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FOURIER SERIES

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The Fourier Theorem

If $f(x)$ is a piecewise continuous function on (α, β) , then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

where $a_n = \frac{1}{L} \int_{\alpha}^{\beta} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$$b_n = \frac{1}{L} \int_{\alpha}^{\beta} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$L = \frac{\beta - \alpha}{2} = \text{half period}$$

Parseval's Identity

$$\frac{1}{L} \int_{\alpha}^{\beta} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

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FOURIER SERIES EXPANSIONS

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Question 1

$$f(x) = x, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{}, \quad f(x) = 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \sin nx}{n} \right]$$

• USING THE STANDARD FOURIER SERIES FORMULA

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{b-a} + b_n \sin \frac{n\pi x}{b-a} \right] \quad (b-a \text{ HALF PERIOD} = \frac{\pi}{2})$$

where $a_n = \frac{1}{b-a} \int_a^b f(x) dx$
 $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n\pi x dx \quad n = 1, 2, 3, \dots$
 $b_n = \frac{1}{b-a} \int_a^b f(x) \sin \frac{n\pi x}{b-a} dx \quad n = 1, 2, 3, \dots$

• USING THE ABOVE RESULTS, WITH $a = -\pi$, $b = \pi$, $f(x) = x$ WE OBTAIN

- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x dx = 0 \quad (\text{ODD INTEGRAND IN A SYMMETRIC DOMAIN})$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx = 0 \quad (\text{ODD INTEGRAND IN A SYMMETRIC DOMAIN})$
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$

• PROCCEED BY INTEGRATION BY PARTS

$$b_n = \frac{2}{\pi} \left[\left[-\frac{x}{n} \cos nx \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx dx \right]$$

$$b_n = \frac{2}{\pi} \left[\left[-\frac{x \cos nx}{n} \right]_0^{\pi} + \left[\frac{1}{n^2} \sin nx \right]_0^{\pi} \right]$$

x	1
$-\frac{1}{n} \cos nx$	$\sin nx$

$\Rightarrow b_n = \frac{2}{\pi} \left[0 - \frac{2\pi \cos n\pi}{n} \right]$
 $\Rightarrow b_n = -\frac{2 \cos n\pi}{n}$
 $\Rightarrow b_n = -\frac{2}{n} (-1)^n$

• FINALLY WE HAVE THE FOURIER SERIES

$$f(x) = \sum_{n=1}^{\infty} \left[-\frac{2}{n} (-1)^n \sin nx \right]$$

$$f(x) = 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \sin nx \right]$$

$$x = 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right]$$

Question 2

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- Find the Fourier series of

$$f(x) = 2x, \quad -\pi \leq x \leq \pi.$$

$$2x = \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n} \sin(nx) \right]$$

a) If $f(x)$ is piecewise continuous on $(-L, L)$, $L > 0$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $n = 0, 1, 2, 3, \dots$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, 4, \dots$$

b) $f(x) = 2x$ is odd $\Rightarrow a_n = 0$ for all n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (2x) \sin \frac{n\pi x}{\pi} dx = \frac{2}{\pi} \int_{-\pi}^{\pi} x \sin nx dx$$

$$= \frac{2}{\pi} \int_{-\pi}^{\pi} x \sin nx dx \dots \text{by parts}$$

$$= \frac{2}{\pi} \left\{ \left[-\frac{1}{n} x \cos nx \right]_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos nx dx \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{\pi}{n} \cos n\pi + \frac{1}{n^2} \left[\sin nx \right]_{-\pi}^{\pi} \right\} = -\frac{2}{n} \cos n\pi$$

$$= -\frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

16. $2x = 4 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \dots \right]$

Question 3

$$f(t) = \begin{cases} 2t+2 & -1 \leq t \leq 0 \\ 0 & 0 \leq t \leq 1 \end{cases}$$

$$f(t) = f(t+2)$$

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{2 \cos[(2n-1)\pi t]}{\pi(2n-1)^2} - \frac{\sin(n\pi t)}{n} \right]$$

Handwritten solution for the Fourier series expansion of $f(t)$. The solution includes the following steps:

- Definition of $f(t)$ and its periodicity: $f(t) = \begin{cases} 2t+2 & -1 \leq t \leq 0 \\ 0 & 0 \leq t \leq 1 \end{cases}$, $f(t) = f(t+2)$.
- Calculation of the average value a_0 : $a_0 = \frac{1}{2} \int_{-1}^1 f(t) dt = \frac{1}{2} \left[\int_{-1}^0 (2t+2) dt + \int_0^1 0 dt \right] = \frac{1}{2} [t^2 + 2t]_{-1}^0 = \frac{1}{2} [0 - (-1)] = \frac{1}{2}$.
- Calculation of the cosine coefficients a_n : $a_n = \frac{1}{2} \int_{-1}^1 f(t) \cos(n\pi t) dt = \frac{1}{2} \int_{-1}^0 (2t+2) \cos(n\pi t) dt$. Integration by parts yields $a_n = \frac{2}{n^2\pi^2} [1 - \cos(n\pi)] = \frac{2}{n^2\pi^2} [1 - (-1)^n]$. This is 0 for even n and $\frac{4}{n^2\pi^2}$ for odd n .
- Calculation of the sine coefficients b_n : $b_n = \frac{1}{2} \int_{-1}^1 f(t) \sin(n\pi t) dt = \frac{1}{2} \int_{-1}^0 (2t+2) \sin(n\pi t) dt$. Integration by parts yields $b_n = -\frac{2}{n\pi} [1 - \cos(n\pi)] = -\frac{2}{n\pi} [1 - (-1)^n]$. This is 0 for even n and $-\frac{4}{n\pi}$ for odd n .
- Final Fourier series expansion: $f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{4 \cos((2n-1)\pi t)}{(2n-1)^2\pi^2} - \frac{4 \sin((2n-1)\pi t)}{(2n-1)\pi} \right]$.

Question 4

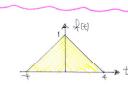
$$f(t) = \begin{cases} 1 + \frac{1}{4}t & -4 \leq t \leq 0 \\ 1 - \frac{1}{4}t & 0 \leq t \leq 4 \end{cases}$$

$$f(t) = f(t+8).$$

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\cos\left[\frac{1}{4}(2n-1)\pi t\right]}{(2n-1)^2} \right]$$

Firstly if $f(x)$ is piece-wise continuous in $(-L, L)$, then
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L})$ with $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $n=0,1,2,\dots$
 $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$ $n=1,2,3,\dots$

• Here $f(t) = \begin{cases} 1 + \frac{1}{4}t & -4 \leq t \leq 0 \\ 1 - \frac{1}{4}t & 0 \leq t \leq 4 \end{cases}$ 

• This $a_0 = \frac{1}{L} \int_{-L}^L f(t) dt = \frac{1}{4} \times \text{AREA OF TRIANGLE} = \frac{1}{4} \times \frac{1}{2} \times 8 \times 1 = 1$

• $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt = \frac{1}{4} \int_{-4}^4 f(t) \cos \frac{n\pi t}{4} dt$ $\therefore \frac{1}{4} \int_{-4}^0 (1 + \frac{1}{4}t) \cos \frac{n\pi t}{4} dt + \frac{1}{4} \int_0^4 (1 - \frac{1}{4}t) \cos \frac{n\pi t}{4} dt$
 $= \frac{1}{4} \left[\frac{1}{n\pi} \left(\cos \frac{n\pi t}{4} \right) - \frac{1}{4} \left(\frac{4}{n\pi} \sin \frac{n\pi t}{4} \right) \right]_{-4}^0 + \frac{1}{4} \left[\frac{1}{n\pi} \left(\cos \frac{n\pi t}{4} \right) - \frac{1}{4} \left(\frac{4}{n\pi} \sin \frac{n\pi t}{4} \right) \right]_0^4$
 $= \frac{1}{4n\pi} [1 - \cos(n\pi)] = \frac{1}{4n\pi} [1 - (-1)^n]$ $\therefore \frac{1}{4n\pi} [1 - (-1)^n]$
 $= \frac{1}{4n\pi} [1 - (-1)^n] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{4n\pi} & \text{if } n \text{ is odd} \end{cases}$ RECALL: $\cos(n\pi) = (-1)^n$

$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{4n\pi} \left(\cos \frac{n\pi t}{4} \right) \right]$ $= \frac{1}{2} + \frac{1}{4\pi} \sum_{n=1}^{\infty} \left[\frac{\cos \frac{n\pi t}{4}}{n} \right]$
 $= \frac{1}{2} + \frac{1}{4\pi} \left[\frac{\cos \frac{\pi t}{4}}{1} + \frac{\cos \frac{3\pi t}{4}}{3} + \frac{\cos \frac{5\pi t}{4}}{5} + \frac{\cos \frac{7\pi t}{4}}{7} + \dots \right]$

Question 5

The “Top Hat” function is defined as

$$f(x) = \begin{cases} 1 & |x| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| \leq \pi \end{cases}$$

for $x \in \mathbb{R}$, $f(x) = f(x + 2\pi)$.

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \cos[(2n-1)x]}{2n-1} \right]$$

Handwritten solution for the Fourier series expansion of the Top Hat function. The solution includes the following steps:

- Definition of the function: $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$ where $L = \pi$ and $a_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L}) dx$, $b_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx$.
- Graph of the function: A plot of $f(x)$ showing a rectangular pulse from $x = -\pi/2$ to $x = \pi/2$ with height 1, and 0 elsewhere.
- Calculation of a_0 : $a_0 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} 1 dx = \frac{2}{\pi} \times \pi = 2$.
- Calculation of b_n : $b_n = 0$ since $f(x)$ is even.
- Calculation of a_n : $a_n = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos(n x) dx = \frac{2}{\pi} [\sin(n x)]_{-\pi/2}^{\pi/2} = \frac{2}{\pi} [\sin(n\pi/2) - \sin(-n\pi/2)] = \frac{4}{\pi} \sin(n\pi/2)$.
- Final series: $f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)}$.
- Verification: $\sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} \ln 2$.

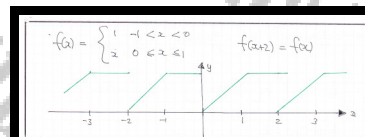
Question 6

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \end{cases}$$

$$f(x+2) = f(x)$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{3}{4} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)\pi x]}{(2n-1)^2} \right] - \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin(n\pi x)}{n} \right]$$



• The function is not odd, not even — Half period is 1

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \int_{-1}^1 f(x) dx = \int_{-1}^0 1 dx + \int_0^1 x dx$$

$$= 1 + \left[\frac{1}{2} x^2 \right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \int_{-1}^1 f(x) \cos(n\pi x) dx$$

$$= \int_{-1}^0 1 \cos(n\pi x) dx + \int_0^1 x \cos(n\pi x) dx$$

By parts

$$= \left[\frac{1}{n\pi} \sin(n\pi x) \right]_{-1}^0 + \left[\frac{x}{n\pi} \sin(n\pi x) - \frac{1}{n^2\pi^2} \cos(n\pi x) \right]_0^1$$

$$= \frac{1}{n^2\pi^2} [\cos(n\pi)]_0^1 = \frac{1}{n^2\pi^2} [\cos(n\pi) - 1] = \frac{1}{n^2\pi^2} [(-1)^n - 1]$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \int_{-1}^1 f(x) \sin(n\pi x) dx$$

$$= \int_{-1}^0 1 \sin(n\pi x) dx + \int_0^1 x \sin(n\pi x) dx$$

By parts

$$= \left[-\frac{1}{n\pi} \cos(n\pi x) \right]_{-1}^0 + \left[-\frac{x}{n\pi} \cos(n\pi x) + \frac{1}{n^2\pi^2} \sin(n\pi x) \right]_0^1$$

$$= -\frac{1}{n\pi} [\cos(n\pi)]_0^1 = -\frac{1}{n\pi} [\cos(n\pi) - 1] = \frac{1}{n\pi} [1 - (-1)^n]$$

$$= \left[-\frac{1}{n\pi} \cos(n\pi) \right]_0^1 + \left[-\frac{x}{n\pi} \cos(n\pi x) + \frac{1}{n^2\pi^2} \sin(n\pi x) \right]_0^1$$

$$= \left[-\frac{1}{n\pi} \cos(n\pi) \right]_0^1 + \left[-\frac{x}{n\pi} \cos(n\pi x) + \frac{1}{n^2\pi^2} \sin(n\pi x) \right]_0^1$$

$$= \frac{1}{n\pi} [\cos(-n\pi) - \cos(n\pi)] + 0 - \frac{1}{n\pi} \cos(n\pi)$$

$$= \frac{1}{n\pi} [\cos(n\pi) - \cos(n\pi)] - \frac{1}{n\pi} \cos(n\pi) = -\frac{1}{n\pi} \cos(n\pi)$$

