INTEGRAL EQUATIONS

Question 1

The non zero function $f(x)$ satisfies the integral equation

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx\,, \qquad f(0)=\frac{1}{4}.$$

Use the substitution $f(x)=\left(\frac{dy}{dx}\right)^2$, to find a simplified expression for $f(x)$.

$\boxed{}$, $\boxed{f(x)=\frac{1}{4}e^{4x}}$

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx \qquad x=0 \quad f(x)=\frac{1}{4}$$

- START WITH THE SUBSTITUTION GIVEN

$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \sqrt{\left(\frac{dy}{dx}\right)^2}\,dx$

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- INTEGRATE THE R.H.S

$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = y + k$

- SQUARING BOTH SIDES

$\Rightarrow \int \left(\frac{dy}{dx}\right)^2 dx = (y+k)^2$

- DIFFERENTIATE BOTH SIDES WITH RESPECT TO x

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 2(y+k)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = 2(y+k)$

$\Rightarrow \frac{1}{y+k}\,dy = 2\,dx$

$\Rightarrow \int \frac{1}{y+k}\,dy = \int 2\,dx$

$\Rightarrow \ln|y+k| = 2x + C$

$\Rightarrow y + k = e^{2x+C}$

$\Rightarrow y + k = Ae^{2x} \quad (A=e^C)$

$\Rightarrow y = Ae^{2x} + k$

- APPLY CONDITION $\quad x=0,\ \left(\frac{dy}{dx}\right)^2 = f(x) = \frac{1}{4}$

$x=0 \quad \frac{dy}{dx} = \frac{1}{2}$

$\Rightarrow \frac{dy}{dx} = 2Ae^{2x}$

$\Rightarrow \frac{1}{2} = 2Ae^{0}$

$\Rightarrow A = \frac{1}{4}$

- FINALLY WE HAVE

$f(x) = \left(\frac{dy}{dx}\right)^2$

$f(x) = \left(2Ae^{2x}\right)^2$

$f(x) = \left(2\times\frac{1}{4}e^{2x}\right)^2$

$f(x) = \left(\frac{1}{2}e^{2x}\right)^2$

$f(x) = \frac{1}{4}e^{4x}$

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Question 2

The non zero functions $u(x)$ and $v(x)$ satisfy the integral equations

$$\int u(x)\,dx = x^2 u(x) \qquad \text{and} \qquad \int u(x)v(x)\,dx = \left[\int u(x)\,dx\right]\left[\int v(x)\,dx\right].$$

Determine, in terms of an arbitrary constant, a simplified expression for $u(x)$ and a similar expression for $\left[v(x)\right]^2$.

$\boxed{\text{SP}}$, $\boxed{u(x) = \dfrac{A\mathrm{e}^{-\frac{1}{x}}}{x^2}}$, $\boxed{\left[v(x)\right]^2 = \dfrac{B}{(1-x)^2\left(1-x^2\right)}}$

- STARTING WITH

$$\int u\,dx = ux^2$$

- DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d}{dx}\int u\,dx = \frac{d}{dx}\left(ux^2\right)$$
$$\Rightarrow u = \frac{du}{dx}x^2 + u(2x)$$
$$\Rightarrow x^2\frac{du}{dx} = u - 2ux$$
$$\Rightarrow x^2\frac{du}{dx} = u(1-2x)$$
$$\Rightarrow \frac{1}{u}\,du = \frac{1-2x}{x^2}\,dx$$
$$\Rightarrow \int \frac{1}{u}\,du = \int \frac{1}{x^2} - \frac{2}{x}\,dx$$
$$\Rightarrow \ln|u| = -\frac{1}{x} - 2\ln|x| + C$$
$$\Rightarrow u = e^{C - \frac{1}{x} + \ln x^{-2}}$$
$$\Rightarrow u = e^{C} \times e^{-\frac{1}{x}} \times e^{\ln\frac{1}{x^2}}$$
$$\Rightarrow u = \frac{A}{x^2}e^{-\frac{1}{x}}$$

- NEXT WE PROCEED WITH

$$\int uv\,dx = \left[\int u\,dx\right]\left[\int v\,dx\right]$$

- DIFFERENTIATING W.R.T x

$$\Rightarrow \frac{d}{dx}\int uv\,dx = \frac{d}{dx}\left[\left[\int u\,dx\right]\left[\int v\,dx\right]\right]$$
$$\Rightarrow uv = \frac{d}{dx}\left[\int u\,dx\right] \times \int v\,dx + \int u\,dx \times \frac{d}{dx}\left[\int v\,dx\right]$$
$$\Rightarrow uv = u\int v\,dx + v\int u\,dx$$
$$\Rightarrow uv = u\int v\,dx + v\left(ux^2\right)$$
$$\Rightarrow v = \int v\,dx + vx^2$$
$$\Rightarrow v - vx^2 = \int v\,dx$$
$$\Rightarrow v(1-x^2) = \int v\,dx$$

DIFFERENTIATE W.R.T x AGAIN

$$\Rightarrow \frac{dv}{dx}(1-x^2) + v(-2x) = \frac{d}{dx}\int v\,dx$$
$$\Rightarrow \frac{dv}{dx}(1-x^2) - 2vx = v$$
$$\Rightarrow \frac{dv}{dx}(1-x^2) = v + 2vx$$
$$\Rightarrow \frac{dv}{dx}(1-x^2) = v(1+2x)$$

- SEPARATING VARIABLES AND INTEGRATING

$$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{2x+1}{1-x^2}\,dx$$
$$\Rightarrow \ln|v| = \int \frac{2x+1}{(1-x)(1+x)}\,dx$$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\Rightarrow \ln|v| = \int \frac{\frac{3}{2}}{1-x} - \frac{\frac{1}{2}}{1+x}\,dx$$
$$\Rightarrow 2\ln|v| = \int \frac{3}{1-x} - \frac{1}{1+x}\,dx$$
$$\Rightarrow \ln v^2 = -3\ln|1-x| - \ln|1+x| + \ln A.$$
$$\Rightarrow \ln v^2 = \ln\left|\frac{B}{(1-x)^3(1+x)}\right|$$
$$\Rightarrow v^2 = \frac{B}{(1-x)^3(1+x)}$$

OR

$$v^2 = \frac{B}{(1-x)^2(1-x^2)}$$