IMPULSE FUNCTION

The Impulse Function / The Dirac Function

1. $\delta(t-c) = \begin{cases} \infty & t = c \\ 0 & t \neq c \end{cases}$, $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

2. $\delta(t) = \dfrac{1}{\pi} \lim_{\varepsilon \to 0} \left[\dfrac{\varepsilon}{\varepsilon^2 + t^2} \right]$

3. $\displaystyle\int_a^b \delta(t-c)\ dt = \begin{cases} 1 & a \leq c \leq b \\ 0 & \text{otherwise} \end{cases}$

4. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt = \begin{cases} f(c) & a \leq c \leq b \\ 0 & \text{otherwise} \end{cases}$

5. $\mathcal{L}\left[\delta(t-c)\right] = \mathrm{e}^{-cs}$

6. $\mathcal{L}\left[f(t)\delta(t-c)\right] = f(c)\mathrm{e}^{-cs}$

7. $\mathcal{F}\left[\delta(x)\right] = \dfrac{1}{\sqrt{2\pi}}$

8. $\mathcal{F}^{-1}\left[\delta(k)\right] = \dfrac{1}{\sqrt{2\pi}}$

9. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right] = \delta(t-c)$

Question 1

Evaluate the following integral

$$\int_0^5 \left(t^2+1\right)\delta(t-1)\ dt .$$

$\boxed{2}$

$\int_0^5 \underbrace{(t^2+1)}_{f(t)} \delta(t-1) = \ldots$ IMPULSE OCCURS AT $t=1$,

• USING

$$\int_a^b f(t)\,\delta(t-c)\,dt = \begin{cases} f(c) & a \leqslant c \leqslant b \\ 0 & \text{OTHERWISE} \end{cases}$$

$= 1^2+1 = 2$

Question 2

Evaluate the following integral

$$\int_0^{\pi} \sin\left(\frac{1}{3}t\right)\delta\left(t-\frac{\pi}{2}\right)\ dt .$$

$\boxed{\frac{1}{2}}$

$\int_0^{\pi} \underbrace{\sin\left(\frac{t}{3}\right)}_{f(t)} \delta\left(t-\frac{\pi}{2}\right) = \ldots$ IMPULSE OCCURS AT $t=\frac{\pi}{2}$

• USING

$$\int_a^b f(t)\,\delta(t-c)\,dt = \begin{cases} f(c) & a \leqslant c \leqslant b \\ 0 & \text{OTHERWISE} \end{cases}$$

$= \sin\frac{\pi/2}{3}$

$= \sin\frac{\pi}{6}$

$= \frac{1}{2}$

Question 3

Find the Laplace transform of $\delta(t-c)$, where c is a positive constant, and hence state the Laplace transform of $\delta(t)$.

$$\boxed{\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}}, \quad \boxed{\mathcal{L}\left[\delta(t)\right]=1}$$

$$\mathcal{L}\left[\delta(t-c)\right] = \int_0^\infty e^{-st}\delta(t-c)\,dt$$

USING

$$\int_a^b f(t)\,\delta(t-c) = \begin{cases} f(c) & a \leq c \leq b \\ 0 & \text{OTHERWISE} \end{cases}$$

$$= e^{-cs}$$

HENCE $\mathcal{L}\left[\delta(t)\right] = e^{-0 \times s} = 1$

Question 4

Given that $F(t)$ is a piecewise continuous function defined for $t \geq 0$, find the Laplace transform of $F(t)\,\delta(t-c)$, where c is a positive constant.

$$\boxed{\mathcal{L}\left[F(t)\,\delta(t-c)\right]=F(c)\mathrm{e}^{-cs}}$$

$$\mathcal{L}\left[F(t)\,\delta(t-c)\right] = \int_0^\infty e^{-st}F(t)\,\delta(t-c)\,dt$$

$$= \int_0^\infty G(t)\,\delta(t-c)\,dt \quad \text{WHERE } G(t) = e^{-st}F(t)$$

$$= G(c)$$

$$= F(c)\,e^{-cs}$$

Question 5

Find the Laplace transform of $\cos 3t \ \delta\left(t-\frac{\pi}{3}\right)$.

