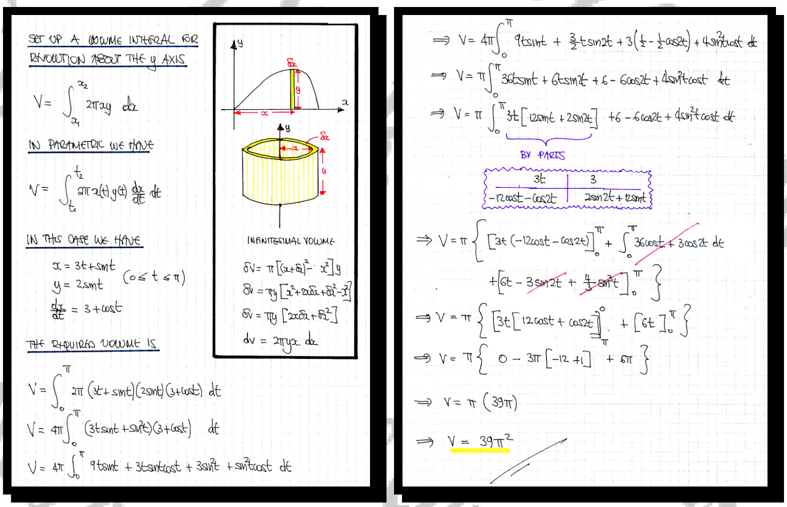
The figure above shows the finite region R, bounded by the coordinate axes and the curve with parametric equations

$$x = 3t + \sin t, \quad y = 2\sin t \ , \quad 0 \le t \le \pi .$$

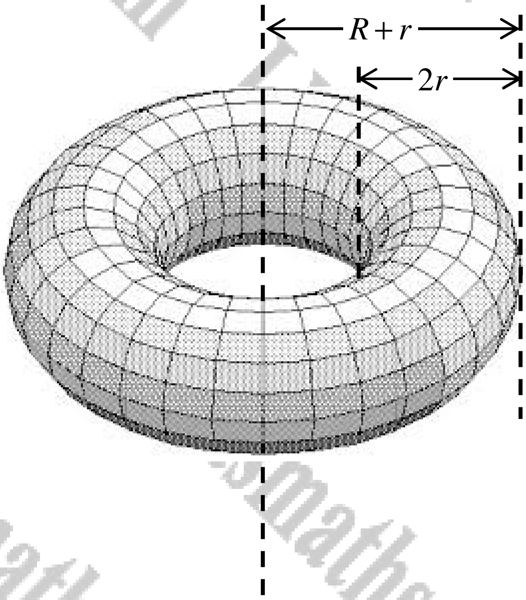
R is fully revolved about the y axis forming a solid of revolution.

Show that the volume of this solid is $39\pi^2$.

[ ], [proof]



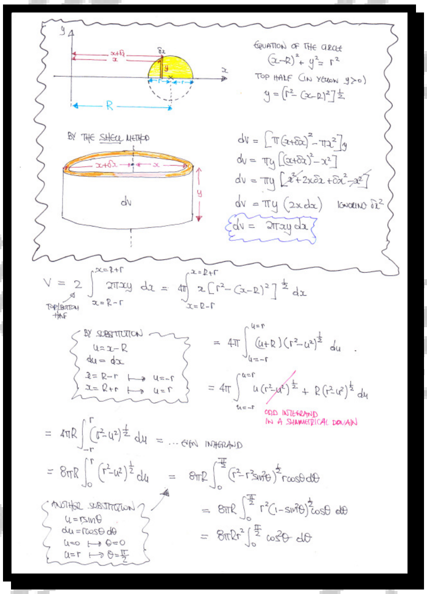
Question 40



Use direct integration in Cartesian coordinates to show the volume V of the circular ring torus, shown in the figure above, is given by

$$V = 2\pi^2 r^2 R, \quad 0 < r < R.$$

proof



EQUATION OF THE CIRCLE

$(x-R)^2 + y^2 = r^2$

TOP HALF (IN YELLOW $y>0$)

$y = \left[r^2 - (x-R)^2\right]^{\frac{1}{2}}$

BY THE SHELL METHOD

$dV = \left[\pi(x+\delta x)^2 - \pi x^2\right] y$

$dV = \pi y\left[(x+\delta x)^2 - x^2\right]$

$dV = \pi y\left[\cancel{x^2} + 2x\delta x + \delta x^2 - \cancel{x^2}\right]$

$dV = \pi y\,(2x\,dx)$ IGNORING δx^2

$dV = 2\pi xy\,dx$

$V = 2\int_{x=R-r}^{x=R+r} 2\pi xy\,dx = 4\pi\int_{x=R-r}^{x=R+r} x\left[r^2 - (x-R)^2\right]^{\frac{1}{2}} dx$

TOP/BOTTOM HALF

BY SUBSTITUTION
$u = x - R$
$du = dx$
$x = R-r \longmapsto u = -r$
$x = R+r \longmapsto u = r$

$= 4\pi\int_{u=-r}^{u=r} (u+R)(r^2-u^2)^{\frac{1}{2}}\,du$

$= 4\pi\int_{u=-r}^{u=r} \cancel{u(r^2-u^2)^{\frac{1}{2}}} + R(r^2-u^2)^{\frac{1}{2}}\,du$

ODD INTEGRAND IN A SYMMETRICAL DOMAIN

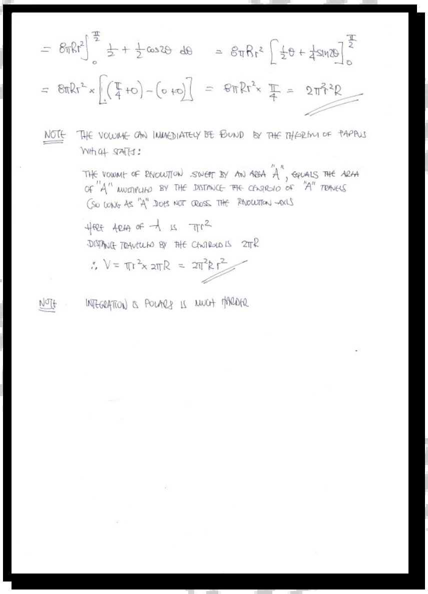
$= 4\pi R\int_{-r}^{r} (r^2-u^2)^{\frac{1}{2}}\,du = \ldots$ EVEN INTEGRAND

$= 8\pi R\int_0^r (r^2-u^2)^{\frac{1}{2}}\,du = 8\pi R\int_0^{\frac{\pi}{2}} (r^2 - r^2\sin^2\theta)^{\frac{1}{2}} r\cos\theta\,d\theta$

ANOTHER SUBSTITUTION
$u = r\sin\theta$
$du = r\cos\theta\,d\theta$
$u = 0 \longmapsto \theta = 0$
$u = r \longmapsto \theta = \frac{\pi}{2}$

$= 8\pi R\int_0^{\frac{\pi}{2}} r^2(1-\sin^2\theta)^{\frac{1}{2}}\cos\theta\,d\theta$

$= 8\pi R r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$



$= 8\pi R r^2\int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta = 8\pi R r^2\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$= 8\pi R r^2 \times \left[\left(\frac{\pi}{4}+0\right) - (0+0)\right] = 8\pi R r^2 \times \frac{\pi}{4} = 2\pi^2 r^2 R$

