

Created by T. Madas

INTEGRAL EQUATIONS

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Question 1

The non zero function $f(x)$ satisfies the integral equation

$$\sqrt{\int f(x) dx} = \int \sqrt{f(x)} dx, \quad f(0) = \frac{1}{4}.$$

Use the substitution $f(x) = \left(\frac{dy}{dx}\right)^2$, to find a simplified expression for $f(x)$.

$$\boxed{}, \quad \boxed{f(x) = \frac{1}{4}e^{4x}}$$

Handwritten solution for Question 1:

Given: $\sqrt{\int f(x) dx} = \int \sqrt{f(x)} dx$, $x=0, f(0) = \frac{1}{4}$

• START WITH THE SUBSTITUTION GIVEN

$$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \sqrt{\left(\frac{dy}{dx}\right)^2} dx$$

$$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = \int \frac{dy}{dx} dx$$

• INTEGRATE THE R.H.S

$$\Rightarrow \sqrt{\int \left(\frac{dy}{dx}\right)^2 dx} = y + k$$

• SQUARE BOTH SIDES

$$\Rightarrow \int \left(\frac{dy}{dx}\right)^2 dx = (y+k)^2$$

• DIFFERENTIATE BOTH SIDES WITH RESPECT TO x

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 2(y+k) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2(y+k)$$

$$\Rightarrow \frac{1}{y+k} dy = 2 dx$$

$$\Rightarrow \int \frac{1}{y+k} dy = \int 2 dx$$

$$\Rightarrow \ln|y+k| = 2x + C$$

• AFTER CONDITIONS $x=0, \left(\frac{dy}{dx}\right)^2 = f(0) = \frac{1}{4}$
 $x=0, \frac{dy}{dx} = \frac{1}{2}$

$$\Rightarrow y + k = e^{2x+C}$$

$$\Rightarrow y + k = Ae^{2x} \quad (A = e^C)$$

$$\Rightarrow y = Ae^{2x} + k$$

• FINALLY WE HAVE

$$f(x) = \left(\frac{dy}{dx}\right)^2$$

$$f(x) = (2Ae^{2x})^2$$

$$f(x) = (2 \times \frac{1}{4} e^{2x})^2$$

$$f(x) = \left(\frac{1}{2} e^{2x}\right)^2$$

$$f(x) = \frac{1}{4} e^{4x}$$

Question 2

The non zero functions $u(x)$ and $v(x)$ satisfy the integral equations

$$\int u(x) dx = x^2 u(x) \quad \text{and} \quad \int u(x) v(x) dx = \left[\int u(x) dx \right] \left[\int v(x) dx \right].$$

Determine, in terms of an arbitrary constant, a simplified expression for $u(x)$ and a similar expression for $[v(x)]^2$.

$$\boxed{\frac{A}{x^2}}, \quad u(x) = \frac{A e^{-\frac{1}{x}}}{x^2}, \quad \boxed{[v(x)]^2 = \frac{B}{(1-x)^2(1-x^2)}}$$

• STARTING WITH

$$\int u dx = ux^2$$

• DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d}{dx} \int u dx = \frac{d}{dx} (ux^2)$$

$$\Rightarrow u = \frac{d}{dx} x^2 + u(2x)$$

$$\Rightarrow x^2 \frac{du}{dx} = u - 2ux$$

$$\Rightarrow x^2 \frac{du}{dx} = u(1-2x)$$

$$\Rightarrow \frac{1}{u} du = \frac{1-2x}{x^2} dx$$

$$\Rightarrow \int \frac{1}{u} du = \int \frac{1}{x^2} - \frac{2}{x} dx$$

$$\Rightarrow \ln|u| = -\frac{1}{x} - 2\ln|x| + C$$

$$\Rightarrow u = e^{-\frac{1}{x} - 2\ln|x| + C}$$

$$\Rightarrow u = e^{-\frac{1}{x}} \cdot e^{-2\ln|x|} \cdot e^C$$

$$\Rightarrow u = \frac{A}{x^2} e^{-\frac{1}{x}}$$

• NEXT WE PROCEED WITH

$$\int uv dx = \left[\int u dx \right] \left[\int v dx \right]$$

• DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d}{dx} \int uv dx = \frac{d}{dx} \left[\left[\int u dx \right] \left[\int v dx \right] \right]$$

