HEAVISIDE FUNCTION

Question 1

The Heaviside function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t \geq 0 \\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)$.

$$\boxed{\mathcal{L}\left(\mathrm{H}(t-c)\right)=\frac{\mathrm{e}^{-cs}}{s}}$$

$$\boxed{H(t-c)=\begin{cases}1 & t \geq c \\ 0 & t<c\end{cases}}$$

$$\mathcal{L}\left[H(t-c)\right]=\int_0^\infty H(t-c)\,e^{-st}\,dt$$

$$=\cancel{\int_0^c 0\times e^{-st}\,dt}+\int_c^\infty 1\times e^{-st}\,dt$$

$$=\left[-\frac{1}{s}e^{-st}\right]_c^\infty=\left[\frac{e^{-st}}{s}\right]_\infty^c=\frac{e^{-cs}}{s}-0$$

$$=\frac{e^{-cs}}{s}$$

Question 2

The Heaviside step function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)\, f(t-c)$, where $f(t)$ is a continuous or piecewise continuous function defined for $t \geq 0$.

$$\boxed{\mathcal{L}\big(\mathrm{H}(t-c)\, f(t-c)\big) = \mathrm{e}^{-cs}\, \mathcal{L}\big(f(t)\big)}$$

$$\mathrm{H}(t-c) = \begin{cases} 1 & t \geqslant c \\ 0 & t < c \end{cases} \quad \& \quad y = f(t) \quad t \geqslant 0$$

$$\mathcal{L}[f(t-c)\,\mathrm{H}(t-c)] = \int_0^\infty e^{-st}\, \mathrm{H}(t-c)\, f(t-c)\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

BY SUBSTITUTION

$T = t - c$

$dT = dt$

$t = c \;,\; T = 0$

$t = \infty \;;\; T = \infty$

$$= \int_0^\infty e^{-s(T+c)} f(T)\, dT = \int_0^\infty e^{-sT} \times e^{-cs} \times f(T)\, dT$$

$$= e^{-cs} \int_0^\infty e^{-sT} f(T)\, dT = e^{-cs}\, \mathcal{L}[f(t)]$$

Question 3

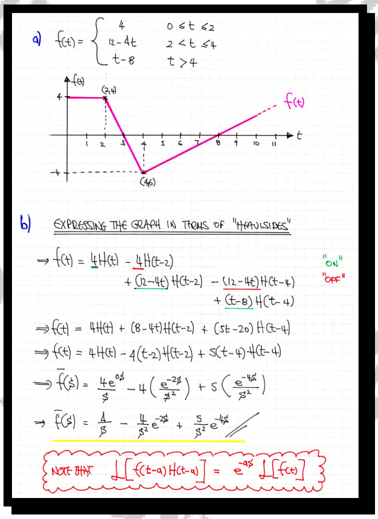
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 4 & 0 \le t \le 2 \\ 12-4t & 2 < t \le 4 \\ t-8 & t > 4 \end{cases}$$

a) Sketch the graph of $f(t)$.

b) Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$\boxed{f(t)=4\mathrm{H}(t)-4(t-2)\mathrm{H}(t-2)+5(t-4)\mathrm{H}(t-4)}$, $\boxed{\mathcal{L}(f(t))=\frac{8}{s}-\frac{4\mathrm{e}^{-2s}}{s^2}+\frac{5\mathrm{e}^{-4s}}{s^2}}$



Question 4

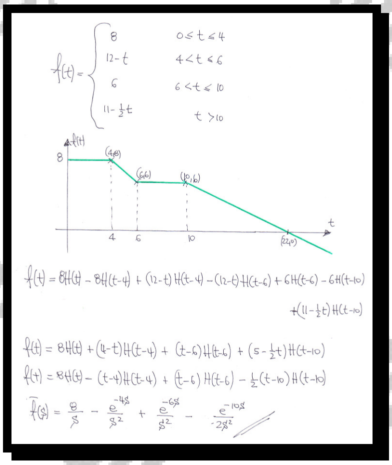
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 8 & 0 \le t \le 4 \\ 12-t & 4 < t \le 6 \\ 6 & 6 < t \le 10 \\ 11-\frac{1}{2}t & t > 10 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=8\mathrm{H}(t)-(t-4)\mathrm{H}(t-4)+(t-4)\mathrm{H}(t-6)-\frac{1}{2}(t-10)\mathrm{H}(t-10)},$$