NOTE THE VOLUME CAN IMMEDIATELY BE FOUND BY THE THEOREM OF PAPPUS WHICH STATES:

THE VOLUME OF REVOLUTION SWEPT BY AN AREA "A", EQUALS THE AREA OF "A" MULTIPLIED BY THE DISTANCE THE CENTROID OF "A" TRAVELS (SO LONG AS "A" DOES NOT CROSS THE REVOLUTION AXIS

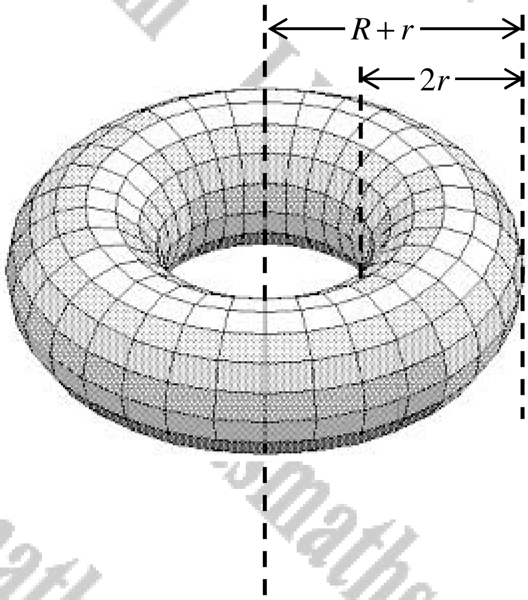
HERE AREA OF A IS πr^2

DISTANCE TRAVELLED BY THE CENTROID IS $2\pi R$

$\therefore V = \pi r^2 \times 2\pi R = 2\pi^2 R r^2$

NOTE INTEGRATION IN POLARS IS MUCH HARDER

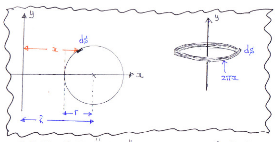
Question 41



Use direct integration in Cartesian coordinates to show the surface area S of the circular ring torus, shown in the figure above, is given by

$$S = (2\pi r)(2\pi R), \ 0 < r < R .$$

proof



- EQUATION OF THE "REVOLVING" CIRCLE IS $(x-R)^2 + y^2 = r^2$

Diff w.r.t x

$\Rightarrow 2(x-R) + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x-R}{y}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + 1 = \frac{(x-R)^2}{y^2} + 1$

$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{(x-R)^2+y^2}}{y} = \frac{\sqrt{(x-R)^2+r^2-(x-R)^2}}{y} = \frac{r}{y}$

$\Rightarrow \boxed{ds = \frac{r}{\left[r^2-(x-R)^2\right]^{\frac{1}{2}}}dx}$

HENCE

$S = \int_{x=R-r}^{x=R+r} 2\pi x \, ds = \int_{x=R-r}^{x=R+r} 2\pi x \times \frac{r}{+(r^2-(x-R)^2)^{\frac{1}{2}}} dx$

TOP HALF ONLY

$= 2\pi r \int_{x=R-r}^{x=R+r} x\left(r^2-(x-R)^2\right)^{-\frac{1}{2}} dx$

BY SUBSTITUTION

$u = x - R$

$du = dx$

$x = u + R$

$x = R - r \longmapsto u = -r$

$x = R + r \longmapsto u = r$

$= 2\pi r \int_{u=-r}^{u=r} (u+R)(r^2-u^2)^{-\frac{1}{2}} du$

$= 2\pi r \int_{u=-r}^{r} \overset{\text{ODD}}{\cancel{u(r^2-u^2)^{-\frac{1}{2}}}} + R\overset{\text{EVEN}}{(r^2-u^2)^{-\frac{1}{2}}} du$

$= 2\pi r R \int_{u=-r}^{u=r} \overset{\text{EVEN}}{(r^2-u^2)^{-\frac{1}{2}}} du$

$= 4\pi r R \int_0^{u=r} (r^2-u^2)^{-\frac{1}{2}} du$

$= 4\pi r R \int_0^{u=r} \frac{1}{\sqrt{r^2-u^2}} du$

THUS THE TOTAL (INCLUDING THE BOTTOM-HALF) IS GIVEN BY

$S = 2 \times 4\pi r R \int_{u=0}^{u=r} \frac{1}{\sqrt{r^2-u^2}} du$

$S = 8\pi r R \left[\arcsin \frac{u}{r}\right]_0^r$

$S = 8\pi r R \left[\arcsin 1 - \cancel{\arcsin 0}\right]$

$S = 8\pi r R \times \frac{\pi}{2}$

$S = 4\pi^2 r R$

$S = (2\pi r)(2\pi R)$

NOTE THE RESULT CAN BE CHECKED BY THE 2ND THEOREM OF PAPPUS

THE SURFACE GENERATED WHEN A CLOSED NON INTERSECTING CURVE ROTATES EQUAL THE LENGTH OF CURVE MULTIPLIED BY THE DISTANCE TRAVELLED BY THE CENTROID OF THE CURVE

HERE THE CURVE IS A CIRCLE RADIUS $r \Rightarrow 2\pi r$
THE CENTROID OF THIS CIRCLE COVERS $2\pi R$
HENCE $2\pi r \times 2\pi R$

Question 42

The circle with equation

$$x^2 + y^2 = 4,$$

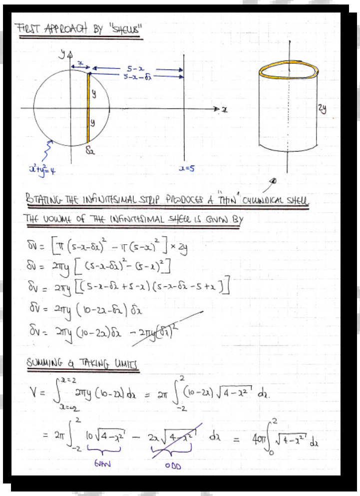
is rotated by 2π radians about the straight line with equation $x = 5$ axis to form a solid of revolution, known as a torus.

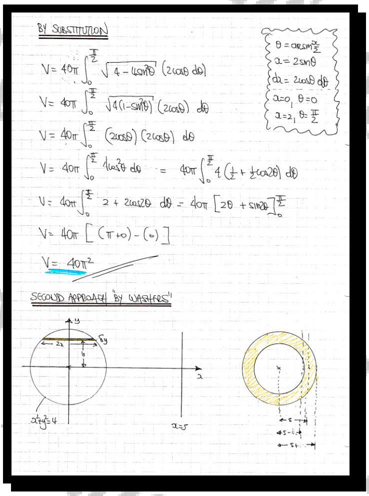
Use integration to show that the volume of the solid is

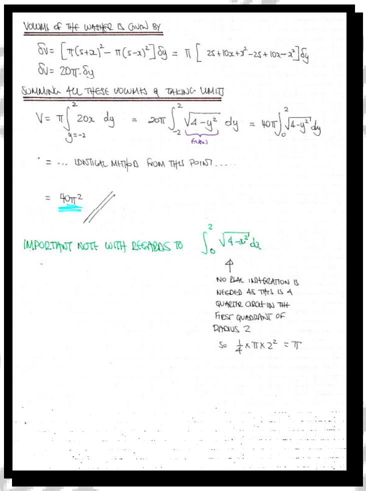
$$40\pi^2.$$

You may not use the formula for the volume of a torus or the theorem of Pappus.