• Then use half range

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$f(x) = \frac{3/2}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} [(-1)^n - 1] \cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x) \right]$$

$$f(x) = \frac{3}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \cos(n\pi x) - \sin(n\pi x) \right]$$

$$f(x) = \frac{3}{4} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x)}{(2n-1)^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}$$

Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- b) Find the Fourier series of

$$f(x) = x^2, \quad -1 \leq x \leq 1.$$

- c) Hence determine the exact value of

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots$$

$$\boxed{\frac{\pi^2}{12}}, \quad x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right], \quad 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots = \frac{\pi^2}{12}$$

a) DERIVED FROM THE DEFINITIONS
 IF $f(x)$ IS PIECEWISE CONTINUOUS IN THE INTERVAL $(-L, L)$, $\pi \neq 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

 WHERE $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$ $n=0,1,2,\dots$
 $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$ $n=1,2,3,\dots$

b) LET US START BY NOTING THAT $f(x) = x^2$, $x \in [-1, 1]$ IS EVEN
 • AS $f(x)$ IS EVEN, ALL $b_n = 0$, AS THE INTEGRAND OF b_n WILL BE ODD IN A SYMMETRICAL DOMAIN
 • $a_0 = \frac{1}{L} \int_{-L}^L x^2 dx = \dots$ EVEN INTEGRAND $\dots 2 \int_0^1 x^2 dx$
 $= \frac{2}{3} [x^3]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$
 • $a_n = \frac{1}{L} \int_{-L}^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \dots$ EVEN INTEGRAND \dots
 $= \int_{-1}^1 x^2 \cos(n\pi x) dx$
INTEGRATION BY PARTS
 $= \left[\frac{x^2}{2\pi} \sin(n\pi x) \right]_{-1}^1 - \frac{1}{\pi} \int_{-1}^1 x \sin(n\pi x) dx$
 $= -\frac{1}{\pi} \int_{-1}^1 x \sin(n\pi x) dx$
INTEGRATION BY PARTS AGAIN
 $= -\frac{1}{\pi} \left[\left[-\frac{x}{n\pi} \cos(n\pi x) \right]_{-1}^1 + \frac{1}{n\pi} \int_{-1}^1 \cos(n\pi x) dx \right]$

$\frac{x^2}{2\pi}$	$\frac{1}{\pi}$
$\frac{x^2}{2\pi} \sin(n\pi x)$	$\cos(n\pi x)$
$\frac{x}{\pi}$	1
$-\frac{x}{\pi} \cos(n\pi x)$	$\sin(n\pi x)$

c) LETTING $a=0$ IN THE ABOVE EXPRESSION
 $\Rightarrow 0^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi \cdot 0) \right]$
 $\Rightarrow 0 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \right]$
 $\Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{3}$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$
 $\Rightarrow -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \dots = -\frac{\pi^2}{12}$
 $\Rightarrow 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \frac{\pi^2}{12}$

Question 8

A function $f(x)$ is defined in an interval $(-\pi, \pi)$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.
- b) Find the Fourier series of

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq -\frac{1}{2}\pi \\ 1 & -\frac{1}{2}\pi < x \leq \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi \leq x \leq \pi \end{cases}$$

- c) Hence determine the exact value of

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$f(x) = \frac{1}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \cos(nx)}{2n-1} \right], \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}$$

a) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$, then

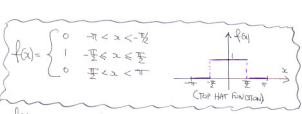
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n=0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad n=1, 2, 3, \dots$$

b)



$f(x) = \begin{cases} 0 & -\pi \leq x \leq -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \leq \pi \end{cases}$

• $f(x)$ is even, so b_n will be zero (even \times odd = odd)

• $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1 \, dx = \frac{1}{\pi} \times \pi = 1$

• $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx$

$$= \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2} = \frac{2}{\pi n} \left[\sin \frac{n\pi}{2} - 0 \right]$$

$= \frac{2}{\pi n} \sin \frac{n\pi}{2}$ if $n=1, 3, 5, \dots$

$= -\frac{2}{\pi n}$ if $n=2, 4, 6, \dots$

c)

Let $x=0$

$$f(x) = \frac{1}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(nx)}{2n-1}$$

$$\Rightarrow 1 = \frac{1}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \frac{1}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$$

Question 9

A function $f(x)$ is defined in an interval $(\alpha, \alpha + 2L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(\alpha, \alpha + 2L)$, giving general expressions for the coefficients of the series.

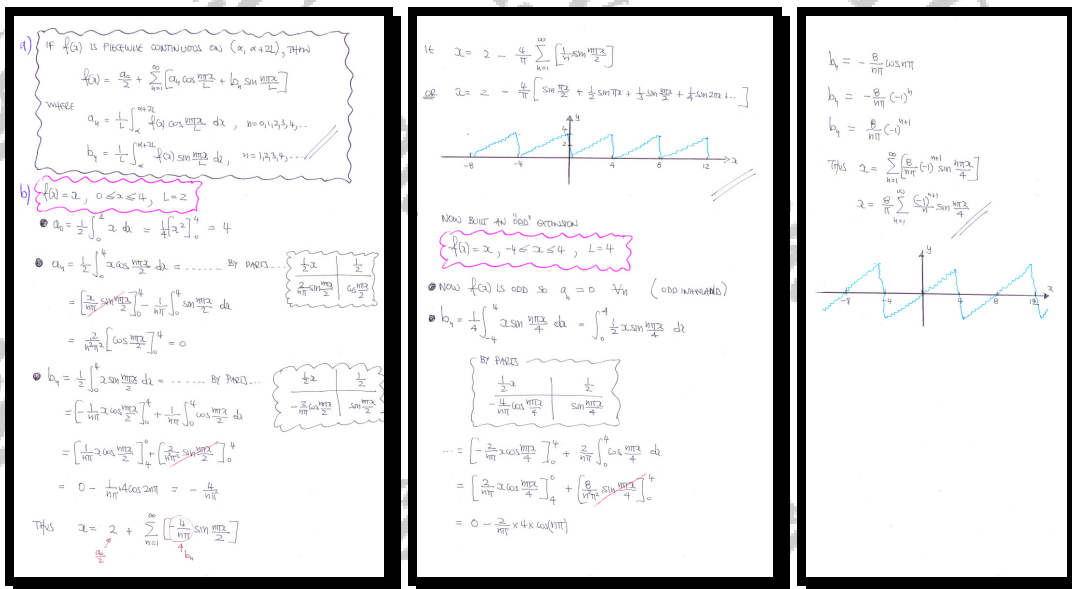
$$f(x) = x, \quad 0 \leq x \leq 4.$$

- b) Find the Fourier series of $f(x)$...

- ... in the interval $0 \leq x \leq 4$, with period 4.
- ... in the interval $0 \leq x \leq 4$, with period 8, by building a suitable "extension" to $f(x)$.

Illustrate the solution in each case with a sketch.

$$x = 2 - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin\left(\frac{1}{2} n \pi x\right) \right], \quad x = \frac{8}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \sin\left(\frac{1}{4} n \pi x\right) \right]$$



Question 10

A function $f(x)$ is defined in an interval $(\alpha, \alpha + 2L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(\alpha, \alpha + 2L)$, giving general expressions for the coefficients of the series.

$$f(x) = x^2, \quad 0 \leq x \leq 1.$$

- b)** Find the Fourier series of $f(x) \dots$
- i.** ... in the interval $0 \leq x \leq 1$, with period 1.
 - ii.** ... in the interval $0 \leq x \leq 1$, with period 2, by building a suitable “extension” to $f(x)$.

Illustrate the solution in each case with a sketch.

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{\cos(2n\pi x)}{n^2 \pi^2} - \frac{\sin(2n\pi x)}{n\pi} \right], \quad x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right]$$

a) If $f(x)$ is periodic with period a and (x_1, x_1+2L) , then

$$f(x) = \frac{a}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$
 where

$$a_n = \frac{1}{L} \int_{x_1}^{x_1+2L} f(x) \cos \frac{n\pi x}{L} dx \quad n=1,2,3,\dots$$

$$b_n = \frac{1}{L} \int_{x_1}^{x_1+2L} f(x) \sin \frac{n\pi x}{L} dx \quad n=1,2,3,\dots$$

b) $f(x) = x^2, 0 \leq x \leq 1, L = \frac{1}{2}$

$$a_0 = \frac{1}{L} \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

$$a_n = \frac{1}{L} \int_0^1 x^2 \cos \left(\frac{n\pi x}{L} \right) dx = \int_0^1 2x^2 \cos(2n\pi x) dx$$

By PARTS ...

$2x^2$	$\frac{1}{\cos(2n\pi x)}$
$\frac{2x^2}{2n\pi \sin(2n\pi x)}$	$\frac{1}{\cos(2n\pi x)}$

$$= \left[\frac{2x^2}{2n\pi \sin(2n\pi x)} + \frac{2}{n\pi} \int_0^1 x \cos(2n\pi x) dx \right]$$

$$= \frac{2}{n\pi} \cos(2n\pi) + \frac{2}{n\pi} \int_0^1 x \cos(2n\pi x) dx$$

By PARTS Again ...

x	$\frac{1}{\cos(2n\pi x)}$
$\frac{x}{2n\pi \sin(2n\pi x)}$	$\frac{1}{\cos(2n\pi x)}$

$$= -\frac{1}{n\pi} + \frac{2}{n\pi} \left[\frac{x \sin(2n\pi x)}{2n\pi} \right]_0^1 - \frac{1}{2n\pi} \int_0^1 \sin(2n\pi x) dx$$

$$= -\frac{1}{n\pi} - \frac{1}{4n^2\pi} \left[\frac{1}{\sin(2n\pi x)} \right]_0^1$$

$b_n = -\frac{1}{n\pi}$

$\therefore x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2\pi^2} \cos(2n\pi x) - \frac{1}{n\pi} \sin(2n\pi x) \right]$

Question 11

$$f(x) = \begin{cases} \pi - x & 0 \leq x \leq \pi \\ \pi + x & -\pi < x \leq 0 \end{cases}$$

for $x \in \mathbb{R}$, $f(x) = f(x + 2\pi)$.

- a)** Determine the Fourier series expansion of $f(x)$.
- b)** Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

- c) Show that

$$\sum_{n=0}^{\infty} \left[\frac{\sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}}{(2n+1)^2} \right] = -\frac{\pi^2}{8\sqrt{2}}$$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)x]}{(2n-1)^2} \right], \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

[illegible]

Question 12

The periodic function f is defined as

$$f(t) = \begin{cases} 0 & -1 \leq t < 0 \\ t^2 & 0 \leq t \leq 1 \end{cases}$$

for $t \in \mathbb{R}$, $f(t) = f(t+2)$.

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{6} + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n \times 2 \cos(n\pi t)}{n^2 \pi^2} + \left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3 \pi^3} [(-1)^n - 1] \right] \sin(n\pi t) \right\}$$

Handwritten solution for the Fourier series expansion of $f(t)$.

Part 1: Finding a_0 and a_n

$f(t) = \begin{cases} 0 & -1 \leq t < 0 \\ t^2 & 0 \leq t \leq 1 \end{cases}$ (Note: The original image has a typo in the definition, it should be $-1 \leq t < 0$ and $0 \leq t \leq 1$).