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{8}{s}-\frac{\mathrm{e}^{-4s}}{s^2}+\frac{\mathrm{e}^{-6s}}{s^2}-\frac{\mathrm{e}^{-10s}}{2s^2}}$$



Question 5

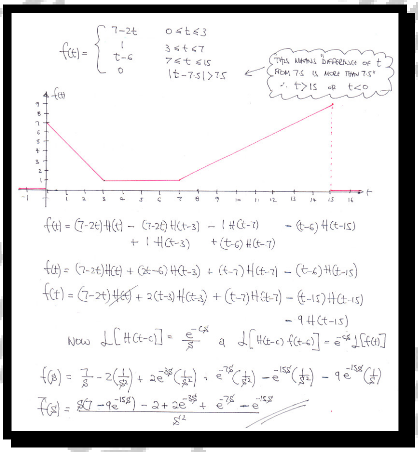
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 7-2t & 0<t\le 3 \\ 1 & 3<t\le 7 \\ t-6 & 7<t\le 15 \\ 0 & |t-7.5|>7.5 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=(7-2t)\mathrm{H}(t)+2(t-3)\mathrm{H}(t-3)+(t-7)\mathrm{H}(t-7)-(t-15)\mathrm{H}(t-15)-9\mathrm{H}(t-15)},$$

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{s\left(7-9\mathrm{e}^{-15s}\right)-2+2\mathrm{e}^{-3s}+\mathrm{e}^{-7s}-\mathrm{e}^{-15s}}{s^2}}$$



Question 6

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left(\frac{2}{s} + \frac{3e^{-s}}{s^2} - \frac{3e^{-3s}}{s^2}\right).$$

Sketch the graph of $f(t)$.

$\boxed{\text{graph}}$

$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s} + \frac{3e^{-s}}{s^2} - \frac{3e^{-3s}}{s^2}\right]$$

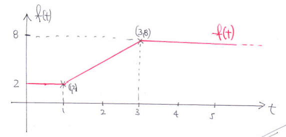
INVERTING INTO HEAVISIDES

$f(t) = 2 + 3(t-1)H(t-1) - 3(t-3)H(t-3)$

$f(t) = 2H(t) + 3(t-1)H(t-1) - 3(t-3)H(t-3)$

INTERVAL	$2H(t)$	$3(t-1)H(t-1)$	$-3(t-3)H(t-3)$	$f(t)$
$0 \leqslant t \leqslant 1$	2	0	0	2
$1 \leqslant t \leqslant 3$	2	$3t-3$	0	$3t-1$
$t \geqslant 3$	2	$3t-3$	$-3t+9$	8.

CHECKING FOR CONTINUITY AT $t=1$ & $t=3$, WE FIND IT IS PIECEWISE CONTINUOUS



Question 7

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\left(3s\,e^{2s} - 4e^{s} + 5\right)\right].$$

a) Determine an expression for $f(t)$.

b) Sketch the graph of $f(t)$.

$$\boxed{f(t) = 3H(t) - 4(t-1)H(t-1) + 5(t-2)H(t-2)}$$

a) $f(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\left(3se^{2s} - 4e^{s} + 5\right)\right] = \mathcal{L}^{-1}\left[\frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{5e^{-2s}}{s^2}\right]$

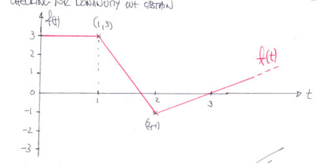
$= 3 - 4(t-1)H(t-1) + 5(t-2)H(t-2)$

$= 3H(t) - 4(t-1)H(t-1) + 5(t-2)H(t-2)$ //

b) TO SKETCH

$0 \le t \le 1$:	3	0	0	$=$	3
$1 \le t \le 2$:	3	$-4(t-1)$	0	$=$	$7-4t$
$t \ge 2$:	3	$-4(t-1)$	$5(t-2)$	$=$	$t-3$
	$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$
	$H(t)$	$-4(t-1)H(t-1)$	$5(t-2)H(t-2)$		$f(t)$