proof







Question 43

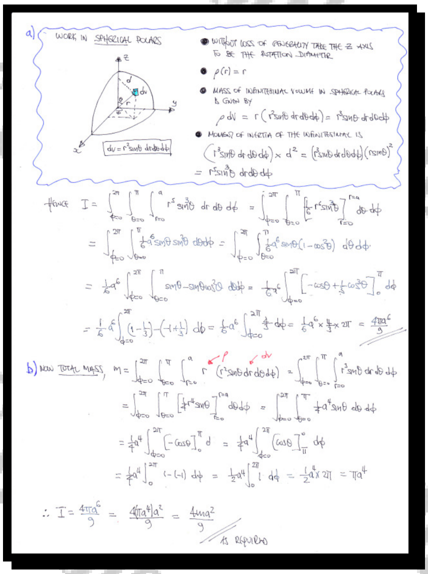
A solid uniform sphere of radius a, has variable density $\rho(r)=r$, where r is the radial distance of a given point from the centre of the sphere.

a) Use spherical polar coordinates, (r,θ,φ), to find the moment of inertia of this sphere I, about one of its diameters.

b) Given that the total mass of the sphere is m, show that

$$I=\frac{4}{9}ma^2.$$

$$\boxed{I=\frac{4}{9}\pi a^6}$$



a) WORK IN SPHERICAL POLARS

$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

- WITHOUT LOSS OF GENERALITY TAKE THE z AXIS TO BE THE ROTATION DIAMETER
- $\rho(r)=r$
- MASS OF INFINITESIMAL VOLUME IN SPHERICAL POLARS IS GIVEN BY

$$\rho\,dV = r\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = r^3\sin\theta\,dr\,d\theta\,d\phi$$

- MOMENT OF INERTIA OF THE INFINITESIMAL IS

$$\left(r^3\sin\theta\,dr\,d\theta\,d\phi\right)\times d^2 = \left(r^3\sin\theta\,dr\,d\theta\,d\phi\right)(r\sin\theta)^2$$

$$= r^5\sin^3\theta\,dr\,d\theta\,d\phi$$

HENCE
$$I=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^5\sin^3\theta\,dr\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{1}{6}r^6\sin^3\theta\right]_{r=0}^{r=a}d\theta\,d\phi$$

$$=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{6}a^6\sin\theta\sin^2\theta\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{6}a^6\sin\theta(1-\cos^2\theta)\,d\theta\,d\phi$$

$$=\frac{1}{6}a^6\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta-\sin\theta\cos^2\theta\,d\theta\,d\phi = \frac{1}{6}a^6\int_{\phi=0}^{2\pi}\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi}d\phi$$

$$=\frac{1}{6}a^6\int_{\phi=0}^{2\pi}\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)d\phi = \frac{1}{6}a^6\int_{\phi=0}^{2\pi}\frac{4}{3}\,d\phi = \frac{1}{6}a^6\times\frac{4}{3}\times 2\pi = \frac{4\pi a^6}{9}$$

b) NOW TOTAL MASS,
$$m=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \overset{\rho}{r}\,\overset{dV}{\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right)} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{1}{4}r^4\sin\theta\right]_{r=0}^{r=a}d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{4}a^4\sin\theta\,d\theta\,d\phi$$

$$=\frac{1}{4}a^4\int_{\phi=0}^{2\pi}\left[-\cos\theta\right]_0^{\pi}d = \frac{1}{4}a^4\int_{\phi=0}^{2\pi}\left[\cos\theta\right]_{\pi}^{0}d\phi$$

$$=\frac{1}{4}a^4\int_0^{2\pi}1-(-1)\,d\phi = \frac{1}{2}a^4\int_0^{2\pi}1\,d\phi = \frac{1}{2}a^4\times 2\pi = \pi a^4$$

$$\therefore I=\frac{4\pi a^6}{9}=\frac{4(\pi a^4)a^2}{9}=\frac{4ma^2}{9}$$

AS REQUIRED

Question 44

A solid sphere has equation

$$x^2 + y^2 + z^2 = a^2, \; a > 0.$$

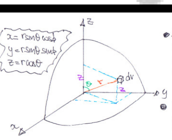
The sphere has variable density ρ, given by

$$\rho = k(a - z), \; k > 0.$$

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the sphere.

$$\boxed{\left(0, 0, -\tfrac{1}{5}a\right)}$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$



- AS THE DENSITY FUNCTION ONLY DEPENDS ON z THE CENTRE OF MASS LIES ON THE z AXIS
- WE MUST FIND THE TOTAL MASS OF THE SPHERE

$$M = \int_V \rho \, dv = \int_V k(a-z)\, dv$$

SWITCH INTO SPHERICAL POLAR CO-ORDINATES

$$M = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} k(a - r\cos\theta)\, r^2\sin\theta \, dr\, d\theta\, d\phi \quad (\text{DO "}\phi\text{" FIRST})$$

$$M = \int_{\theta=0}^{\pi}\int_{r=0}^{a}\int_{\phi=0}^{2\pi} k\left[ar^2\sin\theta - r^3\sin\theta\cos\theta\right] d\phi\, dr\, d\theta$$

$$M = 2\pi k \int_{\theta=0}^{\pi}\int_{r=0}^{a} ar^2\sin\theta - r^3\sin\theta\cos\theta \, dr\, d\theta$$

$$M = 2\pi k \int_{\theta=0}^{\pi} \left[\tfrac{1}{3}ar^3\sin\theta - \tfrac{1}{4}r^4\sin\theta\cos\theta\right]_{r=0}^{a} d\theta$$

$$M = 2\pi k \int_{\theta=0}^{\pi} \tfrac{1}{3}a^4\sin\theta - \tfrac{1}{4}a^4\sin\theta\cos\theta \, d\theta$$

$$M = 2\pi k\left[-\tfrac{1}{3}a^4\cos\theta - \tfrac{1}{4}a^4\sin^2\theta\right]_0^{\pi}$$

$$M = 2\pi k a^4\left[\tfrac{1}{3}\cos\theta + \tfrac{1}{4}\sin^2\theta\right]_{\pi}^{0}$$

$$M = 2\pi k a^4\left[\left(\tfrac{1}{3}+0\right) - \left(-\tfrac{1}{3}+0\right)\right]$$

$$\boxed{M = \tfrac{4}{3}\pi k a^4}$$

NEXT CONSIDER THE MOMENT OF AN INFINITESIMAL VOLUME dv ABOUT THE x-y PLANE (z-CO.ORD)

$$\begin{aligned}(\rho\, dv)z &= k(a-z)z \\ &= k(az - z^2) \\ &= k(ar\cos\theta - r^2\cos^2\theta)\end{aligned}$$

(IN SPHERICAL POLARS)

