FOURIER
SERIES
The Fourier Theorem

If \( f(x) \) is a piecewise continuous function on \((\alpha, \beta)\), then

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right],
\]

where

\[
a_n = \frac{1}{L} \int_{\alpha}^{\beta} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx
\]

\[
b_n = \frac{1}{L} \int_{\alpha}^{\beta} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

\[L = \frac{\beta - \alpha}{2} = \text{half period}\]

Parseval's Identity

\[
\frac{1}{L} \int_{\alpha}^{\beta} \left[ f(x) \right]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left[ a_n^2 + b_n^2 \right]
\]
FOURIER SERIES EXPANSIONS
Question 1

\[ f(x) = x, \ x \in \mathbb{R}, \ -\pi \leq x \leq \pi. \]

\[ f(x) = f(x + 2\pi). \]

Determine the Fourier series expansion of \( f(x) \).

\[ f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}. \]
Question 2

A function \( f(x) \) is defined in an interval \((-L,L)\), \( L > 0 \).

(a) State the general formula for the Fourier series of \( f(x) \) in \((-L,L)\), giving general expressions for the coefficients of the series.

(b) Find the Fourier series of

\[
f(x) = 2x, \ -\pi \leq x \leq \pi.
\]

\[
2x = \sum_{n=1}^{\infty} \frac{A(-1)^{n+1}}{n} \sin(nx)
\]
Question 3

\[ f(t) = \begin{cases} 
2t + 2 & -1 \leq t \leq 0 \\
0 & 0 \leq t \leq 1 
\end{cases} \]

\[ f(t) = f(t + 2). \]

Determine the Fourier series expansion of \( f(t) \):

\[
f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left( \frac{2\cos((2n-1)\pi t)}{(2n-1)^2} - \frac{\sin(n\pi t)}{n} \right).
\]
Question 4

\[ f(t) = \begin{cases} 
1 + \frac{1}{4}t & -4 \leq t \leq 0 \\
1 - \frac{1}{4}t & 0 \leq t \leq 4 
\end{cases} \]

\[ f(t) = f(t+8) \cdot 

Determine the Fourier series expansion of \( f(t) \).

\[ f(t) = \frac{1}{2} + 4 \sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{2}(2n-1)\pi t\right)}{(2n-1)^2} \]
Question 5

The “Top Hat” function is defined as

\[ f(x) = \begin{cases} 
1 & \text{ if } |x| \leq \frac{\pi}{2} \\
0 & \text{ if } \frac{\pi}{2} < |x| \leq \pi 
\end{cases} \]

for \( x \in \mathbb{R} \), \( f(x) = f(x + 2\pi) \).

Determine the Fourier series expansion of \( f(x) \).

\[ f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n+1}}{n} \cos \left( \frac{2n-1}{2} x \right) \right] \]
Question 6

\[ f(x) = \begin{cases} 
1 & -1 \leq x \leq 0 \\
0 & 0 \leq x \leq 1 
\end{cases} 
\]

\[ f(x + 2) = f(x) \]

Determine the Fourier series expansion of \( f(x) \).

\[
f(x) = \frac{3}{4} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left[ \frac{\cos((2n-1)\pi x)}{(2n-1)^2} \right] - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} \]

\[
\sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x)}{(2n-1)^2}
\]

\[
\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}
\]
Question 7

A function \( f(x) \) is defined in an interval \((-L,L), L > 0\).

a) State the general formula for the Fourier series of \( f(x) \) in \((-L,L)\), giving general expressions for the coefficients of the series.

b) Find the Fourier series of

\[ f(x) = x^2, \quad -1 \leq x \leq 1. \]

c) Hence determine the exact value of

\[ 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \ldots \]

\[ x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^2}. \]

\[ 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \ldots = \frac{\pi^2}{12}. \]
Question 8

A function \( f(x) \) is defined in an interval \((-\pi, \pi)\).

a) State the general formula for the Fourier series of \( f(x) \) in \((-\pi, \pi)\), giving general expressions for the coefficients of the series.