CHECKING FOR CONTINUITY WE OBTAIN



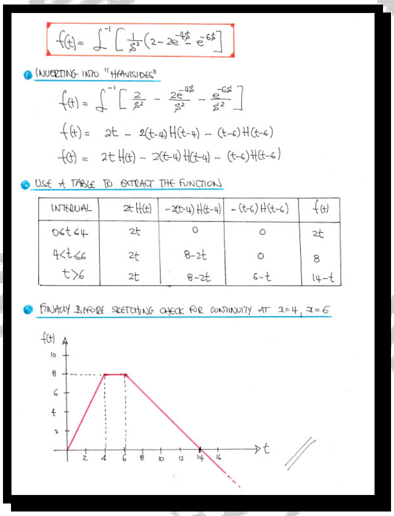
Question 8

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2 - 2e^{-4s} - e^{-6s}\right)\right].$$

Sketch the graph of $f(t)$.

$\boxed{}$, $\boxed{\text{graph}}$



$\boxed{f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2 - 2e^{-4s} - e^{-6s}\right)\right]}$

- INVERTING INTO "HEAVISIDES"

$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2} - \frac{2e^{-4s}}{s^2} - \frac{e^{-6s}}{s^2}\right]$

$f(t) = 2t - 2(t-4)H(t-4) - (t-6)H(t-6)$

$f(t) = 2tH(t) - 2(t-4)H(t-4) - (t-6)H(t-6)$

- USE A TABLE TO EXTRACT THE FUNCTION

INTERVAL	$2tH(t)$	$-2(t-4)H(t-4)$	$-(t-6)H(t-6)$	$f(t)$
$0 \leq t \leq 4$	$2t$	0	0	$2t$
$4 < t \leq 6$	$2t$	$8-2t$	0	8
$t > 6$	$2t$	$8-2t$	$6-t$	$14-t$

- FINALLY BEFORE SKETCHING CHECK FOR CONTINUITY AT $x=4$, $x=6$

Question 9

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{2-3e^{-4s}+e^{-8s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph

$$f(t) = \mathcal{L}^{-1}\left[\frac{2-3e^{-4s}+e^{-8s}}{s^2}\right]$$

INVERT INTO "HEAVISIDES"

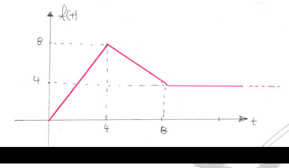
$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2} - \frac{3e^{-4s}}{s^2} + \frac{e^{-8s}}{s^2}\right] = 2t - 3(t-4)H(t-4) + (t-8)H(t-8)$$

$$f(t) = 2tH(t) - 3(t-4)H(t-4) + (t-8)H(t-8)$$

INTERVAL	$2tH(t)$	$-3(t-4)H(t-4)$	$(t-8)H(t-8)$	$f(t)$
$0 \leqslant t \leqslant 4$	$2t$	0	0	$2t$
$4 \leqslant t \leqslant 8$	$2t$	$12-3t$	0	$12-t$
$t \geqslant 8$	$2t$	$12-3t$	$t-8$	4

CHECKING IF THE CURVE IS CONTINUOUS

$$f(t) = \begin{cases} 2t & 0 \leqslant t \leqslant 4 \\ 12-t & 4 \leqslant t \leqslant 8 \\ 4 & t \geqslant 8 \end{cases}$$



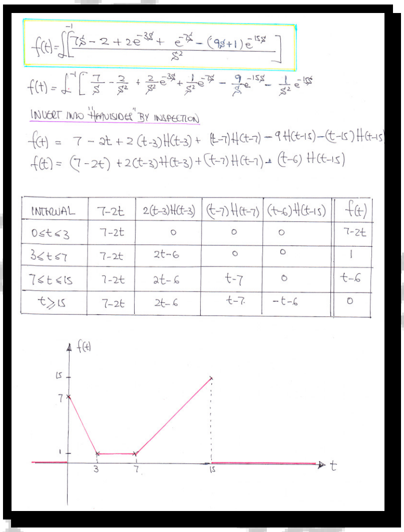
Question 10

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{s\left(7-9e^{-15s}\right)-2+2e^{-3s}+e^{-7s}-e^{-15s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



$$f(t) = \mathcal{L}^{-1}\left[\frac{7s-2+2e^{-3s}+e^{-7s}-(9s+1)e^{-15s}}{s^2}\right]$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{7}{s}-\frac{2}{s^2}+\frac{2}{s^2}e^{-3s}+\frac{1}{s^2}e^{-7s}-\frac{9}{s}e^{-15s}-\frac{1}{s^2}e^{-15s}\right]$$