$$\Rightarrow uv = \frac{d}{dx} \left[\int u dx \right] \times \int v dx + \int u dx \times \frac{d}{dx} \left[\int v dx \right]$$

$$\Rightarrow uv = u \int v dx + v \int u dx$$

$$\Rightarrow uv = u \int v dx + v(ux^2)$$

$$\Rightarrow v = \int v dx + vx^2$$

$$\Rightarrow v - vx^2 = \int v dx$$

$$\Rightarrow v(1-x^2) = \int v dx$$

DIFFERENTIATE W.R.T x AGAIN

$$\Rightarrow \frac{d}{dx} (1-x^2) + v(-2x) = \frac{d}{dx} \int v dx$$

$$\Rightarrow \frac{d}{dx} (1-x^2) - 2vx = v$$

$$\Rightarrow \frac{d}{dx} (1-x^2) = v + 2vx$$

$$\Rightarrow \frac{d}{dx} (1-x^2) = v(1+2x)$$

• SEPARATING VARIABLES AND INTEGRATING

$$\Rightarrow \int \frac{1}{v} dv = \int \frac{2x+1}{(1-x)^2(1+x)} dx$$

$$\Rightarrow \ln|v| = \int \frac{2x+1}{(1-x)^2(1+x)} dx$$

• PARTIAL FRACTIONS BY INSPECTION (CHECKING)

$$\Rightarrow \ln|v| = \int \frac{\frac{3}{1-x} - \frac{1}{1+x}}{(1-x)^2} dx$$

$$\Rightarrow 2\ln|v| = \int \frac{3}{(1-x)^2} - \frac{1}{(1+x)} dx$$

$$\Rightarrow \ln v^2 = -3\ln|1-x| - \ln|1+x| + \ln A$$

$$\Rightarrow \ln v^2 = \ln \left| \frac{A}{(1-x)^3(1+x)} \right|$$

$$\Rightarrow v^2 = \frac{B}{(1-x)^3(1+x)}$$

$$v^2 = \frac{B}{(1-x)^2(1-x^2)}$$

Question 3

The function f satisfies the following relationship.

$$f(x) = \int_1^x [f(t)]^2 dt, \quad f(2) = \frac{1}{2}.$$

Determine the value of $f\left(\frac{1}{2}\right)$.

$$\boxed{}, \quad \boxed{f\left(\frac{1}{2}\right) = \frac{2}{7}}$$

DIFFERENTIATE WITH RESPECT TO x

$$f(x) = \int_1^x [f(t)]^2 dt$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left[\int_1^x [f(t)]^2 dt\right]$$

BY FUNDAMENTAL THEOREM

$$\frac{df}{dx} = [f(x)]^2 \cdot 1 - [f(1)]^2 \cdot 0$$

$$\frac{df}{dx} = f^2$$

SEPARATE VARIABLES AND INTEGRATE OVER THE GIVEN RANGE

$$\Rightarrow \frac{1}{f^2} df = 1 \cdot dx$$

$$\Rightarrow \int_{\frac{1}{2}}^1 \frac{1}{f^2} df = \int_{\frac{1}{2}}^1 1 \cdot dx$$

$$\Rightarrow \left[-\frac{1}{f}\right]_{\frac{1}{2}}^1 = [x]_{\frac{1}{2}}^1$$

$$\Rightarrow -\frac{1}{f} - \left(-\frac{1}{\frac{1}{2}}\right) = \frac{1}{2} - \frac{1}{2}$$

$$\Rightarrow -\frac{1}{f} + 2 = -\frac{1}{2}$$

$$\Rightarrow \frac{1}{f} = \frac{5}{2}$$

$$\Rightarrow f = \frac{2}{5}$$

$\therefore f\left(\frac{1}{2}\right) = \frac{2}{5}$