SUMMING UP & TAKING LIMITS

$$M\bar{z} = \iiint_V k\left[ar\cos\theta - r^2\cos^2\theta\right] r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$M\bar{z} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} k\left[ar^3\sin\theta\cos\theta - r^4\sin\theta\cos^2\theta\right] dr\, d\theta\, d\phi$$

$$\tfrac{4}{3}\cancel{k}\pi a^4\bar{z} = \cancel{k}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\tfrac{1}{4}ar^4\sin\theta\cos\theta - \tfrac{1}{5}r^5\sin\theta\cos^2\theta\right]_{r=0}^{a} d\theta\, d\phi$$

$$\tfrac{4}{3}\pi a^4\bar{z} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \tfrac{1}{4}a^5\sin\theta\cos\theta - \tfrac{1}{5}a^5\sin\theta\cos^2\theta \, d\theta\, d\phi$$

$$\tfrac{4\pi}{3a}\bar{z} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \tfrac{1}{4}\sin\theta\cos\theta - \tfrac{1}{5}\sin\theta\cos^2\theta \, d\theta\, d\phi$$

$$\tfrac{4\pi}{3a}\bar{z} = \int_{\phi=0}^{2\pi}\left[\tfrac{1}{8}\sin^2\theta + \tfrac{1}{15}\cos^3\theta\right]_0^{\pi} d\phi$$

$$\tfrac{4\pi}{3a}\bar{z} = \int_{\phi=0}^{2\pi}\left(0 - \tfrac{1}{15}\right) - \left(0 + \tfrac{1}{15}\right) d\phi$$

$$\tfrac{4\pi}{3a}\bar{z} = \int_{\phi=0}^{2\pi} -\tfrac{2}{15}\, d\phi$$

$$\tfrac{4\cancel{\pi}}{3a}\bar{z} = -\tfrac{2}{15}\times 2\cancel{\pi}$$

$$\bar{z} = -\tfrac{1}{5}a$$

IE BELOW THE x-y PLANE AT $\left(0, 0, -\tfrac{1}{5}a\right)$

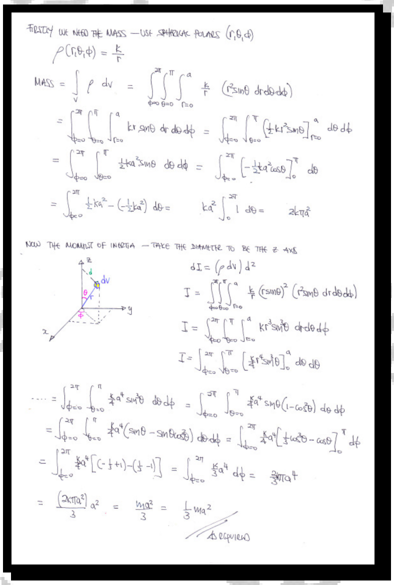
Question 45

A solid sphere has radius a and mass m.

The density ρ at any point in the sphere is inversely proportional to the distance of this point from the centre of the sphere

Show that the moment of inertia of this sphere about one of its diameters is $\frac{1}{3}ma^2$

$\boxed{\text{proof}}$



FIRSTLY WE NEED THE MASS — USE SPHERICAL POLARS (r, θ, ϕ)

$$\rho(r,\theta,\phi) = \frac{k}{r}$$

$$\text{MASS} = \int_V \rho \, dV = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \frac{k}{r}\left(r^2\sin\theta \, dr\,d\theta\,d\phi\right)$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} kr\sin\theta \, dr\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\tfrac{1}{2}kr^2\sin\theta\right]_{r=0}^{a} d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \tfrac{1}{2}ka^2\sin\theta \, d\theta\,d\phi = \int_{\phi=0}^{2\pi}\left[-\tfrac{1}{2}ka^2\cos\theta\right]_0^{\pi} d\theta$$

$$= \int_{\phi=0}^{2\pi} \tfrac{1}{2}ka^2 - \left(-\tfrac{1}{2}ka^2\right) d\theta = ka^2\int_0^{2\pi} 1 \, d\theta = 2k\pi a^2$$

NOW THE MOMENT OF INERTIA — TAKE THE DIAMETER TO BE THE z AXIS

$$dI = (\rho \, dV)\,d^2$$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \frac{k}{r}(r\sin\theta)^2\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right)$$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} kr^3\sin^3\theta \, dr\,d\theta\,d\phi$$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\tfrac{k}{4}r^4\sin^3\theta\right]_0^{a} d\theta\,d\theta$$

$$\ldots = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\tfrac{k}{4}a^4\sin^3\theta\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\tfrac{k}{4}a^4\sin\theta\left(1-\cos^2\theta\right)d\theta\,d\phi$$

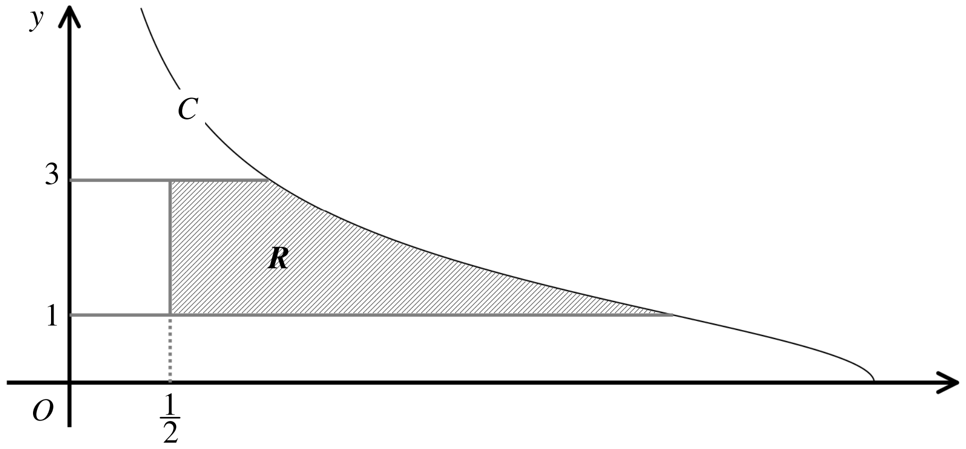
$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\tfrac{k}{4}a^4\left(\sin\theta - \sin\theta\cos^2\theta\right)d\theta\,d\phi = \int_{\phi=0}^{2\pi}\tfrac{k}{4}a^4\left[\tfrac{1}{3}\cos^3\theta - \cos\theta\right]_0^{\pi} d\phi$$

$$= \int_{\phi=0}^{2\pi}\tfrac{k}{4}a^4\left[\left(-\tfrac{1}{3}+1\right)-\left(\tfrac{1}{3}-1\right)\right] = \int_{\phi=0}^{2\pi}\tfrac{k}{3}a^4\,d\phi = \tfrac{2}{3}k\pi a^4$$

$$= \frac{\left(2k\pi a^2\right)}{3}a^2 = \frac{ma^2}{3} = \frac{1}{3}ma^2$$

AS REQUIRED

Question 46



The figure above shows the curve C with parametric equations

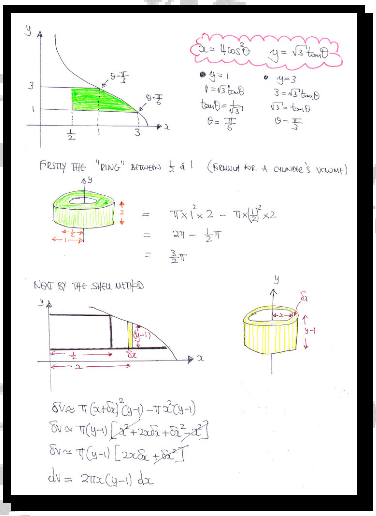
$$x = 4\cos^2\theta, \quad y = \sqrt{3}\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

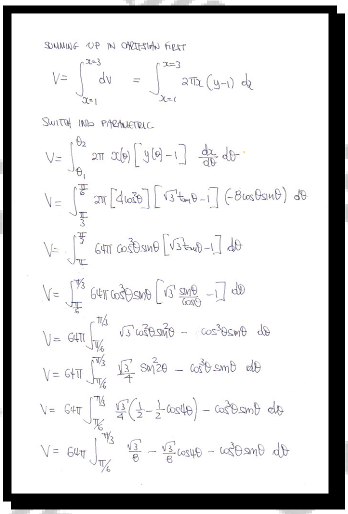
The finite region R shown shaded in the figure, bounded by C, and the straight lines with equations $y = 1$, $y = 3$ and $x = \frac{1}{2}$.