INVERT INTO "HEAVISIDE" BY INSPECTION

$$f(t) = 7-2t+2(t-3)H(t-3)+(t-7)H(t-7)-9H(t-15)-(t-15)H(t-15)$$

$$f(t) = (7-2t)+2(t-3)H(t-3)+(t-7)H(t-7)-(t-6)H(t-15)$$

INTERVAL	$7-2t$	$2(t-3)H(t-3)$	$(t-7)H(t-7)$	$(t-6)H(t-15)$	$f(t)$
$0 \le t \le 3$	$7-2t$	0	0	0	$7-2t$
$3 \le t \le 7$	$7-2t$	$2t-6$	0	0	1
$7 \le t \le 15$	$7-2t$	$2t-6$	$t-7$	0	$t-6$
$t \ge 15$	$7-2t$	$2t-6$	$t-7$	$-t-6$	0

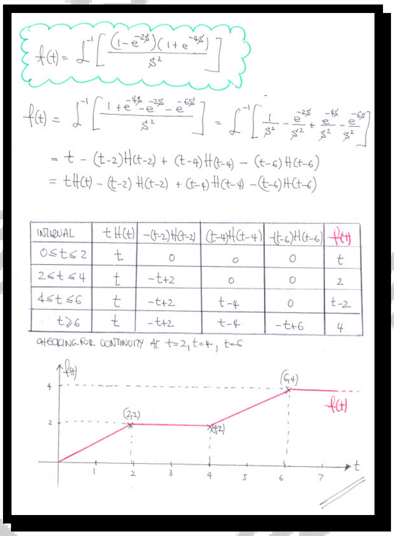
Question 11

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{\left(1-\mathrm{e}^{-2s}\right)\left(1+\mathrm{e}^{-4s}\right)}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



$$f(t) = \mathcal{L}^{-1}\left[\frac{(1-e^{-2s})(1+e^{-4s})}{s^2}\right]$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1+e^{-4s}-e^{-2s}-e^{-6s}}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}-\frac{e^{-2s}}{s^2}+\frac{e^{-4s}}{s^2}-\frac{e^{-6s}}{s^2}\right]$$

$$= t-(t-2)H(t-2)+(t-4)H(t-4)-(t-6)H(t-6)$$

$$= tH(t)-(t-2)H(t-2)+(t-4)H(t-4)-(t-6)H(t-6)$$

INTERVAL	$tH(t)$	$-(t-2)H(t-2)$	$(t-4)H(t-4)$	$-(t-6)H(t-6)$	$f(t)$
$0 \le t \le 2$	t	0	0	0	t
$2 \le t \le 4$	t	$-t+2$	0	0	2
$4 \le t \le 6$	t	$-t+2$	$t-4$	0	$t-2$
$t \ge 6$	t	$-t+2$	$t-4$	$-t+6$	4

CHECKING FOR CONTINUITY AT $t=2$, $t=4$, $t=6$

Question 12

The Heaviside function $\mathrm{H}(x)$ is defined by

$$\mathrm{H}(x)=\begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon x}$ and letting $\varepsilon \to 0$, determine the Fourier transform of $\mathrm{H}(x)$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim_{\varepsilon \to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

$$\boxed{\mathcal{F}[\mathrm{H}(x)]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(k)-\frac{\mathrm{i}}{k}\right]}$$

$$\mathcal{F}(H(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x)^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^\infty e^{-\varepsilon x} e^{-ikx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^\infty e^{x(-\varepsilon-ik)}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{1}{-\varepsilon-ik} e^{x(-\varepsilon-ik)}\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{-\varepsilon+ik}{\varepsilon^2+k^2} e^{-\varepsilon x} e^{-ikx}\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{-\varepsilon+ik}{\varepsilon^2+k^2} e^{-\varepsilon x}(\cos kx - i\sin kx)\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{-\varepsilon+ik}{\varepsilon^2+k^2}(0-1)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon-ik}{\varepsilon^2+k^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right] - ik\lim_{\varepsilon \to 0}\left[\frac{1}{\varepsilon^2+k^2}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\pi \times \underbrace{\frac{1}{\pi}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]}_{\delta(k)} - ik \times \frac{1}{k^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) - \frac{i}{k}\right]$$

Question 13

The function f is defined by

$$f(x) = \operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

a) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, find the Fourier transform of f.

b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, find the Fourier transform of $g(x) = 1$.