$a_0 = \frac{1}{2} \int_{-1}^{1} f(t) dt = \frac{1}{2} \int_0^1 t^2 dt = \frac{1}{2} \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{6}$

$a_n = \frac{1}{2} \int_{-1}^{1} f(t) \cos(n\pi t) dt = \frac{1}{2} \int_0^1 t^2 \cos(n\pi t) dt$

Integration by parts:

$$= \left[\frac{t^2}{2n\pi} \sin(n\pi t) \right]_0^1 - \frac{2}{n\pi} \int_0^1 t \sin(n\pi t) dt$$

Integration by parts again:

$$= \frac{1}{2n\pi} \sin(n\pi) - \frac{2}{n^2\pi^2} \left[-\cos(n\pi t) \right]_0^1 = \frac{1}{2n\pi} \sin(n\pi) + \frac{2}{n^2\pi^2} [\cos(n\pi) - 1]$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$:

$$a_n = \frac{2}{n^2\pi^2} [(-1)^n - 1]$$

Part 2: Finding b_n

$b_n = \frac{1}{2} \int_{-1}^{1} f(t) \sin(n\pi t) dt = \frac{1}{2} \int_0^1 t^2 \sin(n\pi t) dt$

Integration by parts:

$$= \left[-\frac{t^2}{2n\pi} \cos(n\pi t) \right]_0^1 + \frac{2}{n\pi} \int_0^1 t \cos(n\pi t) dt$$

Integration by parts again:

$$= -\frac{1}{2n\pi} \cos(n\pi) + \frac{2}{n^2\pi^2} \left[\sin(n\pi t) \right]_0^1 = -\frac{1}{2n\pi} \cos(n\pi) + \frac{2}{n^2\pi^2} \sin(n\pi)$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$:

$$b_n = -\frac{1}{2n\pi} (-1)^n = \frac{1}{2n\pi} (-1)^{n+1}$$

Final Result:

$$f(t) = \frac{1}{6} + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n \times 2 \cos(n\pi t)}{n^2 \pi^2} + \left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3 \pi^3} [(-1)^n - 1] \right] \sin(n\pi t) \right\}$$

Question 13

$$f(x) = x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2\pi.$$

$$f(x) = f(x + 2\pi).$$

- a) Determine the Fourier series expansion of $f(x)$.
- b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.$$

$$f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$$

④ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{b-a} + b_n \sin \frac{n\pi x}{b-a}$ with $a = \frac{2}{\pi} \int_a^b f(x) dx$ where period $T = 2\pi$
 $a_0 = \frac{2}{\pi} \int_0^{2\pi} f(x) \cos \frac{n\pi x}{2\pi} dx$ $\frac{2\pi}{\pi} = 2$
 $b_n = \frac{2}{\pi} \int_0^{2\pi} f(x) \sin \frac{n\pi x}{2\pi} dx$ $\frac{2}{\pi} = \frac{2}{\pi}$

- $a_0 = \frac{1}{\pi} \int_0^{2\pi} x dx = \frac{1}{\pi} [x^2]_0^{2\pi} = \frac{1}{\pi} [4\pi^2 - 0] = 2\pi$
- $a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos nx dx = \dots$ parts $\dots = \frac{1}{\pi} \left[\frac{1}{n} x \sin nx \right]_0^{2\pi} - \frac{1}{\pi} \int_0^{2\pi} \sin nx dx = 0$
- $b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx = \dots$ parts $\dots = \frac{1}{\pi} \left[-\frac{1}{n} x \cos nx \right]_0^{2\pi} + \frac{1}{\pi} \int_0^{2\pi} \cos nx dx$
 $= \frac{1}{\pi} [-2\pi \cos 2\pi] = -\frac{2\pi}{\pi} = -2$

Thus $f(x) = x = \pi + \sum_{n=1}^{\infty} -\frac{2}{n} \sin nx = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}$
 $= \pi - 2 \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \dots \right]$

⑥ At $x = \frac{\pi}{2}$ $\frac{\pi}{2} = \pi - 2 \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \right]$
 $\frac{\pi}{2} = \pi - 2 \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \right]$
 $-\frac{\pi}{2} = -2 \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \right]$
 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

Question 14

A function $f(x)$ is defined in an interval $(-\pi, \pi)$.

- State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.
- Find the Fourier series of

$$f(x) = 3x^2 - \pi^2, \quad -\pi \leq x \leq \pi.$$

- Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$3x^2 - \pi^2 = 12 \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos nx}{n^2} \right], \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

q) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad n=1,2,3,\dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad n=1,2,3,\dots$$

b) $f(x) = 3x^2 - \pi^2$ (even function)

$\therefore b_n = 0$ due to A.C. since $\int_{-\pi}^{\pi} (3x^2 - \pi^2) \sin nx dx = 0$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (3x^2 - \pi^2) dx = \frac{2}{\pi} \int_0^{\pi} (3x^2 - \pi^2) dx$

$$= \frac{2}{\pi} \left[x^3 - \pi^2 x \right]_0^{\pi} = \frac{2}{\pi} (\pi^3 - \pi^3) = 0$$

$\therefore a_0 = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (3x^2 - \pi^2) \cos nx dx$... BY PARTS ...

$a_n = \frac{1}{\pi} \left[\left(\frac{3x^2}{n} \right) \sin nx - \int_{-\pi}^{\pi} \frac{6x}{n} \sin nx dx \right]$

$a_n = -\frac{6}{n^2} \int_{-\pi}^{\pi} x \cos nx dx$... BY PARTS ...

$a_n = \frac{12}{n^3} \left[x \sin nx - \int_{-\pi}^{\pi} \sin nx dx \right]$

$a_n = \frac{12}{n^3} \left[x \sin nx + \frac{1}{n} \cos nx \right]_{-\pi}^{\pi}$

$a_n = \frac{12}{n^3} \left[\pi \sin n\pi + \frac{1}{n} \cos n\pi - (-\pi \sin n\pi + \frac{1}{n} \cos n\pi) \right]$

$a_n = \frac{12}{n^3} \left[\frac{1}{n} \cos n\pi - \frac{1}{n} \cos n\pi \right] = 0$

$a_n = 0$

$a_1 = \frac{12}{\pi^3} \left[\left(\frac{2 \cos nx}{n} \right)_{-\pi}^{\pi} - \frac{1}{n} \left(\frac{2 \sin nx}{n} \right)_{-\pi}^{\pi} \right]$

$a_1 = \frac{12}{\pi^3} \times \left(\frac{2 \cos n\pi}{n} - 0 \right)$

$a_1 = \frac{12}{\pi^3} (-1)^n$

$\therefore f(x) = 3x^2 - \pi^2 = \sum_{n=1}^{\infty} a_n \cos nx$

$3x^2 - \pi^2 = \sum_{n=1}^{\infty} \frac{12(-1)^n \cos nx}{n^3}$

or $3x^2 - \pi^2 = \frac{12}{\pi^3} \cos x + \frac{12}{\pi^3} \cos 2x - \frac{12}{\pi^3} \cos 3x + \frac{12}{\pi^3} \cos 4x + \dots$

q) If $2=0$

$-\pi^2 = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \cos 0$

$-\pi^2 = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3}$

$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Question 15

$$f(x) = |x|, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

- a) Determine the Fourier series expansion of $f(x)$.
- b) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}, \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

0) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$ with $a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$ (L=Half Period)
 $a_0 = \frac{2}{L} \int_0^L f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$
 $a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$ (Integration by parts)
 $= \frac{2}{\pi} \left[\frac{x \sin nx}{n} - \int \frac{\sin nx}{n} dx \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{\pi \sin n\pi}{n} - \left[-\frac{\cos nx}{n^2} \right]_0^{\pi} \right]$
 $= \frac{2}{\pi} \left[0 - \left[-\frac{\cos n\pi}{n^2} + \frac{\cos 0}{n^2} \right] \right] = \frac{2}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right]$
 $= \frac{2}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$
 $= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$
 $= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$
 $a_{2m-1} = -\frac{4}{\pi (2m-1)^2}$
 $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = 0$ (NOTING IS ODD IN A SYMMETRIC DOMAIN)

Thus $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos nx$
 $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$
 i.e. $|x| = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots \right]$
 b) let $x=0$
 $0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$
 $0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$
 $\frac{4}{\pi} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi}{2}$
 $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$

Question 16

$$f(x) = x, \quad x \in \mathbb{R}, \quad -1 \leq x \leq 1.$$

$$f(x) = f(x+2).$$

- a) Determine the Fourier series expansion of $f(x)$.
- b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n}.$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n} = \frac{1}{2}$$

(a) $f(x) = \frac{x}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2}$ $a_n = \frac{2}{\pi} \int_{-1}^1 f(x) \cos \frac{n\pi x}{2} dx$ $b_n = \frac{2}{\pi} \int_{-1}^1 f(x) \sin \frac{n\pi x}{2} dx$

$a_0 = \frac{1}{\pi} \int_{-1}^1 x dx = 0$ (odd function in a symmetrical domain)

$a_n = \frac{1}{\pi} \int_{-1}^1 x \cos \frac{n\pi x}{2} dx = 0$ (odd function in a symmetrical domain)

$b_n = \frac{1}{\pi} \int_{-1}^1 x \sin \frac{n\pi x}{2} dx = \frac{2}{\pi} \int_0^1 x \sin \frac{n\pi x}{2} dx = \dots$ by parts \dots

$= \frac{2}{\pi} \left[-\frac{1}{n\pi} \cos \frac{n\pi x}{2} + \frac{1}{n\pi} \left[\frac{n\pi x}{2} \sin \frac{n\pi x}{2} \right]_0^1 \right] = \frac{2}{n\pi} (-1)^{n+1}$

Thus $x = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{2}$

(b) $x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin \frac{n\pi x}{2}}{n}$ if $x = \frac{1}{2} \Rightarrow \frac{1}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin \frac{n\pi}{2}}{n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n} = \frac{1}{2}$

Question 17

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad -2 \leq x \leq 2.$$

$$f(x) = f(x+4).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{1}{2} n \pi x\right)$$

$f(x) = x^2, -2 < x < 2 \quad f(x) = f(x+4)$
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}]$, $L = \text{half period}$
 $a_0 = \frac{1}{L} \int_a^{a+2L} f(x) \cos \frac{n\pi x}{L} dx \quad n = 0, 1, 2, 3, \dots$
 $b_n = \frac{1}{L} \int_a^{a+2L} f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, 4, \dots$

• Now $f(x)$ is $C(\infty)$, so b_n are all zero as $\sin \frac{n\pi x}{2}$ is odd.
 • Evaluating a_n - here $L = 2$
 $\Rightarrow a_0 = \frac{1}{2} \int_{-2}^2 x^2 dx = \int_{-2}^2 x^2 dx = \left[\frac{1}{3} x^3 \right]_{-2}^2 = \frac{8}{3}$
 $\Rightarrow a_n = \frac{1}{2} \int_{-2}^2 x^2 \cos \frac{n\pi x}{2} dx = \int_{-2}^2 x^2 \cos \frac{n\pi x}{2} dx$

By parts	
$\frac{x^2}{\frac{2n\pi}{2} \sin \frac{n\pi x}{2}}$	$\frac{2x}{2n \cos \frac{n\pi x}{2}}$

$\Rightarrow a_n = \frac{2}{n\pi} \left[x^2 \sin \frac{n\pi x}{2} \right]_{-2}^2 - \frac{4}{n\pi} \int_{-2}^2 2x \sin \frac{n\pi x}{2} dx$
 $\Rightarrow a_n = -\frac{8}{n\pi} \int_{-2}^2 x \sin \frac{n\pi x}{2} dx$

By parts again	
$\frac{x}{-\frac{2n\pi}{2} \cos \frac{n\pi x}{2}}$	$\frac{1}{-2n \sin \frac{n\pi x}{2}}$

$\Rightarrow a_n = -\frac{1}{n\pi} \left[-\frac{2}{n\pi} \left[2x \cos \frac{n\pi x}{2} \right]_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} dx \right]$
 $\Rightarrow a_n = \frac{8}{n^2 \pi^2} \left(2 \cos(n\pi) \right) - \frac{8}{n^2 \pi^2} \int_0^2 \cos \frac{n\pi x}{2} dx$
 $\Rightarrow a_n = \frac{16(-1)^n}{n^2 \pi^2} - \frac{8}{n^2 \pi^2} \left[\sin \frac{n\pi x}{2} \right]_0^2$
 $\Rightarrow a_n = \frac{16(-1)^n}{n^2 \pi^2}$

• Check the Fourier series of $f(x) = x^2, f(x) = f(x+4), -2 < x < 2$ is
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{16(-1)^n}{n^2 \pi^2} \cos \frac{n\pi x}{2} \right]$
 $f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left(\frac{n\pi x}{2} \right)$