$$\boxed{\mathcal{L}\left[\cos 3t \ \delta\left(t-\frac{\pi}{3}\right)\right] = e^{-\frac{1}{3}\pi s}}$$

$$\mathcal{L}\left[\cos 3t \ \delta\left(t-\tfrac{\pi}{3}\right)\right] = \int_0^\infty e^{-st}\cos 3t \ \delta(t-\pi/3)\ dt$$
$$= e^{-\frac{\pi}{3}s}\cos(\pi)$$
$$= -e^{-\frac{\pi}{3}s}$$

Question 6

Find the Laplace transform of $t^3 \ \delta(t-3)$.

$$\boxed{\mathcal{L}\left[t^3 \ \delta(t-3)\right] = 27e^{-3s}}$$

$$\mathcal{L}\left[t^3 \delta(t-3)\right] = \int_0^\infty e^{-st} t^3 \ \delta(t-3)\ dt$$
$$= e^{-3s} \times 3^3 = 27e^{-3s}$$

Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \delta(t-2),$$

given further that $x = 0$, $\frac{dx}{dt} = 1$ at $t = 0$.

$$\boxed{x = e^{-t}\left[\sin 2t - e^4 \sin(2t-4)\,H(t-2)\right]}$$

$\ddot{x} + 2\dot{x} + 5x = \delta(t-2)$ $\quad t=0,\ x_0 = 0,\ \dot{x}_0 = 1$

TAKING LAPLACE TRANSFORMS

$\Rightarrow \left[s^2\bar{x} - s x_0 - \dot{x}_0\right] + 2\left[s\bar{x} - x_0\right] + 5\bar{x} = \mathcal{L}\left[\delta(t-2)\right]$

$\Rightarrow s^2\bar{x} - 1 + 2s\bar{x} + 5\bar{x} = e^{-2s}$

$\Rightarrow \bar{x}(s^2 + 2s + 5) = 1 - e^{-2s}$

$\Rightarrow \bar{x} = \frac{1 - e^{-2s}}{s^2 + 2s + 5}$

$\Rightarrow \bar{x} = \frac{1 - e^{-2s}}{(s+1)^2 + 4}$

$\Rightarrow \bar{x} = \frac{1}{(s+1)^2 + 4} - \frac{e^{-2s}}{(s+1)^2 + 4}$

INVERTING

$\Rightarrow x = e^{-t}\sin 2t - e^{-(t-2)}\sin 2(t-2)\,H(t-2)$

$\Rightarrow x = e^{-t}\sin 2t - e^{-t}e^{4}\sin(2t-4)\,H(t-2)$

Question 8

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\delta(t-6),$$

given further that $x=0$, $\frac{dx}{dt}=2$ at $t=0$.

$$\boxed{x=\mathrm{e}^{-3t}\left[\mathrm{e}^{2t}-1\right]+\mathrm{e}^{-3t}\,\mathrm{e}^{6}\left[\mathrm{e}^{12}-\mathrm{e}^{2t}\right]\mathrm{H}(t-6)}$$

$\ddot{x}+4\dot{x}+3x = 2\delta(t-6)$ SUBJECT TO $t=0$, $x_0=0$, $\dot{x}_0=2$

TAKING LAPLACE TRANSFORMS

$\Rightarrow \left[s^2\bar{x}-sx_0-\dot{x}_0\right]+4\left[s\bar{x}-x_0\right]+3\bar{x} = \mathcal{L}\left[2\delta(t-6)\right]$

$\Rightarrow s^2\bar{x}-2+4s\bar{x}+3\bar{x} = 2e^{-6s}$

$\Rightarrow \bar{x}(s^2+4s+3) = 2-2e^{-6s}$

$\Rightarrow \bar{x} = \frac{2(1-e^{-6s})}{s^2+4s+3}$

$\Rightarrow \bar{x} = 2(1-e^{-6s})\times\frac{1}{(s+1)(s+3)}$ ← PARTIAL FRACTIONS

$\Rightarrow \bar{x} = 2(1-e^{-6s})\times\left[\frac{\frac{1}{2}}{s+1}-\frac{\frac{1}{2}}{s+3}\right]$

$\Rightarrow \bar{x} = \frac{1-e^{-6s}}{s+1}-\frac{1-e^{-6s}}{s+3}$

$\Rightarrow \bar{x} = \frac{1}{s+1}-\frac{e^{-6s}}{s+1}-\frac{1}{s+3}+\frac{e^{-6s}}{s+3}$

INVERTING...