You may assume that $\delta(t) = \dfrac{1}{\pi} \lim\limits_{\varepsilon \to 0}\left[\dfrac{\varepsilon}{\varepsilon^2 + t^2}\right]$.

c) Hence determine the Fourier transform of the Heaviside function $\mathrm{H}(x)$,

$$\mathrm{H}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\boxed{\mathcal{F}[\operatorname{sign}(x)] = -\frac{\mathrm{i}}{k}\sqrt{\frac{1}{\pi}}}, \quad \boxed{\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k)}, \quad \boxed{\mathcal{F}[\mathrm{H}(x)] = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) - \frac{\mathrm{i}}{k}\right]}$$

a)
$$\begin{aligned}
\mathcal{F}[\operatorname{sign}(x)] &= \lim_{\varepsilon \to 0}\left[e^{-\varepsilon|x|}\operatorname{sign} x\right] \\
&= \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{e^{-\varepsilon|x|}}_{\text{EVEN}}\,\underbrace{\operatorname{sign} x}_{\text{ODD}}\; e^{-ikx}\,dx\right] \\
&= \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_0^{\infty} 2e^{-\varepsilon x} \times 1 \times (-i\sin kx)\,dx\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_0^{\infty} e^{-\varepsilon x}\sin kx\,dx\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\int_0^{\infty} e^{-\varepsilon x}e^{ikx}\,dx\right]\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\int_0^{\infty} e^{x(-\varepsilon+ik)}\,dx\right]\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{1}{-\varepsilon+ik}e^{x(-\varepsilon+ik)}\right]_0^{\infty}\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}e^{-\varepsilon x}(\cos kx + i\sin kx)\right]_0^{\infty}\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}(0-1)\right]\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Im}\left[\frac{\varepsilon+ik}{\varepsilon^2+k^2}\right]\right] \\
&= \frac{-2i}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left(\frac{k}{\varepsilon^2+k^2}\right) \\
&= \frac{-2i}{\sqrt{2\pi}}\,\frac{k}{k^2} \\
&= -\frac{i}{k}\sqrt{\frac{2}{\pi}}
\end{aligned}$$

b)
$$\begin{aligned}
\mathcal{F}[1] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[\int_{-\infty}^{\infty} e^{-\varepsilon|x|}e^{-ikx}\,dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon \to 0}\left[2\int_0^{\infty} e^{-\varepsilon x}\cos kx\,dx\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\int_0^{\infty} e^{-\varepsilon x}e^{ikx}\,dx\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\int_0^{\infty} e^{x(-\varepsilon+ik)}\,dx\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\frac{1}{-\varepsilon+ik}e^{x(-\varepsilon+ik)}\right]_0^{\infty}\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}e^{-\varepsilon x}[\cos kx + i\sin kx]\right]_0^{\infty}\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\left[\frac{-\varepsilon-ik}{\varepsilon^2+k^2}(0-1)\right]\right] \\
&= \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\operatorname{Re}\frac{\varepsilon+ik}{\varepsilon^2+k^2}\right] = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left(\frac{\varepsilon}{\varepsilon^2+k^2}\right) \\
&= \sqrt{\frac{2}{\pi}} \times \pi \times \underbrace{\frac{1}{\pi}\lim_{\varepsilon \to 0}\left[\frac{\varepsilon}{\varepsilon^2+k^2}\right]}_{\delta(k)} = \sqrt{2\pi}\,\delta(k)
\end{aligned}$$

c)
$$\begin{aligned}
\mathcal{F}[\mathrm{H}(x)] &= \mathcal{F}\left[\tfrac{1}{2}(1+\operatorname{sign} x)\right] \\
&= \tfrac{1}{2}\left[\mathcal{F}[1] + \mathcal{F}[\operatorname{sign} x]\right] \\
&= \tfrac{1}{2}\left[\sqrt{2\pi}\,\delta(k) - i\sqrt{\frac{2}{\pi}}\frac{1}{k}\right] \\
&= \frac{1}{\sqrt{2}}\left[\sqrt{\pi}\,\delta(k) - \frac{i}{\sqrt{\pi}}\frac{1}{k}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) - \frac{i}{k}\right]
\end{aligned}$$