Question 18

A function $f(x)$ is defined in the interval $(-\pi, \pi)$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.
- b) Find the Fourier series of

$$f(x) = x, \quad -\pi \leq x \leq \pi.$$

- c) Hence determine the exact value of

$$g(x) = x^2, \quad -\pi \leq x \leq \pi.$$

$$f(x) = 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \sin nx}{n} \right], \quad g(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos nx}{n^2} \right]$$

1) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$

2) Now $f(x) = x$, $-\pi < x < \pi$ is odd

• Hence $a_0 = 0$ as the integrand will be odd in a symmetric interval

• $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$

... by parts ...

$$= \frac{2}{\pi} \left[-\frac{1}{n} x \cos nx + \frac{1}{n} \int \cos nx dx \right]_0^{\pi} = \frac{2}{\pi} \left[-\frac{\pi \cos n\pi}{n} + \frac{1}{n^2} \sin n\pi \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[0 - \pi \cos n\pi \right] = \frac{2}{\pi} \pi (-1)^{n+1} = \frac{2}{\pi} (-1)^{n+1}$$

hence $f(x) = x = \frac{2}{\pi} \sin x - \frac{2}{\pi} \sin 3x + \frac{2}{3\pi} \sin 5x - \frac{2}{5\pi} \sin 7x + \dots$

$$f(x) = x = \frac{2}{\pi} \left(\sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{7} \sin 7x + \frac{1}{9} \sin 9x - \dots \right)$$

3) $g(x) = x^2$

$$\frac{d}{dx} g(x) = 2x$$

$$\frac{d}{dx} \left(\frac{g(x)}{n^2} \right) = 2 \frac{f(x)}{n}$$

$$g(x) = 2 \int f(x) dx$$

4) $g(x) = 2 \int \left(\sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{7} \sin 7x + \dots \right) dx$

$$= 2 \left(-\cos x + \frac{1}{9} \cos 3x - \frac{1}{25} \cos 5x + \frac{1}{49} \cos 7x - \dots \right) + C$$

To find the constant

• Consider $x=0$

$$0 = -4 \left(1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \dots \right) + C$$

$$0 = -4 \times \frac{\pi^2}{12} + C$$

$$0 = -\frac{\pi^2}{3} + C$$

$$C = \frac{\pi^2}{3}$$

• or directly

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2}{3\pi} \left[\pi^3 \right]$$

$$= \frac{2}{3\pi} \times \pi^3 = \frac{2\pi^2}{3}$$

• $\therefore \frac{a_0}{2} = \frac{\pi^2}{3}$

4) hence $g(x) = x^2 = \frac{\pi^2}{3} - \frac{4}{\pi^2} \left(\cos x - \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x - \frac{1}{49} \cos 7x + \dots \right)$

$$g(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

Question 19

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 1.$$

Determine the Fourier series of $f(x)$ as

a) ... as half range cosine expansion.

b) ... as half range sine expansion.

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right],$$

$$f(x) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{2}{n^3} \left[(2 - n^2 \pi^2) (-1)^n - 2 \right] \sin(n\pi x) \right]$$

a) $f(x) = x^2 \quad 0 \leq x \leq 1$

• As $f(x)$ is even $a_0 \neq 0$ $a_1, a_2, a_3, \dots \neq 0$ $b_1, b_2, b_3, \dots = 0$

• $a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{1} \int_{-1}^1 x^2 dx = 2 \int_0^1 x^2 dx$
 $= \frac{2}{3} [x^3]_0^1 = \frac{2}{3}$

• $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{1} \int_{-1}^1 x^2 \cos n\pi x dx$
 $a_n = 2 \int_0^1 x^2 \cos n\pi x dx$... BY PARTS

x^2	$2x$
$\frac{1}{\pi^2} \sin n\pi x$	$\cos n\pi x$

$a_n = 2 \left[\frac{x^2}{\pi^2} \sin n\pi x \right]_0^1 - \frac{2}{\pi^2} \int_0^1 x \sin n\pi x dx$

$a_n = -\frac{4}{\pi^2} \left[\frac{x}{n\pi} \cos n\pi x \right]_0^1 + \frac{4}{\pi^2} \int_0^1 \cos n\pi x dx$

$a_n = -\frac{4}{\pi^2} \left[\frac{1}{n\pi} \cos n\pi + \frac{1}{n^2 \pi^2} \sin n\pi x \right]_0^1$

$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{1} \int_{-1}^1 x^2 \cos n\pi x dx$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$

$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{\pi^2 n^2} (-1)^n \cos(n\pi x) \right]$

$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right]$

b) $f(x) = x^2 \quad 0 \leq x \leq 1$

• AS THE FUNCTION IS ODD $a_0, a_1, a_2, a_3, \dots = 0$

• $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{1} \int_{-1}^1 x^2 \sin n\pi x dx$
 $= \int_0^1 2x^2 \sin n\pi x dx$... BY PARTS

x^2	$2x$
$-\frac{1}{\pi^2} \sin n\pi x$	$\cos n\pi x$

$b_n = \left[\frac{2x^2}{\pi^2} \cos n\pi x \right]_0^1 - \frac{4}{\pi^2} \int_0^1 x \cos n\pi x dx$

$= -\frac{2 \cos n\pi}{\pi^2} + \frac{4}{\pi^2} \int_0^1 x \cos n\pi x dx$

$= -\frac{2 \cos n\pi}{\pi^2} + \frac{4}{\pi^2} \left[\frac{x}{n\pi} \sin n\pi x \right]_0^1$

BY PARTS AGAIN

x	1
$\frac{1}{\pi^2} \sin n\pi x$	$\cos n\pi x$

$= -\frac{2 \cos n\pi}{\pi^2} + \frac{4}{\pi^2} \left[\frac{x}{n\pi} \sin n\pi x \right]_0^1 - \frac{4}{\pi^2} \int_0^1 \sin n\pi x dx$

$= -\frac{2 \cos n\pi}{\pi^2} - \frac{4}{\pi^2} \left[\frac{1}{n\pi} \cos n\pi x \right]_0^1$

$= -\frac{2 \cos n\pi}{\pi^2} + \frac{4}{\pi^2} \left[\cos n\pi x - 1 \right]_0^1$

$= -\frac{2 \cos n\pi}{\pi^2} + \frac{4}{\pi^2} \left[\cos n\pi - 1 \right]$

$= \frac{(2 \cos n\pi + 4) (-1)^n - 4}{\pi^2} = \frac{2}{\pi^2} [(2 - \cos n\pi) (-1)^n - 2]$

$\therefore x^2 = \sum_{n=1}^{\infty} \left[\frac{2}{\pi^2 n^3} (2 - \cos n\pi) (-1)^n - 2 \right] \sin n\pi x$

Question 20

$$f(x) = \begin{cases} \pi - x & 0 \leq x \leq \pi \\ 0 & \pi \leq x \leq 2\pi \end{cases}, \quad x \in \mathbb{R}.$$

$$f(x) = f(x + 2\pi).$$

- a)** Determine the Fourier series expansion of $f(x)$.

- b)** Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$f(x) = \frac{\pi}{4} + \sum_{m=1}^{\infty} \left[\frac{2 \cos[(2m-1)x]}{\pi(2m-1)^2} + \frac{\sin mx}{m} \right], \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

4) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n \cos nx + b_n \sin nx}{2} \right)$ where $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$ where $L = \pi$ (here period)

$(L = \text{half period})$

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$ $f(x) = \begin{cases} x - \pi & 0 \leq x < \pi \\ 0 & \pi \leq x < 2\pi \end{cases}$

$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$

• $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} (x - \pi) dx = \frac{1}{\pi} \left[\frac{x^2}{2} - \pi x \right]_0^{\pi} = \frac{1}{\pi} \left[\frac{\pi^2}{2} - \pi^2 \right] = \frac{1}{\pi} \times \pi^2 = \pi$

• $a_n = \frac{1}{\pi} \int_0^{\pi} (x - \pi) \cos nx dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 0 \cos nx dx$

= by parts ...

$\begin{matrix} \text{u} & \text{dv} \\ x - \pi & \cos nx \end{matrix}$

$= \frac{1}{\pi} \left\{ \frac{(x - \pi) \sin nx}{n} + \frac{1}{n} \int \sin nx dx \right\}$

$= \frac{1}{n\pi} \int_0^{\pi} \sin nx dx$

$= \frac{1}{n\pi} \left[-\cos nx \right]_0^{\pi}$

$= \frac{1}{n\pi} [\cos nx]_0^{\pi}$

$= \frac{1}{n\pi} [1 - 1] = 0$ if $\sin \pi = 0$

$\neq 0$ if $\sin \pi \neq 0$

$\therefore a_n = 0$ when $\sin \pi = 0$

$\therefore \dots$ by parts ...

$\begin{matrix} \text{u} & \text{dv} \\ x - \pi & \sin nx \end{matrix}$

$= \frac{1}{\pi} \left\{ \frac{(x - \pi) \cos nx}{n} - \frac{1}{n} \int \cos nx dx \right\}$

$= \frac{1}{\pi} \left\{ \frac{(x - \pi) \cos nx}{n} - \frac{1}{n^2} \sin nx \right\}_0^{\pi}$

$= \frac{1}{\pi} \times \frac{\pi^2}{n^2}$

$= \frac{\pi}{n^2}$

$\therefore f(x) = \frac{\pi}{2} - \frac{\pi}{n^2} \sin nx$

$$\begin{aligned} \text{Th 5} \\ (a) &= \frac{a^2}{2} + \sum_{k=1}^{\infty} \left[\frac{1}{2k} \cos k\pi x + \frac{1}{2k+1} \sin k\pi x \right] \quad (L=1) \\ (b) &= \frac{a^2}{2} + \sum_{k=1}^{\infty} \left[\frac{1}{2k} \cos \left(\frac{(2k-1)\pi}{2} x \right) + \frac{\sin k\pi x}{k+1} \right] \end{aligned}$$

Question 21

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

- a) Determine the Fourier series expansion of $f(x)$.
- b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Handwritten solution for Question 21:

a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T}$ where $a_0 = \frac{2}{T} \int_{-\pi}^{\pi} f(x) dx$, $a_n = \frac{2}{T} \int_{-\pi}^{\pi} f(x) \cos \frac{2n\pi x}{T} dx$, $b_n = \frac{2}{T} \int_{-\pi}^{\pi} f(x) \sin \frac{2n\pi x}{T} dx$. For $f(x) = x^2$, $T = 2\pi$, $\frac{2\pi}{T} = 1$, $\frac{T}{2\pi} = \frac{1}{2}$.

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{1}{3} x^3 \right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3} = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$ (parts ... $\frac{2}{\pi} \left[\left(\frac{1}{n} x^2 \sin nx \right)_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \sin nx dx \right]$
 $= -\frac{4}{n\pi} \int_0^{\pi} x \sin nx dx$ (parts again ... $-\frac{4}{n\pi} \left[\left(-\frac{1}{n} x \cos nx \right)_0^{\pi} + \frac{4}{n^2} \int_0^{\pi} \cos nx dx \right]$
 $= -\frac{4}{n^2\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi} = \frac{4}{n^2\pi} (\cos n\pi - 1) = \frac{4}{n^2\pi} (-1)^n - \frac{4}{n^2\pi}$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx dx = 0$ (odd function in the integrand in a symmetric domain)

$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi} (-1)^n \cos nx = \frac{\pi^2}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$
 $= \frac{\pi^2}{3} + \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \frac{\cos 4x}{16} - \dots \right]$

b) Let $x=0$: $0^2 = \frac{\pi^2}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos 0}{n^2}$
 $0 = \frac{\pi^2}{3} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
 $\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Question 22

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2\pi.$$

$$f(x) = f(x + 2\pi).$$

- a) Determine the Fourier series expansion of $f(x)$.
- b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$f(x) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n} \right], \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Handwritten solution for part (a):

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$ where $a_0 = \frac{1}{L} \int_0^{2\pi} f(x) dx$ and $a_n = \frac{1}{L} \int_0^{2\pi} f(x) \cos \frac{n\pi x}{L} dx$, $b_n = \frac{1}{L} \int_0^{2\pi} f(x) \sin \frac{n\pi x}{L} dx$.