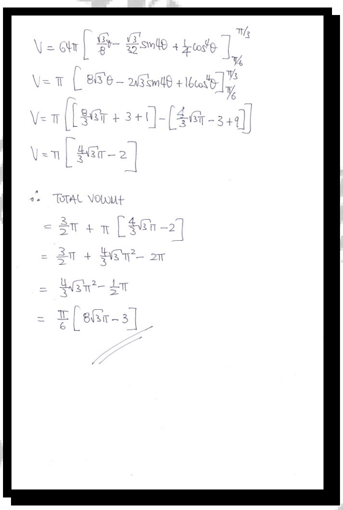
Use integration in parametric to find an exact value for the volume of the solid formed when R is fully revolved about the y axis.

[*you may only use the shell method in parametric in this question*]

$$\boxed{V = \frac{\pi}{6}\left[8\pi\sqrt{3} - 3\right]}$$







Question 47

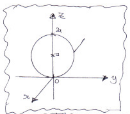
A solid uniform sphere has mass M and radius a.

Use spherical polar coordinates, (r,θ,φ), and direct calculus methods, to show that the moment of inertial of this sphere about one of its tangents is $\frac{7}{5}Ma^2$.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

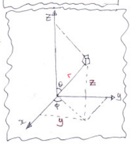
proof

- IN ORDER TO USE <u>DIRECT</u> INTEGRATION WE SHALL TRANSLATE THE SPHERE WITH EQUATION $x^2+y^2+z^2=a^2$ "UP" BY a SO THAT THE xy PLANE IS A TANGENT PLANE AND THE ROTATION AXIS IS THE x AXIS



$$x^2+y^2+(z-a)^2=a^2$$
$$x^2+y^2+z^2-2az+a^2=a^2$$
$$x^2+y^2+z^2=2az$$

- USE SPHERICAL POLARS (r,θ,ϕ)



- LET $\rho = \frac{m}{\frac{4}{3}\pi a^3} = \frac{3m}{4\pi a^3}$ = MASS PER UNIT VOLUME
- AN INFINITESIMAL VOLUME ELEMENT HAS MASS

$$\rho\, dV = \rho\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

- THE EQUATION OF THE SPHERE IN SPHERICAL POLARS IS

$$x^2+y^2+z^2 = 2az$$
$$r^2 = 2a(r\cos\theta)$$
$$r^2 = 2ar\cos\theta$$
$$\boxed{r = 2a\cos\theta}$$

- THE DISTANCE OF THE INFINITESIMAL FROM THE x AXIS IS $\sqrt{y^2+z^2}$, SO THE MOMENT OF INERTIA OF THE INFINITESIMAL ABOUT THE x AXIS IS

$$I = m\times d^2 = (\rho r^2\sin\theta\, dr\, d\theta\, d\phi)\times(y^2+z^2)$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$\left[y^2+z^2 = r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\right]$ so $I = md^2$

BECOMES:

$$md^2 = (\rho r^2\sin\theta\, dr\, d\theta\, d\phi)(r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)$$
$$= \rho r^4(\sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta)\, dr\, d\theta\, d\phi$$

SUMMING UP: $\displaystyle\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=2a\cos\theta} \rho r^4(\sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta)\, dr\, d\theta\, d\theta$

(NOTE THAT $0\le\theta\le\frac{\pi}{2}$ AS SPHERE IS TRANSLATED)

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left[r^5\right]_{r=0}^{2a\cos\theta}(\sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta)\, d\theta\, d\phi$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} 32a^5\cos^5\theta\sin^3\theta\sin^2\phi + 32a^5\cos^7\theta\sin\theta\ \ d\theta\, d\phi$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\left\{16a^5\sin^2\phi\int_{\theta=0}^{\frac{\pi}{2}}2\cos^5\theta\sin^3\theta\, d\theta + \left[-4a^5\cos^8\theta\right]_0^{\frac{\pi}{2}}\right\}d\phi$$

$$B(3,2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{2!\times 1!}{4!} = \frac{1}{12}$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\left\{\frac{4}{3}a^5\sin^2\phi + [0+4a^5]\right\}d\phi$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\frac{4}{3}a^5\sin^2\phi + 4a^5\, d\phi = \frac{1}{5}\rho a^5\int_{\phi=0}^{2\pi}\frac{4}{3}\sin^2\phi + 4\ d\phi$$

$$I = \frac{1}{5}\rho a^5\int_0^{2\pi}\frac{4}{3}\left(\frac{1}{2}-\frac{1}{2}\cos 2\phi\right) + 4\ d\phi$$

$$I = \frac{1}{5}\rho a^5\int_0^{2\pi}\frac{2}{3} - \cancel{\frac{2}{3}\cos 2\phi} + 4\ d\phi$$

$$I = \frac{1}{5}\rho a^5\times\int_0^{2\pi}\frac{14}{3}\, d\phi$$

$$I = \frac{14}{15}\rho a^5\int_0^{2\pi}1\ d\phi$$

$$I = \frac{14}{15}\rho a^5\times 2\pi$$

$$I = \frac{28}{15}\rho a^5$$

$$I = \frac{28}{15}\left(\frac{3m}{4\pi a^3}\right)a^5$$

$$I = \frac{7}{5}ma^2$$

AS REQUIRED

Question 48

$$\mathbf{F}(x,y) \equiv \left(-\frac{y}{x^2+y^2}\right)\mathbf{i} + \left(\frac{x}{x^2+y^2}\right)\mathbf{j}.$$

By considering the line integral of $\mathbf{F}$ over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta = \frac{2\pi}{ab},$$

where a and b are real constants.

You may assume without proof that the line integral of $\mathbf{F}$ yields the same value over any simple closed curve which contains the origin.