$x(t) = e^{-t}-e^{-(t-6)}H(t-6)-e^{-3t}+e^{-3(t-6)}H(t-6)$

$x(t) = e^{-t}-e^{-3t}+e^{-3t}e^{18}H(t-6)-e^{-t}e^{6}H(t-6)$

$x(t) = e^{-3t}\left[e^{2t}-1\right]+e^{-3t}e^{6}H(t-6)\left[e^{12}-e^{2t}\right]$ //

Question 9

The function f is defined as

$$f(x) = \frac{1}{\pi}\left[\frac{\varepsilon}{\varepsilon^2 + x^2}\right],$$

where ε is a positive parameter.

a) Show that $\lim_{\varepsilon \to 0}\left[f(x)\right] = \delta(x)$.

The function g is defined as

$$g(x) = \frac{\lambda}{\sqrt{\pi}}\mathrm{e}^{-\lambda^2 x^2},$$

where λ is a positive parameter.

b) Show that $\lim_{\lambda \to \infty}\left[g(x)\right] = \delta(x)$.

proof

a) $\boxed{f(x) = \frac{1}{\pi}\left(\frac{\varepsilon}{\varepsilon^2+x^2}\right)}$

- IF $x=0$

$$\lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+\cancel{x^2}}\right] = \lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\times\frac{1}{\varepsilon}\right] = \infty$$

- IF $x \neq 0$

$$\lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\frac{\varepsilon}{\cancel{\varepsilon^2}+x^2}\right] = \lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\pi x^2}\right] = 0$$

- $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{\varepsilon}{\pi}\frac{1}{\varepsilon^2+x^2}\,dx = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{1}{\varepsilon^2+x^2}\,dx$$
$$= \frac{\varepsilon}{\pi}\times\frac{1}{\varepsilon}\left[\arctan\left(\frac{x}{\varepsilon}\right)\right]_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$
$$= 1$$

$\therefore$ $f(x)$ IS AN "INFINITE HEIGHT" SPIKE AT $x=0$, WITH AREA 1

$\therefore$ $\delta(x) = \lim_{\varepsilon \to 0}\left[\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+x^2}\right]$ //

b) $\boxed{g(x) = \frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2}}$

- IF $x=0$

$$\lim_{\lambda \to \infty}\left[\frac{\lambda}{\sqrt{\pi}}\right] = \infty$$

- IF $x \neq 0$

$$\lim_{\lambda \to \infty}\left[\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2}\right] = 0 \quad \text{SINCE } e^{-\lambda^2 x^2} \to 0 \text{ FASTER THAN } \frac{\lambda}{\sqrt{\pi}} \to \infty$$

- $$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2}\,dx = \frac{\lambda}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(\lambda x)^2}\,dx$$

BY SUBSTITUTION
$u = \lambda x$
$\frac{du}{dx} = \lambda$
$dx = \frac{du}{\lambda}$
LIMITS UNCHANGED

$$= \frac{\lambda}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,\frac{du}{\lambda}$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,du$$
$$= \frac{1}{\sqrt{\pi}}\times\sqrt{\pi}$$
$$= 1$$

$\therefore$ $g(x)$ IS A SPIKE OF INFINITE HEIGHT IF $x=0$, WITH AREA 1

$\therefore$ $\delta(x) = \lim_{\lambda \to \infty}\left[\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2 x^2}\right]$ //

Question 10

The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine

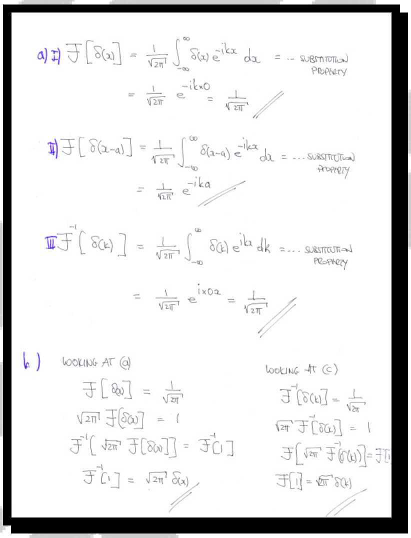
i. … $\mathcal{F}\left[\delta(x)\right]$.

ii. … $\mathcal{F}\left[\delta(x-a)\right]$, where a is a positive constant.

iii. … $\mathcal{F}^{-1}\left[\delta(k)\right]$.

b) Use the above results to deduce $\mathcal{F}[1]$ and $\mathcal{F}^{-1}[1]$.