$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \frac{1}{2\pi} [x^3]_0^{2\pi} = \frac{8\pi^3}{3\pi} = \frac{8\pi^2}{3}$

$a_n = \frac{1}{2\pi} \int_0^{2\pi} x^2 \cos nx dx$ (by parts) ... $= \frac{1}{2\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{2\pi} = \frac{1}{2\pi} \left[\frac{4\pi^2 \sin 2\pi}{n} + \frac{4\pi \cos 2\pi}{n^2} - \frac{2 \sin 2\pi}{n^3} \right] = \frac{1}{2\pi} \left[\frac{4\pi}{n^2} \right] = \frac{2}{n^2}$

$b_n = \frac{1}{2\pi} \int_0^{2\pi} x^2 \sin nx dx$ (by parts) ... $= \frac{1}{2\pi} \left[-\frac{x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right]_0^{2\pi} = \frac{1}{2\pi} \left[-\frac{4\pi^2 \cos 2\pi}{n} + \frac{4\pi \sin 2\pi}{n^2} + \frac{2 \cos 2\pi}{n^3} \right] = \frac{1}{2\pi} \left[-\frac{4\pi^2}{n} + \frac{2}{n^3} \right] = -\frac{2\pi}{n} + \frac{1}{n^3}$

Handwritten solution for part (b):

Let $x = \pi$

$f(\pi) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{\cos n\pi}{n^2} - \frac{\pi \sin n\pi}{n} \right]$

$\pi^2 = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\pi^2 - \frac{4\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$-\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Question 23

It is given that for $x \in \mathbb{R}$, $-\pi \leq x \leq \pi$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}, \quad |x| = |x + 2\pi|.$$

- Use the above Fourier series expansion to deduce the Fourier series expansion of $\operatorname{sgn}(x)$.
- Verify the answer of part (a) by obtaining directly the Fourier series expansion of $\operatorname{sgn}(x)$.
- Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1}$$

$$\boxed{}, \quad \operatorname{sgn}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{(2n-1)^2}, \quad \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1} = \frac{\pi}{4}$$

a) $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}$

$$\operatorname{sgn}(x) = \frac{d}{dx} |x| = \frac{d}{dx} \left[\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2} \right]$$

$$= -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-(2n-1) \sin[(2n-1)x]}{(2n-1)^2}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{(2n-1)}$$

b) NOW FIND THE FOURIER EXPANSION DIRECTLY

$$f(x) = \operatorname{sgn}(x), \quad -\pi < x < \pi, \quad f(x+2\pi) = f(x)$$

- $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x) dx = 0$ (ODD FUNCTION IN A SYMMETRIC INTERVAL)
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x) \cos(nx) dx = 0$ (ODD FUNCTION IN A SYMMETRIC INTERVAL)
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x) \sin(nx) dx$

$$= \frac{1}{\pi} \times 2 \int_0^{\pi} \sin(nx) dx = \frac{2}{\pi} \left[-\frac{1}{n} \cos(nx) \right]_0^{\pi}$$

$$= \frac{2}{\pi n} [\cos(n\pi) - 1] = \frac{2}{\pi n} [1 - \cos(n\pi)] = \frac{2}{\pi n} [1 - (-1)^n]$$

$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$

HENCE WE REWRITE THE COEFFICIENT AS

$$b_n = \frac{4}{\pi(2n-1)}, \quad n=1,2,3,\dots$$

HENCE WE CAN SUBSTITUTE INTO THE FOURIER FORMULA

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{b-a}\right) + b_n \sin\left(\frac{n\pi x}{b-a}\right) \right]$$

$$\operatorname{sgn}(x) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi(2n-1)} \sin\left(\frac{(2n-1)\pi x}{2\pi}\right) \right]$$

$$\operatorname{sgn}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{(2n-1)}$$

c) SUBSTITUTING $x = \frac{\pi}{2}$ INTO THE ABOVE FORMULA GIVES

$$\operatorname{sgn}\left(\frac{\pi}{2}\right) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left[\frac{(2n-1)\pi}{2}\right]}{(2n-1)}$$

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} = \frac{\pi}{4}$$

OR $\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

Question 24

$$f(x) = \begin{cases} -x & 0 \leq x \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = f(x + \pi).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = -\frac{\pi}{8} + \sum_{n=1}^{\infty} \frac{\cos[(4n-2)x]}{(2n-1)^2 \pi} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \sin 2nx}{n}$$

Handwritten solution for the Fourier series expansion of $f(x)$.

Given $f(x) = \begin{cases} -x & 0 \leq x \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$, with period $T = \pi$.

Calculations:

- $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \left[-\frac{x^2}{2} \right]_0^{\pi/2} = -\frac{\pi}{8}$
- $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi/2} -x \cos nx dx = \frac{1}{\pi} \int_0^{\pi/2} x \cos nx dx$
- $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi/2} -x \sin nx dx = -\frac{1}{\pi} \int_0^{\pi/2} x \sin nx dx$

Final series expansion:

$$f(x) = -\frac{\pi}{8} + \sum_{n=1}^{\infty} \frac{\cos[(4n-2)x]}{(2n-1)^2 \pi} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \sin 2nx}{n}$$

Question 25

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- b) Determine the Fourier series of

$$f(x) = e^x, \quad -\pi \leq x \leq \pi.$$

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [\cos(nx) - n \sin(nx)]}{1+n^2} \right]$$

a) If $f(x)$ is periodic continuous and $[-L, L]$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{b-a} + b_n \sin \frac{n\pi x}{b-a}]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{b-a} dx$ for $n=1, 2, 3, 4, \dots$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{b-a} dx$$
 for $n=1, 2, 3, 4, \dots$

b) $f(x) = e^x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(n+x) dx = \frac{1}{\pi(n+i)} \left[e^x \sin(n+x) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1+i)} \left[e^{\pi} \sin(n+\pi) - e^{-\pi} \sin(n-\pi) \right]$$

$$= \frac{1-i}{\pi(1+i)} \left[\pi e^{\pi} - e^{-\pi} e^{-i\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1+i} \right] \left[\pi (e^{\pi} + 1) - e^{-\pi} (e^{\pi} - 1) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1+i} \right] \left[\pi (e^{\pi} + 1) - e^{-\pi} (e^{\pi} + 1) + e^{-\pi} (e^{\pi} - 1) \right]$$

$$= \frac{1}{\pi(1+i)} (1-i) \left[(e^{\pi} + 1) (1 - e^{-\pi}) + i \sin \pi (e^{\pi} + e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1+i)} (1-i) \left[2 \sinh \pi \cosh \pi + i \cosh \pi \sinh \pi \right]$$

$$= \frac{1}{\pi(1+i)} \left[2 \sinh \pi \frac{e^{\pi} + 1}{2} + 2 \cosh \pi \frac{e^{\pi} - 1}{2} \right] + i \left[-2 \sinh \pi \frac{e^{\pi} - 1}{2} + 2 \cosh \pi \frac{e^{\pi} + 1}{2} \right]$$

$$= \frac{2}{\pi(1+i)} \left[(1-i) \sinh \pi - i(1-i) \cosh \pi \right]$$

c) $f(x) = \cos x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x+n) dx = \frac{1}{\pi(n+i)} \left[\sin(x+n) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1+i)} \left[\sin(\pi+n) - \sin(-\pi+n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1+i} \right] \left[\sin(\pi+n) - \sin(-\pi+n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1+i} \right] \left[\sin(\pi+n) - \sin(\pi-n) \right]$$

$$= \frac{1}{\pi(1+i)} (1-i) \left[\sin \pi (e^{\pi} + e^{-\pi}) + i \sin \pi (e^{\pi} - e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1+i)} (1-i) \left[2 \sinh \pi \cosh \pi + i \cosh \pi \sinh \pi \right]$$

$$= \frac{1}{\pi(1+i)} \left[2 \sinh \pi \frac{e^{\pi} + 1}{2} + 2 \cosh \pi \frac{e^{\pi} - 1}{2} \right] + i \left[-2 \sinh \pi \frac{e^{\pi} - 1}{2} + 2 \cosh \pi \frac{e^{\pi} + 1}{2} \right]$$

$$= \frac{2}{\pi(1+i)} \left[(1-i) \sinh \pi - i(1-i) \cosh \pi \right]$$

d) $f(x) = \sin x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x+n) dx = \frac{1}{\pi(n+i)} \left[-\cos(x+n) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1+i)} \left[-\cos(\pi+n) + \cos(-\pi+n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1+i} \right] \left[-\cos(\pi+n) + \cos(-\pi+n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1+i} \right] \left[-\cos(\pi+n) + \cos(\pi-n) \right]$$

$$= \frac{1}{\pi(1+i)} (1-i) \left[-\cos \pi (e^{\pi} + e^{-\pi}) + i \sin \pi (e^{\pi} - e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1+i)} (1-i) \left[-2 \cosh \pi \cosh \pi + i \sinh \pi \sinh \pi \right]$$

$$= \frac{1}{\pi(1+i)} \left[-2 \cosh \pi \frac{e^{\pi} + 1}{2} + 2 \sinh \pi \frac{e^{\pi} - 1}{2} \right] + i \left[-2 \cosh \pi \frac{e^{\pi} - 1}{2} + 2 \sinh \pi \frac{e^{\pi} + 1}{2} \right]$$

$$= \frac{2}{\pi(1+i)} \left[(-1-i) \cosh \pi - i(-1-i) \sinh \pi \right]$$

e) $f(x) = \sin x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x-n) dx = \frac{1}{\pi(n-i)} \left[-\cos(x-n) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1-i)} \left[-\cos(\pi-n) + \cos(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[-\cos(\pi-n) + \cos(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[-\cos(\pi-n) + \cos(\pi+n) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[-\cos \pi (e^{\pi} + e^{-\pi}) + i \sin \pi (e^{\pi} - e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[-2 \cosh \pi \cosh \pi + i \sinh \pi \sinh \pi \right]$$

$$= \frac{1}{\pi(1-i)} \left[-2 \cosh \pi \frac{e^{\pi} + 1}{2} + 2 \sinh \pi \frac{e^{\pi} - 1}{2} \right] + i \left[-2 \cosh \pi \frac{e^{\pi} - 1}{2} + 2 \sinh \pi \frac{e^{\pi} + 1}{2} \right]$$

$$= \frac{2}{\pi(1-i)} \left[(-1+i) \cosh \pi - i(-1+i) \sinh \pi \right]$$

f) $f(x) = \cos x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x-n) dx = \frac{1}{\pi(n-i)} \left[\sin(x-n) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1-i)} \left[\sin(\pi-n) - \sin(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[\sin(\pi-n) - \sin(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[\sin(\pi-n) - \sin(\pi+n) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[\sin \pi (e^{\pi} - e^{-\pi}) + i \sin \pi (e^{\pi} + e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[2 \sinh \pi \sinh \pi + i \cosh \pi \cosh \pi \right]$$

$$= \frac{1}{\pi(1-i)} \left[2 \sinh \pi \frac{e^{\pi} - 1}{2} + 2 \cosh \pi \frac{e^{\pi} + 1}{2} \right] + i \left[-2 \sinh \pi \frac{e^{\pi} + 1}{2} + 2 \cosh \pi \frac{e^{\pi} - 1}{2} \right]$$

$$= \frac{2}{\pi(1-i)} \left[(1+i) \sinh \pi + i(1+i) \cosh \pi \right]$$

g) $f(x) = \sin x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x-n) dx = \frac{1}{\pi(n-i)} \left[-\cos(x-n) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1-i)} \left[-\cos(\pi-n) + \cos(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[-\cos(\pi-n) + \cos(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[-\cos(\pi-n) + \cos(\pi+n) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[-\cos \pi (e^{\pi} + e^{-\pi}) + i \sin \pi (e^{\pi} - e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[-2 \cosh \pi \cosh \pi + i \sinh \pi \sinh \pi \right]$$

$$= \frac{1}{\pi(1-i)} \left[-2 \cosh \pi \frac{e^{\pi} + 1}{2} + 2 \sinh \pi \frac{e^{\pi} - 1}{2} \right] + i \left[-2 \cosh \pi \frac{e^{\pi} - 1}{2} + 2 \sinh \pi \frac{e^{\pi} + 1}{2} \right]$$

$$= \frac{2}{\pi(1-i)} \left[(-1+i) \cosh \pi - i(-1+i) \sinh \pi \right]$$

h) $f(x) = \cos x, x \in (-\pi, \pi)$

Compute the Fourier

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x-n) dx = \frac{1}{\pi(n-i)} \left[\sin(x-n) \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1-i)} \left[\sin(\pi-n) - \sin(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[\sin(\pi-n) - \sin(-\pi-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{1-i}{1-i} \right] \left[\sin(\pi-n) - \sin(\pi+n) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[\sin \pi (e^{\pi} - e^{-\pi}) + i \sin \pi (e^{\pi} + e^{-\pi}) \right]$$

$$= \frac{1}{\pi(1-i)} (1-i) \left[2 \sinh \pi \sinh \pi + i \cosh \pi \cosh \pi \right]$$

$$= \frac{1}{\pi(1-i)} \left[$$

Question 26

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- Show that

$$\int_{-\pi}^{\pi} e^{ax} e^{inx} dx = \frac{2(a - in)(-1)^n \sinh(a\pi)}{a^2 + n^2}$$