$\boxed{}$, $\boxed{\text{proof}}$

SETTING UP THE LINE INTEGRAL OVER A CLOSED PATH C, WHICH CONTAINS O

$$\oint_C \underline{F}\cdot d\underline{r} = \oint_C \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\cdot(dx,dy) = \oint \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

PARAMETRIZE OVER A UNIT CIRCLE WHICH CONTAINS THE ORIGIN

- $x = \cos\theta$
- $y = \sin\theta$
- $dx = -\sin\theta\, d\theta$
- $dy = \cos\theta\, d\theta$
- θ RUNS FROM 0 TO 2π

$$= \int_{\theta=0}^{2\pi} \frac{-\sin\theta}{\cos^2\theta+\sin^2\theta}(-\sin\theta\, d\theta) + \frac{\cos\theta}{\cos^2\theta+\sin^2\theta}(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} \frac{\sin^2\theta}{\cos^2\theta+\sin^2\theta}\,d\theta + \frac{\cos^2\theta}{\cos^2\theta+\sin^2\theta}\,d\theta = \int_0^{2\pi} \cancel{\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta+\sin^2\theta}}\,d\theta$$

$$= \int_0^{2\pi} 1\, d\theta = 2\pi$$

NEXT PARAMETRIZE OVER AN ELLIPSE

- $x = a\cos\theta$
- $y = b\sin\theta$
- $dx = -a\sin\theta\, d\theta$
- $dy = b\cos\theta\, d\theta$
- θ RUNS FROM 0 to 2π

HENCE WE NOW HAVE

$$\Rightarrow \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi$$

$$\Rightarrow \int_{\theta=0}^{2\pi} \frac{-b\sin\theta}{a^2\cos^2\theta+b^2\sin^2\theta}(-a\sin\theta\, d\theta) + \frac{a\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}(b\cos\theta\, d\theta) = 2\pi$$

$$\Rightarrow \int_{\theta=0}^{2\pi} \frac{ab\sin^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta + \frac{ab\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{ab\sin^2\theta+ab\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{ab(\sin^2\theta+\cos^2\theta)}{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta = \frac{2\pi}{ab}$$

A LITTLE ASIDE INTO THIS QUESTION

$$\underline{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

$$\underline{\nabla}\wedge\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left[0-0,\ 0-0,\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]$$

LOOKING AT THE k COMPONENT

$$\frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right] = \frac{(x^2+y^2)\times 1 - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)\times 1 - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2} = 0$$

YET THE INTEGRATION OVER A CLOSED PATH DID NOT YIELD ZERO!

LOOK FURTHER

$$\underset{\substack{\uparrow \\ \frac{-y}{x^2+y^2}}}{\frac{\partial f}{\partial x}}\,dx + \underset{\substack{\uparrow \\ \frac{x}{x^2+y^2}}}{\frac{\partial f}{\partial y}}\,dy$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{x}{x^2+y^2}\right] = \frac{2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} = 0$$

... IT SATISFIES LAPLACE'S EQUATION, SO HARMONIC ...

... COMPLEX NUMBERS, RESIDUES ETC COME TO MIND ....

$f(z) = \frac{1}{z}$, AND THE INTEGRATION OVER A UNIT CIRCLE AT THE ORIGIN

$$\oint_C \frac{1}{z}\,dz = 2\pi i \quad \text{(STANDARD)}$$

$$\oint_C \frac{1}{x+iy}(dx+i\,dy) = 2\pi i$$

$$\oint_C \frac{x-iy}{x^2+y^2}(dx+i\,dy) = 2\pi i$$

$$\oint_C \left(\frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i\right)(dx+i\,dy) = 2\pi i$$

$$\oint_C \frac{1}{x^2+y^2}\left[x\,dx + ix\,dy - iy\,dx + y\,dy\right] = 2\pi i$$

$$\oint_C \underbrace{\left[\frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy\right]}_{\text{THIS YIELDS ZERO}} + \underbrace{\left[\frac{-y}{x^2+y^2} + \frac{x}{x^2+y^2}\right]}_{\text{THIS ALWAYS YIELDS } 2\pi \text{, OVER ANY CLOSED PATH THAT CONTAINS THE ORIGIN}} i = 2\pi i$$

Question 49

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

It is asserted that

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos\left(2x^2\right)\, dx = \sqrt{\frac{\pi\phi}{5}}.$$

By considering the real part of a suitable function, use double integration in plane polar coordinates to prove the validity of the above result.

You may assume the principal value in any required complex evaluation.

$\boxed{}$, $\boxed{\text{proof}}$

CONSIDER THE FOLLOWING INTEGRAL

$$\Rightarrow I = \int_{-\infty}^{\infty} e^{-x^2} e^{i(2x^2)}\, dx = \int_{-\infty}^{\infty} e^{-x^2(1-2i)}\, dx$$

OUR INTEGRAL IS THE REAL PART OF "I"

$$\Rightarrow I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2(1-2i)}\, dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2(1-2i)}\, dy\right]$$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2(1-2i)} \times e^{-y^2(1-2i)}\, dx\, dy$$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(1-2i)(x^2+y^2)}\, dx\, dy$$

THE AREA OF INTEGRATION IS THE ENTIRE x-y PLANE SO VERY EASY TO TRANSFORM INTO POLARS

$$\Rightarrow I^2 = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-(1-2i)r^2} (r\, dr\, d\theta)$$

IN POLARS
$x^2+y^2=r^2$
$dx\,dy = r\,dr\,d\theta$

CARRY OUT THE θ INTEGRATION FIRST

$$\Rightarrow I^2 = 2\pi \int_0^{\infty} r e^{-(1-2i)r^2}\, dr$$

$$\Rightarrow I^2 = 2\pi \left[-\frac{1}{2} \times \frac{1}{1-2i} \times e^{-(1-2i)r^2}\right]_0^{\infty}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[e^{-(1-2i)r^2}\right]_{\infty}^{0}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[e^{-r^2} \times e^{2r^2 i}\right]_{\infty}^{0}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[e^{-r^2}\left(\cos 2r^2 + i \sin 2r^2\right)\right]_{\infty}^{0}$$

$$\Rightarrow I^2 = \frac{\pi}{1-2i}\left[1(1+0i) - 0\left(\text{"}\cos\infty + i\sin\infty\text{"}\right)\right]$$

FINITE

$$\Rightarrow I^2 = \frac{\pi}{1-2i} = \frac{\pi(1+2i)}{5}$$

$$\Rightarrow I = \sqrt{\frac{\pi}{5}}\sqrt{1+2i}$$

& RECALL WE WANT THE REAL PART OF THIS AT THE END

NOW WE NEED THE PRINCIPAL VALUE OF $\sqrt{1+2i}$

$$\Rightarrow z^2 = 1+2i$$

$$\Rightarrow z^2 = |1+2i|\, e^{i\arg(1+2i)}$$

← PRINCIPAL ONLY

$$\Rightarrow z^2 = \sqrt{5}\, e^{i\theta}$$

WHERE $\theta = \arctan 2$

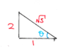
$$\Rightarrow z^2 = 5^{\frac{1}{2}} e^{i\theta}$$

$$\Rightarrow z = 5^{\frac{1}{4}} e^{i\frac{\theta}{2}}$$

RETURNING TO THE INTEGRAL

$$\Rightarrow I = \sqrt{\frac{\pi}{5}} \times 5^{\frac{1}{4}} e^{i\frac{\theta}{2}} \qquad [\theta = \arctan 2]$$