b) Find the Fourier series of

\[
f(x) =
\begin{cases} 
0 & -\pi \leq x \leq -\frac{\pi}{2} \\
1 & -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\
0 & \frac{\pi}{2} \leq x \leq \pi 
\end{cases}
\]

c) Hence determine the exact value of

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots
\]

\[
f(x) = \frac{1}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \cos(nx) \right] \frac{1}{2n-1} = \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}
\]
Question 9

A function \( f(x) \) is defined in an interval \((\alpha, \alpha + 2L), \ L > 0\).

a) State the general formula for the Fourier series of \( f(x) \) in \((\alpha, \alpha + 2L)\), giving general expressions for the coefficients of the series.

\[
f(x) = x, \quad 0 \leq x \leq 4.
\]

b) Find the Fourier series of \( f(x) \) ...

i. \( f(x) \) in the interval \( 0 \leq x \leq 4\), with period 4.

ii. \( f(x) \) in the interval \( 0 \leq x \leq 4\), with period 8, by building a suitable "extension" to \( f(x) \).

Illustrate the solution in each case with a sketch.

\[
x = 2 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n \pi x}{2} \right), \quad x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{4n \pi x}{4} \right)
\]
Question 10

A function \( f(x) \) is defined in an interval \((\alpha, \alpha + 2L), \ L > 0\).

a) State the general formula for the Fourier series of \( f(x) \) in \((\alpha, \alpha + 2L)\), giving general expressions for the coefficients of the series.

\[
f(x) = x^2, \ 0 \leq x \leq 1.
\]

b) Find the Fourier series of \( f(x) \)...

i. \( f(x) \) in the interval \( 0 \leq x \leq 1 \), with period 1.

ii. \( f(x) \) in the interval \( 0 \leq x \leq 1 \), with period 2, by building a suitable “extension” to \( f(x) \).

Illustrate the solution in each case with a sketch.

\[
x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{\cos(2\pi nx) - \sin(2\pi nx)}{n^2 \pi^2}.
\]

\[
x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^2}.
\]
Question 11

\[ f(x) = \begin{cases} 
\pi - x & 0 \leq x \leq \pi \\
\pi + x & -\pi < x \leq 0 
\end{cases} \]

for \( x \in \mathbb{R} \), \( f(x) = f(x + 2\pi) \).

a) Determine the Fourier series expansion of \( f(x) \).

b) Hence determine the exact value of

\[ \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} \cdot \]

c) Show that

\[ \sum_{n=0}^{\infty} \left[ \sin \frac{nx}{2} - \cos \frac{nx}{2} \right] \left( \frac{1}{(2n+1)^2} \right) = -\frac{\pi^2}{8\sqrt{2}} \]
Question 12

The periodic function $f$ is defined as

$$f(t) = \begin{cases} 
   0 & -1 \leq t < 0 \\
   t^2 & 0 \leq t \leq 1 
\end{cases}$$

for $t \in \mathbb{R}$, $f(t) = f(t + 2)$.

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{6} + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n \times 2 \cos(n\pi t)}{n^2 \pi^2} + \frac{(-1)^n + 1}{n \pi} + \frac{2}{n^3 \pi^3} \left[ (-1)^n - 1 \right] \sin(n\pi t) \right\}$$
Question 13

\[ f(x) = x, \ x \in \mathbb{R}, \ 0 \leq x \leq 2\pi. \]

\[ f(x) = f(x + 2\pi). \]

a) Determine the Fourier series expansion of \( f(x) \).

b) Hence determine the exact value of

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}. \]

\[ f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \pi \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \]
Question 14

A function $f(x)$ is defined in an interval $(-\pi, \pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = 3x^2 - \pi^2, \quad -\pi \leq x \leq \pi.$$ 

c) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx.$$ 

$$3x^2 - \pi^2 = 12 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$ 

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Question 15

\( f(x) = |x|, \ x \in \mathbb{R}, \ -\pi \leq x \leq \pi. \)