- Determine the Fourier series of

$$f(x) = e^{ax}, \quad a > 0, \quad -\pi \leq x \leq \pi.$$

- Hence find the Fourier series of $\cosh(ax)$ and $\sinh(ax)$, for $-\pi \leq x \leq \pi$.

$$e^{ax} = \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [a \cos(nx) - n \sin(nx)]}{a^2 + n^2} \right],$$

$$\cosh(ax) = \frac{\sinh(a\pi)}{a\pi} + \frac{2a \sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos(nx)}{a^2 + n^2} \right],$$

$$\sinh(ax) = \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1} \sin(nx)}{a^2 + n^2} \right]$$

a) If $f(x)$ is piecewise continuous on $[-L, L]$, then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \quad \text{where } a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

b) $\int_{-\pi}^{\pi} e^{ax} e^{inx} dx = \int_{-\pi}^{\pi} e^{(a+in)x} dx = \left[\frac{e^{(a+in)x}}{a+in} \right]_{-\pi}^{\pi} = \frac{e^{(a+in)\pi} - e^{-(a+in)\pi}}{a+in} = \frac{e^{a\pi} e^{in\pi} - e^{-a\pi} e^{-in\pi}}{a+in} = \frac{e^{a\pi} (-1)^n - e^{-a\pi} (-1)^n}{a+in} = \frac{(-1)^n (e^{a\pi} - e^{-a\pi})}{a+in} = \frac{2(-1)^n \sinh(a\pi)}{a+in}$

c) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx = \frac{1}{\pi} \left[\frac{e^{ax}}{a} \right]_{-\pi}^{\pi} = \frac{1}{\pi a} (e^{a\pi} - e^{-a\pi}) = \frac{2\sinh(a\pi)}{\pi a}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \cos nx + n \sin nx) \right]_{-\pi}^{\pi} = \frac{1}{\pi} \times \frac{2a(-1)^n \sinh(a\pi)}{a^2 + n^2} = \frac{2a(-1)^n \sinh(a\pi)}{\pi(a^2 + n^2)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx dx = \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \sin nx - n \cos nx) \right]_{-\pi}^{\pi} = \frac{1}{\pi} \times \frac{-2n(-1)^{n+1} \sinh(a\pi)}{a^2 + n^2} = \frac{2n(-1)^{n+1} \sinh(a\pi)}{\pi(a^2 + n^2)}$$

$$\therefore f(x) = e^{ax} = \frac{\sinh(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n \sinh(a\pi)}{\pi(a^2 + n^2)} \cos nx + \frac{2n(-1)^{n+1} \sinh(a\pi)}{\pi(a^2 + n^2)} \sin nx \right]$$

d) $\cosh(ax) = \frac{e^{ax} + e^{-ax}}{2}$

$$= \frac{1}{2} \left(\frac{2\sinh(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n \sinh(a\pi)}{\pi(a^2 + n^2)} \cos nx + \frac{2n(-1)^{n+1} \sinh(a\pi)}{\pi(a^2 + n^2)} \sin nx \right] + \frac{2\cosh(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n \cosh(a\pi)}{\pi(a^2 + n^2)} \cos nx + \frac{2n(-1)^{n+1} \cosh(a\pi)}{\pi(a^2 + n^2)} \sin nx \right] \right)$$

$$= \frac{\sinh(a\pi)}{a\pi} + \frac{\cosh(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n (\sinh(a\pi) + \cosh(a\pi))}{\pi(a^2 + n^2)} \cos nx + \frac{2n(-1)^{n+1} (\sinh(a\pi) + \cosh(a\pi))}{\pi(a^2 + n^2)} \sin nx \right]$$

$$= \frac{\sinh(a\pi) + \cosh(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n e^{a\pi}}{\pi(a^2 + n^2)} \cos nx + \frac{2n(-1)^{n+1} e^{a\pi}}{\pi(a^2 + n^2)} \sin nx \right]$$

$$= \frac{e^{a\pi}}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n e^{a\pi}}{\pi(a^2 + n^2)} \cos nx + \frac{2n(-1)^{n+1} e^{a\pi}}{\pi(a^2 + n^2)} \sin nx \right]$$

Question 27

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- Determine the Fourier series of

$$f(x) = e^x, \quad -\pi \leq x \leq \pi.$$

- Hence find the Fourier series of $\sinh x$ and $\cosh x$, for $-\pi \leq x \leq \pi$.

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [\cos(nx) - n \sin(nx)]}{1+n^2} \right],$$

$$\sinh x = \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1} \sin(nx)}{1+n^2} \right],$$

$$\cosh x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos(nx)}{1+n^2} \right],$$

Handwritten solutions for Question 27:

Left page: Derives the Fourier series for $f(x) = e^x$ on $(-\pi, \pi)$. It starts with the general formula for the Fourier series, then calculates the coefficients a_n and b_n using integration by parts. The final result is
$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [\cos(nx) - n \sin(nx)]}{1+n^2} \right]$$

Right page: Derives the Fourier series for $\sinh x$ and $\cosh x$ using the results from the first part. It uses the identity $e^x = \cosh x + \sinh x$ and the Fourier series for e^x to find the series for $\cosh x$ and $\sinh x$. The final results are
$$\sinh x = \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1} \sin(nx)}{1+n^2} \right]$$
 and
$$\cosh x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos(nx)}{1+n^2} \right]$$

Question 28

A function f is defined by

$$f(t) = V|\cos \omega t|, \quad t \in \mathbb{R},$$

where V and ω are positive constants.

Show that the Fourier series of f is given by

$$f(t) = \frac{2V}{\pi} + \frac{4V}{\pi} \left[\frac{1}{3} \cos(2\omega t) - \frac{1}{15} \cos(4\omega t) + \frac{1}{35} \cos(6\omega t) + \dots \right]$$

proof

$f(t) = V|\cos \omega t|$
 LOOKING AT THE GRAPH, CREATE
 $f(t)$ THIS NATURAL PERIOD $\frac{\pi}{\omega}$
 (HALF PERIOD $L = \frac{\pi}{2\omega}$) AND IT
 IS EVEN

$y = \cos \omega t$
 $y = |\cos \omega t|$

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L}]$

- $b_n = 0$ (As $f(t)$ is even)
- $a_0 = \frac{1}{L} \int_{-L}^L f(t) dt = \frac{1}{\frac{\pi}{2\omega}} \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} V|\cos \omega t| dt = \frac{2\omega}{\pi} \times 2 \int_0^{\frac{\pi}{2\omega}} V \cos \omega t dt$
 $= \frac{4\omega V}{\pi} \int_0^{\frac{\pi}{2\omega}} \cos \omega t dt = \frac{4\omega V}{\pi} \times \frac{1}{\omega} [\sin \omega t]_0^{\frac{\pi}{2\omega}}$
 $= \frac{4V}{\pi} [\sin \frac{\pi}{2} - 0] = \frac{4V}{\pi}$
- $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt = \frac{1}{\frac{\pi}{2\omega}} \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} V|\cos \omega t| \cos \left(\frac{n\pi t}{\frac{\pi}{2\omega}} \right) dt$
 $= \frac{2\omega}{\pi} \times 2 \int_0^{\frac{\pi}{2\omega}} V \cos \omega t \cos (2n\omega t) dt = \frac{4\omega V}{\pi} \int_0^{\frac{\pi}{2\omega}} \cos \omega t \cos (2n\omega t) dt$

Now $\cos(2n\omega t) \cos \omega t = \frac{1}{2} [\cos(2n\omega t + \omega t) + \cos(2n\omega t - \omega t)]$
 $\cos(2n\omega t) \cos \omega t = \frac{1}{2} [\cos((2n+1)\omega t) + \cos((2n-1)\omega t)]$
 $= \frac{2\omega V}{\pi} \int_0^{\frac{\pi}{2\omega}} \cos((2n+1)\omega t) + \cos((2n-1)\omega t) dt$

$= \frac{2\omega V}{\pi} \left[\frac{1}{(2n+1)\omega} \sin((2n+1)\omega t) + \frac{1}{(2n-1)\omega} \sin((2n-1)\omega t) \right]_0^{\frac{\pi}{2\omega}}$
 $= \frac{2V}{\pi} \left[\frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2}\right) + \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi}{2}\right) \right]$
 $= \frac{2V}{\pi} \left[\frac{1}{2n+1} (-1)^n + \frac{1}{2n-1} (-1)^{n-1} \right]$
 $= \frac{2V(-1)^n}{\pi} \left[\frac{1}{2n+1} - \frac{1}{2n-1} \right]$
 $= \frac{2V(-1)^n}{\pi} \left[\frac{2n-1 - (2n+1)}{(2n+1)(2n-1)} \right]$
 $= \frac{4V(-1)^n}{\pi(1-4n^2)}$

$\therefore f(t) = \frac{4V}{\pi} + \sum_{n=1}^{\infty} \left[\frac{4V(-1)^n}{\pi(1-4n^2)} \cos \frac{n\pi t}{\frac{\pi}{2\omega}} \right]$
 $f(t) = \frac{2V}{\pi} + \frac{4V}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{1-4n^2} \cos(2n\omega t) \right]$
 $f(t) = \frac{2V}{\pi} + \frac{4V}{\pi} \left[\frac{1}{3} \cos 2\omega t - \frac{1}{15} \cos 4\omega t + \frac{1}{35} \cos 6\omega t + \dots \right]$

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PARSEVAL'S IDENTITY

Created by T. Madas

Question 1

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = |x|, \quad -\pi \leq x \leq \pi.$$

c) State Parseval's identity for the Fourier series of $f(x)$ from part (a).