PROCEED AS FOLLOWS



$\tan\theta = 2$
$\cos\theta = \frac{1}{\sqrt{5}}$

$$\Rightarrow \cos 2A = 2\cos^2 A - 1$$

$$\Rightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow \frac{1}{\sqrt{5}} = 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow 1 + \frac{1}{\sqrt{5}} = 2\cos^2\frac{\theta}{2}$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{2}\left(\frac{\sqrt{5}+1}{\sqrt{5}}\right)$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{\sqrt{5}}\phi$$

$$\Rightarrow \cos\frac{\theta}{2} = +\sqrt{\frac{\phi}{\sqrt{5}}}$$

RETURN TO THE MAIN QUESTION

$$\Rightarrow I = \sqrt{\frac{\pi}{5}} \times 5^{\frac{1}{4}} \times e^{i\frac{\theta}{2}}$$

$$\Rightarrow I = \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}} \times \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\cos(2x^2)\, dx = \mathrm{Re}\left[\frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]\right]$$

$$= \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}} \times \cos\frac{\theta}{2}$$

$$= \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times 5^{\frac{1}{4}} \times \sqrt{\frac{\phi}{\sqrt{5}}}$$

$$= \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{2}}} \times \cancel{5^{\frac{1}{4}}} \times \frac{\phi^{\frac{1}{2}}}{\cancel{5^{\frac{1}{4}}}}$$

$$= \frac{\pi^{\frac{1}{2}}\phi^{\frac{1}{2}}}{5^{\frac{1}{2}}}$$

$$= \sqrt{\frac{\pi\phi}{5}}$$

AS REQUIRED

Question 50

The point $S\left[x_1, f(x_1)\right]$ and the point $T\left[x_2, f(x_2)\right]$ lie on the curve C with Cartesian equation $y = f(x)$.

The straight line L has equation $y = mx + c$, where m and c are constants.

The finite region R is bounded by C, L, and perpendicular straight line segments from S to L and from T to L.

A solid is formed by revolving R about L, by a complete turn.

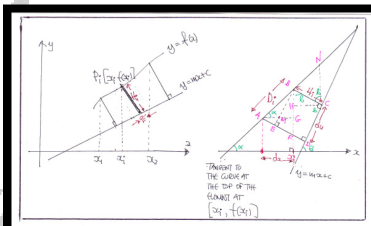
a) Show that the area of R is given by

$$\frac{1}{m^2+1}\int_{x_1}^{x_2} \left[f(x) - mx - c\right]\left[1 + m f'(x)\right]\,dx.$$

b) Show that the volume of the solid of revolution is given by

$$\frac{\pi}{\left(m^2+1\right)^{\frac{3}{2}}}\int_{x_1}^{x_2} \left[f(x) - mx - c\right]^2\left[1 + m f'(x)\right]\,dx.$$

proof



AUXILIARIES

$\tan\beta = m$ I.E. THE CONSTANT GRADIENT OF THE LINE

$\tan\alpha = f'(x_i)$ I.E. THE GRADIENT OF THE TANGENT AT $P_i(x_i, f(x_i))$

LOOKING AT AN ENLARGED VERSION OF THE INFINITESIMAL STRIP ON THE DIAGRAM ON THE RIGHT

$\frac{dx}{D_i} = \cos\alpha$

$\boxed{D_i = \frac{dx}{\cos\alpha}}$

NEXT LOADS OF ANGLES TRYING TO RELATE du to dx

$H\hat{C}D = \beta$ (ALTERNATE)

$\Rightarrow B\hat{C}H = 90 - \beta$

$\Rightarrow H\hat{B}C = \beta$

$G\hat{A}B = \alpha$ (CORRESPONDING)

$\Rightarrow A\hat{B}G = 90 - \alpha$

IF $\overline{BE}$ IS PARALLEL TO $\overline{CD}$, THEN

$\Rightarrow B\hat{M}G = \beta$

$\Rightarrow MBG = 90 - \beta$

THUS IF $A\hat{B}G = 90 - \alpha$

$M\hat{B}G = 90 - \beta$

$A\hat{B}E = A\hat{B}G - M\hat{B}G$

$A\hat{B}E = (90 - \alpha) - (90 - \beta)$

$A\hat{B}E = \beta - \alpha$

THUS PUTTING THINGS TOGETHER

$du = |CD| = |BE| = D_i\cos(\beta - \alpha)$

$du = \frac{dx}{\cos\alpha}\left[\cos\beta\cos\alpha + \sin\beta\sin\alpha\right]$

$du = dx\left[\cos\beta + \tan\alpha\sin\beta\right]$

$du = dx\left[\cos\beta + f'(x_i)\sin\beta\right]$

NEXT WE NEED TO EXPRESS $\cos\beta$ & $\sin\beta$ IN TERMS OF $\tan\beta$ WHICH IS ESSENTIALLY A CONSTANT m

$\cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+m^2}}$

$\sin\beta = \frac{\sin\beta}{\cos\beta}\cos\beta = \tan\beta\cos\beta = \frac{\tan\beta}{\sqrt{1+\tan^2\beta}} = \frac{m}{\sqrt{1+m^2}}$

RETURNING TO THE ELEMENT

$\boxed{du = dx\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}f'(x_i)\right]}$

NEXT WE NEED THE HEIGHT OF THE STRIP H_i

$H_i = |NC|\cos\beta$

DIFFERENCE IN y CO-ORDS

$H_i = \left[f(x_i) - (mx_i + c)\right]\cos\beta$

$\boxed{H_i = \frac{f(x_i) - mx_i - c}{\sqrt{1+m^2}}}$

TO FIND AREA WE SUM ALL THE STRIPS

$A = \int_{x_1}^{x_2} H(x)\,du$

$A = \int_{x_1}^{x_2} \frac{f(x) - mx - c}{\sqrt{1+m^2}}\,dx\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}f'(x)\right]$

$A = \frac{1}{m^2+1}\int_{x_1}^{x_2}\left[f(x) - mx - c\right]\left[1 + mf'(x)\right]dx$

TO FIND THE VOLUME WE SUM INFINITESIMAL DISCS OF RADIUS H_i AND THICKNESS du

$V = \pi\int_{x_1}^{x_2}\left[H(x)\right]^2 du$

$V = \pi\int_{x_1}^{x_2}\left[\frac{f(x) - mx - c}{\sqrt{1+m^2}}\right]^2 dx\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}f'(x)\right]$

$V = \frac{\pi}{(1+m^2)^{\frac{3}{2}}}\int_{x_1}^{x_2}\left(f(x) - mx - c\right)^2\left(1 + mf'(x)\right)dx$