\( f(x) = f(x+2\pi). \)

a) Determine the Fourier series expansion of \( f(x). \)

b) Hence determine the exact value of

\[ \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}. \]

\[ f(x) = \frac{\pi}{2} - 4 \sum_{n=1}^{\infty} \cos((2n-1)x) \frac{\cos((2n-1)x)}{(2n-1)^2}, \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8} \]
Question 16

\[ f(x) = x, \quad x \in \mathbb{R}, \quad -1 \leq x \leq 1. \]

\[ f(x) = f(x+2). \]

a) Determine the Fourier series expansion of \( f(x) \).

b) Hence determine the exact value of

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n}. \]

\[ f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n \pi x}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n} = \frac{1}{2} \]
Question 17

\[ f(x) = x^2, \ x \in \mathbb{R}, \ -2 \leq x \leq 2. \]

\[ f(x) = f(x + 4). \]

Determine the Fourier series expansion of \( f(x) \).

\[ f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{1}{2} n\pi x \right) \]
Question 18

A function \( f(x) \) is defined in the interval \((-\pi, \pi)\).

a) State the general formula for the Fourier series of \( f(x) \) in \((-\pi, \pi)\), giving general expressions for the coefficients of the series.

b) Find the Fourier series of

\[ f(x) = x, \quad -\pi \leq x \leq \pi. \]

c) Hence determine the exact value of

\[ g(x) = x^2, \quad -\pi \leq x \leq \pi. \]

\[
f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n},
\]

\[
g(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}.
\]
Question 19

\[ f(x) = x^2, \ x \in \mathbb{R}, \ 0 \leq x \leq 1. \]

Determine the Fourier series of \( f(x) \) as

a) \( \ldots \) as half range cosine expansion.

b) \( \ldots \) as half range sine expansion.

\[
f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{n^2} \cos(n\pi x) \right),
\]

\[
f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \frac{2}{n^3} \left(2 - n^2\pi^2\right)(-1)^n - 2 \sin(n\pi x) \right].
\]
Question 20

\[ f(x) = \begin{cases} \pi - x & 0 \leq x \leq \pi \\ 0 & \pi \leq x \leq 2\pi \end{cases}, \quad x \in \mathbb{R}. \]

\[ f(x) = f(x + 2\pi). \]

a) Determine the Fourier series expansion of \( f(x) \).

b) Hence determine the exact value of

\[ \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}. \]

\[ f(x) = \frac{\pi}{4} + \sum_{m=1}^{\infty} \frac{2\cos \left( \frac{(2m-1)x}{\pi} \right) + \sin mx}{\pi (2m-1)^2} m. \]

\[ \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}. \]
Question 21

\[ f(x) = x, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi. \]

\[ f(x) = f(x + 2\pi). \]

a) Determine the Fourier series expansion of \( f(x) \).

b) Hence determine the exact value of

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \]

\[ f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = -\frac{\pi^2}{12}. \]
Question 22

\[ f(x) = x^2, \ x \in \mathbb{R}, \ 0 \leq x \leq 2\pi. \]

\[ f(x) = f(x + 2\pi). \]

a) Determine the Fourier series expansion of \( f(x) \).

b) Hence determine the exact value of

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \]

\[
\begin{align*}
  f(x) &= \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[ \frac{\cos nx - \pi \sin nx}{n^2} \right], \\
  \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= -\frac{\pi^2}{12}.
\end{align*}
\]
Question 23
It is given that for \( x \in \mathbb{R}, -\pi \leq x \leq \pi, \)

\[
|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}, \quad |x| = |x + 2\pi|.
\]

a) Use the above Fourier series expansion to deduce the Fourier series expansion of \( \text{sgn}(x) \).

b) Verify the answer of part (a) by obtaining directly the Fourier series expansion of \( \text{sgn}(x) \).

c) Hence determine the exact value of

\[
\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1} = \frac{\pi}{4}.
\]
Question 24

\[ f(x) = \begin{cases} 
-x & 0 \leq x \leq \frac{\pi}{2} \\
0 & \frac{\pi}{2} < x < \pi 
\end{cases} \]

\[ f(x) = f(x+\pi). \]

Determine the Fourier series expansion of \( f(x) \).