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)x]}{(2n-1)^2} \right]$$

i) If $f(x)$ is piecewise continuous on $(-L, L)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

 where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $n=0,1,2,\dots$
 $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$ $n=1,2,3,\dots$

ii) $f(x) = |x|$ $-\pi < x < \pi$ is an even function
 Hence $b_n = 0$ for all n , since $|x| \sin nx$ is odd
 $\bullet a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} 2x dx = \frac{1}{\pi} \int_0^{\pi} 2x dx$
 $= \frac{1}{\pi} \left[x^2 \right]_0^{\pi} = \frac{1}{\pi} (\pi^2 - 0) = \pi$ (i.e. $a_0 = \pi$)

$\bullet a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$

Integration by parts

$\frac{2}{\pi} \int_0^{\pi} x \cos nx dx$	$\frac{1}{\pi} \int_0^{\pi} \cos nx dx$
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$\dots = \frac{2}{\pi} \left[\frac{1}{n^2} \cos nx \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx$
 $= \frac{2}{\pi} \left[\frac{1}{n^2} \cos nx \right]_0^{\pi} = \frac{2}{\pi n^2} [\cos n\pi - 1] = \frac{2}{\pi n^2} [(-1)^n - 1]$
 $= \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$
 $\therefore a_n = \begin{cases} -\frac{4}{\pi (2n-1)^2} & n=1,2,3,\dots \\ 0 & n=2,4,6,\dots \end{cases}$

Thus $|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{-4}{\pi (2n-1)^2} \cos[(2n-1)x] \right]$
 $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}$

c) If the conditions/requirements of part (a) are satisfied then Parseval's identity states
 $\frac{1}{L} \int_{-L}^L |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$

d) $\frac{1}{\pi} \int_{-\pi}^{\pi} |x|^2 dx = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \left[\frac{16}{\pi (2n-1)^4} \right]$
 $\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \frac{16}{\pi (2n-1)^4}$
 $\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^3}{3} + \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$
 $\Rightarrow \frac{\pi^3}{3} = \frac{\pi^3}{3} + \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$
 $\Rightarrow \frac{\pi^3}{3} = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$

Question 2

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = \text{sign}(x), \quad -\pi \leq x \leq \pi.$$

c) Prove Parseval's identity for the Fourier series of $f(x)$ in $(-\pi, \pi)$.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$$\text{sign}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin[(2n-1)x]}{(2n-1)} \right]$$

a) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$, $n=1,2,3,\dots$
 $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$, $n=1,2,3,\dots$

b) $f(x) = \text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$
 (odd function) $\Rightarrow a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{sign}(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi} = -\frac{2}{\pi n} [\cos n\pi - 1]$$

$$= \frac{2}{\pi n} [(-1)^n + 1]$$

IF $n = \text{even}$ $\Rightarrow (-1)^n = 1 \Rightarrow b_n = 0$
 IF $n = \text{odd}$ $\Rightarrow (-1)^n = -1 \Rightarrow b_n = \frac{4}{\pi n}$

$\therefore \text{sign}(x) = \sum_{n=1}^{\infty} \frac{4}{\pi (2n-1)} \sin[(2n-1)x]$

c) Parseval's identity
 Given that the series of part (a) converges, then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Parseval

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

Multiply through by $\frac{1}{\pi} f(x)$

$$\frac{1}{\pi} [f(x)]^2 = \frac{1}{\pi} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 \cos^2 nx + b_n^2 \sin^2 nx] \right]$$

Integrate both sides from $-\pi$ to π , noting that $\int_{-\pi}^{\pi} \cos^2 nx \, dx = \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{a_0^2}{2} \cdot \pi + \sum_{n=1}^{\infty} [a_n^2 \cdot \pi + b_n^2 \cdot \pi]$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

d) $f(x) = \text{sign}(x) \Rightarrow [f(x)]^2 = 1$
 $\Rightarrow a_0 = 0$
 $b_n^2 = \left(\frac{4}{\pi (2n-1)} \right)^2 = \frac{16}{\pi^2 (2n-1)^2}$

Thus

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \, dx = \sum_{n=1}^{\infty} \frac{16}{\pi^2 (2n-1)^2}$$

$$\Rightarrow \frac{2}{\pi} \int_0^{\pi} 1 \, dx = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{2}{\pi} \cdot \pi = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Question 3

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = \frac{1}{2} + \frac{1}{2} \operatorname{sign}(x), \quad -\pi \leq x \leq \pi.$$

c) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$$\boxed{}, \quad \frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin[(2n-1)x]}{(2n-1)} \right]$$

a) STATING THE FOURIER SERIES THEOREM

If $f(x)$ is piecewise continuous on $(-L, L)$, $L > 0$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

where

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

b) FOURIER SERIES OF $f(x) = \frac{1}{2} + \frac{1}{2} \operatorname{sign}(x)$

NOTE: $\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) \right) dx = \frac{1}{\pi} \times \pi = 1$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) \right) \cos nx dx = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \cos nx dx + \int_{-\pi}^{\pi} \operatorname{sign}(x) \cos nx dx \right]$
 $= \frac{1}{2\pi} \left[\sin nx - 0 \right]_{-\pi}^{\pi} = 0$
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) \right) \sin nx dx = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \sin nx dx + \int_{-\pi}^{\pi} \operatorname{sign}(x) \sin nx dx \right]$
 $= -\frac{1}{2\pi} \left[\cos nx - (-1) \right]_{-\pi}^{\pi} = \frac{1 - \cos nx}{\pi} = \frac{1 - (-1)^n}{\pi}$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) \right) dx = \frac{1}{\pi} \times \pi = 1$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) \right) \cos nx dx = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \cos nx dx + \int_{-\pi}^{\pi} \operatorname{sign}(x) \cos nx dx \right]$
 $= \frac{1}{2\pi} \left[\sin nx - 0 \right]_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \operatorname{sign}(x) \right) \sin nx dx = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \sin nx dx + \int_{-\pi}^{\pi} \operatorname{sign}(x) \sin nx dx \right]$
 $= -\frac{1}{2\pi} \left[\cos nx - (-1) \right]_{-\pi}^{\pi} = \frac{1 - \cos nx}{\pi} = \frac{1 - (-1)^n}{\pi}$

WHICH IS PARSEVAL'S IDENTITY

d) USING PARSEVAL'S IDENTITY WITH $f(x) = \frac{1}{2} + \frac{1}{2} \operatorname{sign}(x)$ IN THE INTERVAL $(-\pi, \pi)$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} 1^2 dx = \frac{1}{2} + \sum_{n=1}^{\infty} \left[0^2 + \left(\frac{1 - (-1)^n}{\pi} \right)^2 \right]$$

$$\Rightarrow \frac{1}{\pi} \times \pi = \frac{1}{2} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{2}$$

$$\Rightarrow 1 = \frac{1}{2} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{2}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

WHICH IS PARSEVAL'S IDENTITY

Question 4

$$f(x) = x, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Handwritten solution for Question 4:

Given $f(x) = x$, $x \in (-\pi, \pi]$, $f(x) = f(x + 2\pi)$.

Fourier coefficients:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = -\frac{2}{n^2} (-1)^n = \frac{2}{n^2} (-1)^{n+1}$$

Parseval's identity:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left(\frac{\pi^3}{3} - \left(-\frac{\pi^3}{3} \right) \right) = \frac{\pi^2}{3}$$

$$= \frac{1}{2\pi} \left(a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right) = \frac{1}{2\pi} \left(0 + \sum_{n=1}^{\infty} \left(0 + \left(\frac{2}{n^2} \right)^2 \right) \right) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{4}{n^4} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Equating the two expressions for the integral:

$$\frac{\pi^2}{3} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^4} \implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{6}$$

Question 5

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Handwritten solution for Question 5:

Given $f(x) = x^2$, $x \in [-\pi, \pi]$, $T = 2\pi$, $\frac{2\pi}{T} = \frac{1}{\pi}$.

Parseval's identity: $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{T} \int_{-\pi}^{\pi} f(x)^2 dx$

Calculations:

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{3} \pi^2$
- $a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos x dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos x dx = \frac{2}{\pi} \left[x^2 \sin x - 2x \cos x + 2 \sin x \right]_0^{\pi} = \frac{2}{\pi} [0 - 2\pi(-1) + 2(0)] = \frac{4}{\pi}$
- $b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin x dx = 0$ (odd function)

Parseval's identity:

$$\frac{1}{2} \left(\frac{2}{3} \pi^2 \right)^2 + \sum_{n=1}^{\infty} \left(\frac{4}{\pi} \right)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx$$

$$\frac{2}{9} \pi^4 + \sum_{n=1}^{\infty} \frac{16}{\pi} = \frac{2}{2\pi} \left[\frac{x^5}{5} \right]_{-\pi}^{\pi} = \frac{2}{5} \pi^4$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Question 6

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.
- Find the Fourier series of

$$f(x) = x^2, \quad -\pi \leq x \leq \pi.$$

- Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos nx \right]$$

a) $f(x)$ is piecewise continuous on $(-L, L)$, $L > 0$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ for $n=1, 2, 3, \dots$
 $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$ for $n=1, 2, 3, \dots$

b) Starting from the result of part (a)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

multiply through by $\frac{1}{L} f(x)$ with x from $-L$ to L

$$\Rightarrow \frac{1}{L} \int_{-L}^L f(x) dx = \frac{a_0}{2} \int_{-L}^L \frac{1}{L} dx + \sum_{n=1}^{\infty} \left[\frac{1}{L} \int_{-L}^L a_n f(x) \cos \frac{n\pi x}{L} dx + \frac{1}{L} \int_{-L}^L b_n f(x) \sin \frac{n\pi x}{L} dx \right]$$

$$\Rightarrow \frac{1}{L} \int_{-L}^L f(x) dx = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{L} \int_{-L}^L a_n f(x) \cos \frac{n\pi x}{L} dx + \frac{1}{L} \int_{-L}^L b_n f(x) \sin \frac{n\pi x}{L} dx \right]$$

$$\Rightarrow \frac{1}{L} \int_{-L}^L f(x) dx = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{L} \int_{-L}^L a_n f(x) \cos \frac{n\pi x}{L} dx + \frac{1}{L} \int_{-L}^L b_n f(x) \sin \frac{n\pi x}{L} dx \right]$$

valid if known as Parseval's identity

c) $f(x) = x^2$

$f(x)$ is continuous on $(-\pi, \pi)$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx dx = 0 \quad \forall n$$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$

By parts

$u = x^2$	$\frac{du}{dx} = 2x$
$v = \cos nx$	$\frac{dv}{dx} = -n \sin nx$

$$= \frac{2}{\pi} \left[\left(\frac{x^2}{n} \cos nx - \int \frac{2x}{n} \sin nx dx \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{x^2}{n} \cos nx + \frac{2}{n^2} \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{n} \cos n\pi + \frac{2}{n^2} \cos n\pi - \left(\frac{0}{n} \cos 0 + \frac{2}{n^2} \cos 0 \right) \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{n} (-1)^n + \frac{2}{n^2} (-1)^n - \frac{2}{n^2} \right]$$

Thus $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2} \cos nx \right]$$

d) By Parseval's identity

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{(\frac{2\pi^2}{3})^2}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{4(-1)^n}{n^2} \right)^2 \right]$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{2\pi^4}{5} = \frac{2\pi^4}{3} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \frac{2\pi^4}{5} - \frac{2\pi^4}{3} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- State and prove Parseval's identity for the Fourier series of $f(x)$ in $(-L, L)$.
- By considering the Fourier series of

$$f(x) = x^3, \quad -\pi \leq x \leq \pi,$$

show that
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

You may use without proof the following results.

$$\bullet \int x^3 \sin nx \, dx = \frac{1}{n^4} \left[nx(6 - n^2 x^2) \cos nx + 3(n^2 x^2 - 2) \sin nx \right] + C$$

$$\bullet \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\bullet \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

proof

The image shows two pages of handwritten mathematical work. The left page details the Fourier series expansion of $f(x) = x^3$ on the interval $(-\pi, \pi)$. It starts by noting that $f(x)$ is periodic and continuous on $(-\pi, \pi)$, so it can be expanded as a Fourier series. The coefficients a_n and b_n are calculated using integration by parts. The right page continues the derivation, showing the final Fourier series for x^3 and then applying Parseval's identity to derive the sum $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$. The work includes various intermediate steps, such as calculating $\int_{-\pi}^{\pi} x^6 \cos nx \, dx$ and using the known sums for $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^4}$.

FOURIER SERIES

Complex Expansions

Question 1

A periodic function $f(t)$ is defined in the interval $(-L, L)$, $L > 0$, $f(t + 2L) = f(t)$.