$\boxed{\mathcal{F}\left[\delta(x)\right]=\frac{1}{\sqrt{2\pi}}}$, $\boxed{\mathcal{F}\left[\delta(x-a)\right]=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}ka}}$, $\boxed{\mathcal{F}^{-1}\left[\delta(k)\right]=\frac{1}{\sqrt{2\pi}}}$,
$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)}$, $\boxed{\mathcal{F}^{-1}[1]=\sqrt{2\pi}\,\delta(x)}$



a) I) $\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \dots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{-ik\times 0} = \frac{1}{\sqrt{2\pi}}$

II) $\mathcal{F}[\delta(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-a)e^{-ikx}\,dx = \dots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{-ika}$

III) $\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(k)e^{ikx}\,dk = \dots$ SUBSTITUTION PROPERTY

$= \frac{1}{\sqrt{2\pi}}e^{i\times 0x} = \frac{1}{\sqrt{2\pi}}$

b) LOOKING AT (a)

$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}$

$\sqrt{2\pi}\,\mathcal{F}[\delta(x)] = 1$

$\mathcal{F}^{-1}[\sqrt{2\pi}\,\mathcal{F}[\delta(x)]] = \mathcal{F}^{-1}[1]$

$\mathcal{F}^{-1}[1] = \sqrt{2\pi}\,\delta(x)$

LOOKING AT (c)

$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}$

$\sqrt{2\pi}\,\mathcal{F}^{-1}[\delta(k)] = 1$

$\mathcal{F}[\sqrt{2\pi}\,\mathcal{F}^{-1}(\delta(k))] = \mathcal{F}[1]$

$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)$

Question 11

The impulse function $\delta(x)$ is defined by

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

a) Determine the inverse Fourier transform of the impulse function $\mathcal{F}^{-1}\left[\delta(k)\right]$, and use it to deduce the Fourier transform of $f(x) = 1$.

b) Find directly the Fourier transform of $f(x) = 1$, by introducing the converging factor $e^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$.

$$\boxed{\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)}$$

a) CONSIDER THE INVERSE FOURIER TRANSFORM OF $\delta(k)$

$$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(k)\, e^{ikx}\, dk = \text{SUBSTITUTION PROPERTY}$$

$$= \frac{1}{\sqrt{2\pi}} e^{i0x} = \frac{1}{\sqrt{2\pi}}$$

NOW

$$\mathcal{F}^{-1}[\delta(k)] = \frac{1}{\sqrt{2\pi}}$$

$$\sqrt{2\pi}\, \mathcal{F}^{-1}[\delta(k)] = 1$$

$$\mathcal{F}\left[\sqrt{2\pi}\, \mathcal{F}^{-1}[\delta(k)]\right] = \mathcal{F}[1]$$

$$\sqrt{2\pi}\, \delta(k) = \mathcal{F}[1]$$

$$\mathcal{F}[1] = \sqrt{2\pi}\, \delta(k)$$

b)

$$\mathcal{F}[1] = \lim_{\varepsilon \to 0} \left[\mathcal{F}\left[1 \times e^{-\varepsilon|x|}\right]\right] =$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\varepsilon|x|} e^{-ikx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to 0} \left[\int_{0}^{\infty} 2e^{-\varepsilon x} \cos kx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\text{Re} \int_{0}^{\infty} e^{-\varepsilon x} e^{ikx}\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\text{Re}\left[\int_{0}^{\infty} e^{x(-\varepsilon+ik)}\, dx\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\text{Re}\left[\frac{1}{-\varepsilon+ik} e^{x(-\varepsilon+ik)}\right]_{0}^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\text{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2} e^{-\varepsilon x}(\cos kx + i \sin kx)\right]_{0}^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\text{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}(0-1)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\text{Re}\, \frac{\varepsilon+ik}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \times \pi \times \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]$$

$$= \sqrt{2\pi}\, \delta(k)$$

NOTE $\delta(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[\frac{\varepsilon}{x^2+\varepsilon^2}\right]$