\[ f(x) = \frac{\pi}{8} + \sum_{n=1}^{\infty} \cos\left(\frac{4n-2}{2n-1}\pi x\right) + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \sin 2nx \]
Question 25

A function $f(x)$ is defined in an interval $(-L, L), \ L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Determine the Fourier series of

$$f(x) = e^x, \ -\pi \leq x \leq \pi.$$
Question 26

A function \( f(x) \) is defined in an interval \((-L, L)\), \( L > 0 \).

a) State the general formula for the Fourier series of \( f(x) \) in \((-L, L)\), giving general expressions for the coefficients of the series.

b) Show that

\[
\int_{-\pi}^{\pi} e^{ax} e^{inx} \, dx = \frac{2(a-ni)(-1)^n}{a^2 + n^2} \sinh(a\pi)
\]

c) Determine the Fourier series of

\[ f(x) = e^{ax} , \ a > 0 , \ -\pi \leq x \leq \pi \]

d) Hence find the Fourier series of \( \cosh(ax) \) and \( \sinh(ax) \), for \(-\pi \leq x \leq \pi\).
Question 27

A function \( f(x) \) is defined in an interval \((-L, L), \ L > 0\). 

a) State the general formula for the Fourier series of \( f(x) \) in \((-L, L)\), giving general expressions for the coefficients of the series.

b) Determine the Fourier series of

\[
f(x) = e^x, \quad -\pi \leq x \leq \pi.
\]

c) Hence find the Fourier series of \( \sinh x \) and \( \cosh x \), for \(-\pi \leq x \leq \pi\).

\[
e^x = \frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \left[ \cos(nx) - n\sin(nx) \right]}{1+n^2}
\]

\[
\sinh x = \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{n(-1)^{n+1} \sin(nx)}{1+n^2}
\]

\[
cosh x = \frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{1+n^2}
\]
Question 28

A function $f$ is defined by

$$f(t) = V\cos \omega t, \quad t \in \mathbb{R},$$

where $V$ and $\omega$ are positive constants.

Show that the Fourier series of $f$ is given by

$$f(t) = \frac{2V}{\pi} + \frac{4V}{\pi} \left[ \frac{1}{3} \cos(2\omega t) - \frac{1}{15} \cos(4\omega t) + \frac{1}{35} \cos(6\omega t) + \ldots \right]$$

proof
PARSEVAL’S
IDENTITY
Question 1

A function \( f(x) \) is defined in an interval \((-L,L)\), \( L > 0 \).

a) State the general formula for the Fourier series of \( f(x) \) in \((-L,L)\), giving general expressions for the coefficients of the series.

b) Find the Fourier series of

\[ f(x) = |x|, \quad -\pi \leq x \leq \pi. \]

c) State Parseval’s identity for the Fourier series of \( f(x) \) from part (a).

d) Hence show that

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}. \]
Question 2

A function \( f(x) \) is defined in an interval \((-L, L), \ L > 0\).

a) State the general formula for the Fourier series of \( f(x) \) in \((-L, L)\), giving general expressions for the coefficients of the series.

b) Find the Fourier series of

\[
f(x) = \text{sign}(x), \ -\pi \leq x \leq \pi.
\]

c) Prove Parseval’s identity for the Fourier series of \( f(x) \) in \((-\pi, \pi)\).

d) Hence show that

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},
\]

\[
\text{sign}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)}.
\]
Question 3
A function $f(x)$ is defined in an interval $(-L,L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L,L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = \frac{1}{2} + \frac{1}{2} \text{sign}(x), \quad -\pi \leq x \leq \pi.$$  

c) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L,L)$.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$
Question 4

\[ f(x) = x, \ x \in \mathbb{R}, \ -\pi \leq x \leq \pi. \]

\[ f(x) = f(x + 2\pi). \]

Use Parseval’s identity for the Fourier coefficients of \( f(x) \) to determine the exact value of

\[ \sum_{n=1}^{\infty} \frac{1}{n^2}. \]
Use Parseval’s identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$
Question 6

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Prove the validity of Parseval’s identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.

c) Find the Fourier series of

$$f(x) = x^2, \quad -\pi \leq x \leq \pi.$$ 

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$ 

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$ 

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Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) State and prove Parseval’s identity for the Fourier series of $f(x)$ in $(-L, L)$.

c) By considering the Fourier series of

$$f(x) = x^3, \ -\pi \leq x \leq \pi,$$

show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

You may use without proof the following results.