Question 51

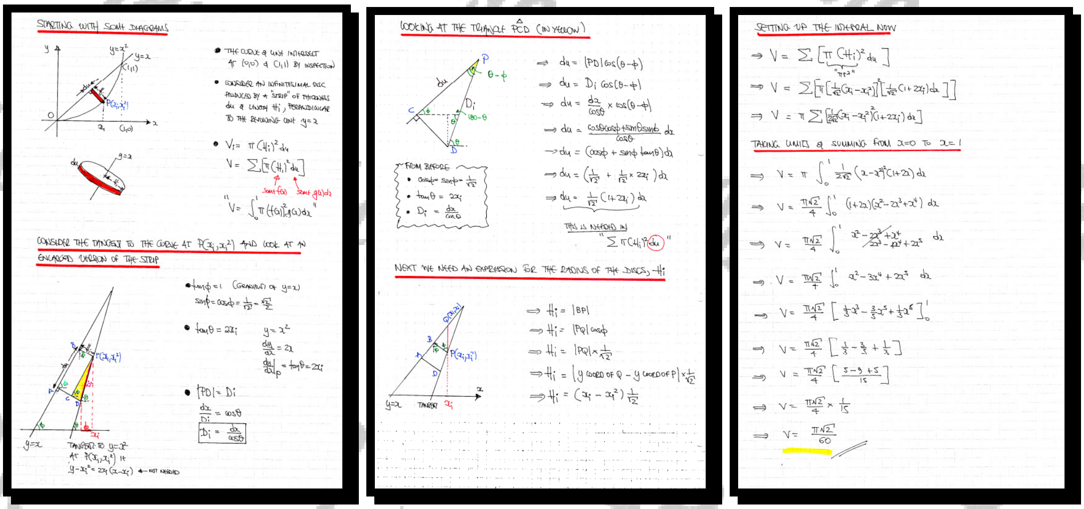
A curve C and a straight line L have respective equations

$$y = x^2 \quad \text{and} \quad y = x\,.$$

The finite region bounded by C and L is rotated around L by a full turn, forming a solid of revolution S.

Find, in exact form, the volume of S.

$\boxed{}$, $\boxed{\dfrac{\pi\sqrt{2}}{60}}$



Question 52

A curve C has equation

$$y = (x-2)(6-x), \ x \in \mathbb{R}.$$

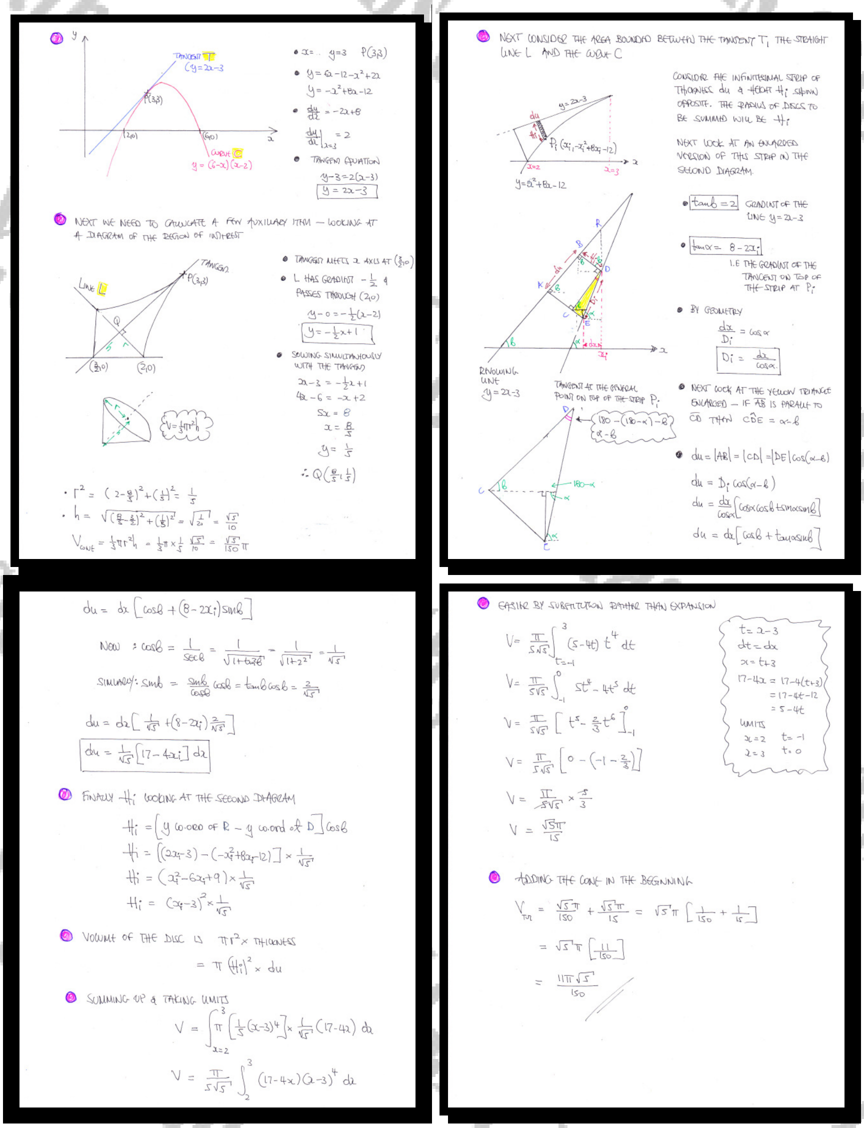
The straight line T is the tangent to C at the point where $x = 3$.

The finite region R is bounded by C, T, and the x axis.

A solid S is formed by revolving R about T, by a complete turn.

Find, in exact form, the volume of S.

$$\boxed{\frac{11\pi\sqrt{5}}{150}}$$



Question 53

Find the general solution of the following equation

$$\frac{d}{dx}\left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin\left(t^2\right)+\cos\left(2t^2\right)\ dx\right] = -\sqrt{\frac{2}{x}}, \quad x \in \mathbb{R}.$$

$\boxed{x = \frac{1}{4}\pi\left(4k-1\right) \quad k \in \mathbb{Z}}$

PROCEED BY LEIBNITZ INTEGRAL RULE & NOTE $\frac{d}{dx}\left(\frac{\pi}{6}\right)=0$

$$\frac{d}{dx}\int_{\frac{\pi}{6}}^{\sqrt{2x}} \sin(t^2)+\cos(2t^2)\ dt = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow \sin\left(\sqrt{2x}\right)^2 \times \frac{d}{dx}\left(\sqrt{2x}\right) + \cos\left[2\left(\sqrt{2x}\right)^2\right] = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow \left[\sin 2x + \cos 4x\right] \times \frac{d}{dx}\left[\sqrt{2}x^{\frac{1}{2}}\right] = -\frac{\sqrt{2}}{\sqrt{x}}$$

$$\Rightarrow \left(\sin 2x + \cos 4x\right) \times \frac{1}{2}\cancel{\sqrt{2}}\cancel{x^{-\frac{1}{2}}} = -\cancel{\sqrt{2}}\cancel{x^{-\frac{1}{2}}}$$

$$\Rightarrow \sin 2x + \cos 4x = -2$$

NO IDENTITIES NEEDED HERE – JUST NEED A COMMON SOLUTION

- $\sin 2x = -1$

  $2x = -\frac{\pi}{2} \pm 2n\pi \quad n = 0,1,2,3,\ldots$

  $2x = -\frac{\pi}{2}\left[1 \pm 4n\right]$

  $x = \frac{1}{4}\pi\left(-1 \pm 4n\right) \qquad x = \ldots -\frac{9\pi}{4}, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$

- $\cos 4x = -1$

  $4x = \pi \pm 2n\pi \quad n = 0,1,2,3,\ldots$

  $4x = \pi\left(1 \pm 2n\right)$

  $x = \frac{1}{4}\pi\left(1 \pm 2n\right) \qquad x = \ldots -\frac{5}{4}\pi, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

THUS COMMON SOLUTIONS ARE THOSE OF THE SINE

$$\therefore x = \frac{1}{4}\pi\left(4k-1\right) \quad k \in \mathbb{Z}$$