- $\int x^3 \sin nx \, dx = \frac{1}{n^4} \left[ nx (6-n^2 x^2) \cos nx + 3 (n^2 x^2 - 2) \sin nx \right] + C$

- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

- $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$
FOURIER SERIES
Complex Expansions
Question 1

A periodic function \( f(t) \) is defined in the interval \((-L, L)\), \( L > 0 \), \( f(t + 2L) = f(t) \).

It is further given that \( f(t) \) is continuous or piecewise continuous in \((-L, L)\) and has Fourier series

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi t}{L} \right) + b_n \sin \left( \frac{n\pi t}{L} \right) \right],
\]

where \( a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \left( \frac{n\pi t}{L} \right) \, dt \), \( n = 0, 1, 2, 3, \ldots \)

and \( b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \left( \frac{n\pi t}{L} \right) \, dt \), \( n = 1, 2, 3, 4, \ldots \)

Show that the complex Fourier series expansion of \( f(t) \) is

\[
f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi nt}{L}},
\]

where \( c_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{\frac{i\pi nt}{L}} \, dt \), \( n \in \mathbb{Z} \)

______, proof
Question 2

\[ f(t) = \begin{cases} 
1 & 2 \leq t \leq 2 \\
0 & 2 < t < 6 
\end{cases}, \quad f(t + 8) = f(t). \]

Determine the complex Fourier series expansion of \( f(t) \).

\[
f(t) = \frac{1}{2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ e^{\frac{4\pi i}{n\pi}} \sin c \left( \frac{1}{n\pi} \right) \right]
\]
Question 3

\( f(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & a < t < T \\ \frac{1}{2}T, & f(t+T) = f(t) \end{cases} \)

Determine the complex Fourier series expansion of \( f(t) \).

\[
f(t) = \frac{a}{T} + \frac{a}{T} \sum_{n=-\infty}^{\infty} \exp \left( \frac{n\pi(2t-a)}{T} \right) \text{sinc} \left( \frac{n\pi a}{T} \right) \]

\[
\text{sinc}(x) = \frac{\sin(x)}{x}.
\]
Determine the complex Fourier series expansion of \( f(t) \).

\[
 f(t) = \frac{1}{2} + \frac{i}{2\pi} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi i nt}}{n}
\]
Question 5

\( f(t) = e^{2\pi t}, \quad 0 \leq t < 2, \quad f(t + 2) = f(t). \)

Determine the complex Fourier series expansion of \( f(t). \)

\[
f(t) = \frac{e^{2\pi} - 1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(1 + in)e^{2\pi n t}}{(n^2 + 1)}
\]
Question 6

\[ f(t) = \cos(\pi t), \quad -\frac{1}{2} \leq t < \frac{1}{2}, \quad f(t+1) = f(t). \]

Determine the complex Fourier series expansion of \( f(t) \).

\[
f(t) = \frac{2}{\pi} \sum_{n=\infty}^\infty \frac{(-1)^{n+1} e^{2\pi i n t}}{n^2 - 1}.
\]
Question 7

\[ f(t) = \sin(\pi t), \quad 0 \leq t < 1, \quad f(t+1) = f(t). \]

Determine the complex Fourier series expansion of \( f(t) \).

\[
f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n t}}{4n^2 - 1}.
\]
Question 8

The function \( f \) is defined as

\[
f(t) = V \cos \left( \frac{\pi t}{T} \right), \quad -\frac{1}{2} T \leq t < \frac{1}{2} T, \quad f(t) = f(t + T),
\]

where \( V \) and \( T \) are positive constants.

Determine the complex Fourier series expansion of \( f(t) \).

\[
f(t) = \frac{2V}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{2n\pi i}}{1 - 4n^2} \]