It is further given that $f(t)$ is continuous or piecewise continuous in $(-L, L)$ and has Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right],$$

where $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$, $n = 0, 1, 2, 3, \dots$

and $b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt$, $n = 1, 2, 3, 4, \dots$

Show that the complex Fourier series expansion of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \left[c_n e^{\frac{in\pi t}{L}} \right],$$

where $c_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-\frac{in\pi t}{L}} dt$, $n \in \mathbb{Z}$

, proof

START WITH THE DEFINITION OF A FOURIER SERIES IN t , $-L < t < L$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right]$$

- $a_0 = \frac{1}{L} \int_{-L}^L f(t) dt$ $n=0, 1, 2, 3, \dots$
- $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$ $n=1, 2, 3, \dots$
- $b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt$ $n=1, 2, 3, \dots$

BY MANIPULATING EULER'S FORMULA & SUBSTITUTING INTO THE ABOVE

- $\cos\left(\frac{n\pi t}{L}\right) = \frac{1}{2} \left[e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}} \right]$
- $\sin\left(\frac{n\pi t}{L}\right) = \frac{1}{2i} \left[e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}} \right]$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}} \right) + \frac{b_n}{2i} \left(e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}} \right) \right]$$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi t}{L}} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-i\frac{n\pi t}{L}} \right]$$

- Let $C_0 = \frac{a_0}{2}$ & $\frac{1}{2}(a_1 + ib_1)$ with $n=0$
- Let $C_n = \frac{1}{2}(a_n + ib_n)$
- Let $\bar{C}_n = \frac{1}{2}(a_n - ib_n)$, As C_n & \bar{C}_n ARE COMPLEXES

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} + \bar{C}_n e^{-i\frac{n\pi t}{L}} \right]$$

NOW FOR NOTATIONAL CONVENIENCE LET US WRITE THE COMPLEXES AS EQUATIONS

$$C_n = \frac{1}{2}(a_n + ib_n)$$

$$\Rightarrow C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-i\frac{n\pi t}{L}} dt \Rightarrow \bar{C}_n = \frac{1}{2L} \int_{-L}^L f(t) e^{i\frac{n\pi t}{L}} dt$$

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right] + \sum_{n=1}^{\infty} \left[\bar{C}_n e^{-i\frac{n\pi t}{L}} \right]$$

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right] + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right]$$

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right] + C_1 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right]$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} \right]$$

with $C_0 = \frac{1}{2L} \int_{-L}^L f(t) dt$

$$\Rightarrow C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-i\frac{n\pi t}{L}} dt$$

$$\Rightarrow C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-i\frac{n\pi t}{L}} dt$$

$$\Rightarrow C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-i\frac{n\pi t}{L}} dt, \quad n \in \mathbb{Z}$$

Question 2

$$f(t) = \begin{cases} 1 & -2 \leq t \leq 2 \\ 0 & 2 < t < 6 \end{cases}, \quad f(t+8) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[e^{\frac{1}{4}n\pi t i} \operatorname{sinc}\left(\frac{1}{2}n\pi\right) \right]$$

$f(t) = \begin{cases} 1 & -2 \leq t \leq 2 \\ 0 & 2 < t < 6 \end{cases}$
 $f(t+8) = f(t)$

• START BY DRAWING A GRAPH

• CONSIDER A SYMMETRICAL INTERVAL OF PERIOD $T=8$, FROM -4 TO 4 ($L=4$ = HALF PERIOD)

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i \frac{n\pi t}{L}} \quad \text{WITH } C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-i \frac{n\pi t}{L}} dt$$

• EVALUATE THE COMPLEX COEFFICIENTS

$$C_0 = \frac{1}{8} \int_{-4}^4 f(t) e^{-i \frac{n\pi t}{4}} dt$$

$$C_0 = \frac{1}{8} \int_{-4}^4 f(t) \left[\cos \frac{n\pi t}{4} - i \sin \frac{n\pi t}{4} \right] dt$$

$$C_0 = \frac{1}{8} \int_{-4}^4 f(t) \cos \frac{n\pi t}{4} dt$$

$$C_0 = \frac{1}{8} \int_{-2}^2 1 \times \cos \frac{n\pi t}{4} dt$$

$$C_0 = \frac{1}{4} \times \frac{4}{n\pi} \left[\sin \frac{n\pi t}{4} \right]_{-2}^2 \quad n \neq 0$$

$\Rightarrow C_0 = \frac{1}{n\pi} \left[\sin \frac{n\pi}{2} - 0 \right], n \neq 0$
 $\Rightarrow C_0 = \frac{1}{n\pi} \sin \frac{n\pi}{2}, n \neq 0$

• THE SPECIAL CASE WHERE $n=0$ WE CALCULATE SEPARATELY

$$C_0 = C_n = \frac{1}{8} \int_{-4}^4 f(t) e^{-i \frac{n\pi t}{4}} dt = \frac{1}{8} \int_{-4}^4 1 dt = \frac{1}{2}$$

• THERE WE CAN THEN THE FOURIER EXPANSION BE $f(t)$

$$f(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{i \frac{n\pi t}{L}}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{\sin \frac{n\pi}{2}}{n\pi} e^{i \frac{n\pi t}{4}} \right]$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{1}{2} \left(\frac{\sin \frac{n\pi}{2}}{n\pi} \right) e^{i \frac{n\pi t}{4}} \right]$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) e^{i \frac{n\pi t}{4}} \right]$$

Question 3

$$f(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & a < t < T \end{cases}, \quad a < \frac{1}{2}T, \quad f(t+T) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{a}{T} + \frac{a}{T} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\exp\left(\frac{n\pi(2t-a)}{T}\right) \operatorname{sinc}\left(\frac{n\pi a}{T}\right) \right]$$

$f(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & a < t < T \end{cases}$
 $f(t+T) = f(t)$

• Sketch with a quick sketch

• This function is neither odd nor even
 — Period is T
 — Half period $\frac{T}{2} = L$

$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n t}{T}}$ where $c_n = \frac{1}{T} \int_0^T f(t) e^{-j \frac{2\pi n t}{T}} dt$

• Evaluate the complex coefficients

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-j \frac{2\pi n t}{T}} dt$$

$$c_n = \frac{1}{T} \int_0^a 1 \times e^{-j \frac{2\pi n t}{T}} dt$$

$$c_n = \frac{1}{T} \times \left[\frac{T}{2\pi n} e^{-j \frac{2\pi n t}{T}} \right]_0^a, \quad n \neq 0$$

$$c_n = \frac{1}{2\pi n} \left[e^{-j \frac{2\pi n a}{T}} - 1 \right], \quad n \neq 0$$

$$c_n = \frac{1}{2\pi n} \left[1 - e^{-j \frac{2\pi n a}{T}} \right], \quad n \neq 0$$

$\Rightarrow c_n = \frac{1}{2\pi n} e^{-j \frac{2\pi n a}{T}} \left[e^{j \frac{2\pi n a}{T}} - e^{-j \frac{2\pi n a}{T}} \right], \quad n \neq 0$

$\Rightarrow c_n = \frac{e^{-j \frac{2\pi n a}{T}}}{\pi n} \times \frac{1}{2i} \left[e^{j \frac{2\pi n a}{T}} - e^{-j \frac{2\pi n a}{T}} \right], \quad n \neq 0$

$\Rightarrow c_n = \frac{e^{-j \frac{2\pi n a}{T}}}{\pi n} \sin\left(\frac{2\pi n a}{T}\right) \times \frac{1}{n\pi}, \quad n \neq 0$

$\Rightarrow c_n = \frac{e^{-j \frac{2\pi n a}{T}}}{\pi n} \sin\left(\frac{2\pi n a}{T}\right) \times \frac{1}{n\pi}, \quad n \neq 0$

$\Rightarrow c_n = \frac{a}{T} e^{-j \frac{2\pi n a}{T}} \operatorname{sinc}\left(\frac{n\pi a}{T}\right), \quad n \neq 0$

• The special case where $n=0$, we handle separately

$$c_0 = \frac{1}{T} \int_0^T 1 \times e^0 dt = \frac{1}{T} \int_0^T 1 dt = \frac{a}{T}$$

• Finally we have a complex Fourier series

$$f(t) = c_0 + \sum_{n=-\infty, n \neq 0}^{\infty} c_n e^{j \frac{2\pi n t}{T}}$$

$$f(t) = \frac{a}{T} + \sum_{n=-\infty, n \neq 0}^{\infty} \left[\frac{a}{T} e^{-j \frac{2\pi n a}{T}} \operatorname{sinc}\left(\frac{n\pi a}{T}\right) \right] e^{j \frac{2\pi n t}{T}}$$

$$f(t) = \frac{a}{T} + \sum_{n=-\infty, n \neq 0}^{\infty} \left[e^{j \frac{2\pi n (t-a)}{T}} \operatorname{sinc}\left(\frac{n\pi a}{T}\right) \right]$$

Question 4

$$f(t) = t, \quad 0 \leq t < 1, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{i}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{e^{2n\pi i}}{n} \right]$$

$f(t) = \begin{cases} t & 0 \leq t < 1 \\ f(t+1) = f(t) \end{cases}$

• START WITH A QUICK SKETCH

• THE FUNCTION IS NEITHER ODD NOR EVEN WITH $T=1$, $L=\frac{1}{2}$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i 2\pi n t} \quad \text{where} \quad C_n = \frac{1}{2\pi} \int_0^1 f(t) e^{-i 2\pi n t} dt$$
 or
$$C_n = \frac{1}{T} \int_0^T f(t) e^{-i 2\pi n t} dt$$

• CALCULATE THE COEFFICIENTS C_n , $n \neq 0$

$$\Rightarrow C_n = \frac{1}{1} \int_0^1 t e^{-i 2\pi n t} dt = \int_0^1 t e^{-i 2\pi n t} dt$$

BY PARTS

t	1
$-\frac{1}{i 2\pi n} e^{-i 2\pi n t}$	$-\frac{1}{i 2\pi n} e^{-i 2\pi n t}$

$$\Rightarrow C_n = \left[-\frac{t e^{-i 2\pi n t}}{i 2\pi n} \right]_0^1 + \int_0^1 \frac{1}{i 2\pi n} e^{-i 2\pi n t} dt$$

$$\Rightarrow C_n = \left[-\frac{t e^{-i 2\pi n t}}{i 2\pi n} - \frac{1}{i 2\pi n} e^{-i 2\pi n t} \right]_0^1$$

$$\Rightarrow C_n = \left[-\frac{1}{i 2\pi n} e^{-i 2\pi n t} - \frac{t}{i 2\pi n} e^{-i 2\pi n t} \right]_0^1$$

$$\Rightarrow C_n = \left[-\frac{1}{i 2\pi n} e^{-i 2\pi n} - \frac{1}{i 2\pi n} e^{-i 2\pi n} \right] - \left[-\frac{1}{i 2\pi n} - 0 \right]$$

$$\Rightarrow C_n = \frac{1}{i 2\pi n} (e^{-i 2\pi n} - 1) - \frac{1}{i 2\pi n} (e^{-i 2\pi n} - 1)$$

$$\Rightarrow C_n = \frac{1}{i 2\pi n} - \frac{1}{i 2\pi n} = \frac{1}{i 2\pi n}$$

For $n=0$, C_0

$$C_0 = \frac{1}{1} \int_0^1 t e^0 dt = \int_0^1 t dt = \left[\frac{1}{2} t^2 \right]_0^1 = \frac{1}{2}$$

FINALLY THE COMPLEX FOURIER SERIES

$$f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{i 2\pi n t}$$

$$f(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{i 2\pi n} e^{i 2\pi n t}$$

$$f(t) = \frac{1}{2} + \frac{i}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i 2\pi n t}}{n}$$

Question 5

$$f(t) = e^{\pi t}, \quad 0 \leq t < 2, \quad f(t+2) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{(e^{2\pi} - 1)}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(1 + in)e^{2n\pi i}}{(n^2 + 1)} \right]$$

$f(t) = \begin{cases} e^{\pi t} & 0 \leq t < 2 \\ f(t+2) = f(t) \end{cases}$

THE FUNCTION IS NEITHER ODD NOR EVEN $\rightarrow T=2, L=1$

$f(t) = \sum_{n=-\infty}^{\infty} [C_n e^{in\pi t}]$, where $C_n = \frac{1}{2L} \int_0^L f(t) e^{-in\pi t} dt$
(or from a to $a+2L$)

CALCULATE THE COEFFICIENTS C_n

$$\Rightarrow C_n = \frac{1}{2 \times 1} \int_0^2 e^{\pi t} e^{-in\pi t} dt$$

$$\Rightarrow C_n = \frac{1}{2} \int_0^2 e^{\pi t(1-in)} dt$$

$$\Rightarrow C_n = \frac{1}{2\pi(1-in)} \left[e^{\pi t(1-in)} \right]_0^2$$

$$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)} \left[e^{2\pi(1-in)} - 1 \right]$$

$$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)} \left[e^{2\pi} \left[\cos(2n\pi) - i\sin(2n\pi) \right] - 1 \right]$$

$$\Rightarrow C_n = \frac{1+in}{2\pi(n^2+1)} [e^{2\pi} - 1]$$

FINALLY WE HAVE

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1+in}{2\pi(n^2+1)} (e^{2\pi} - 1) e^{-in\pi t} \right]$$

Question 6

$$f(t) = \cos(\pi t), \quad -\frac{1}{2} \leq t < \frac{1}{2}, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^{n+1} e^{2n\pi i}}{4n^2 - 1} \right]$$

$f(t) = \begin{cases} \cos \pi t & -\frac{1}{2} \leq t < \frac{1}{2} \\ f(t+1) = f(t) \end{cases}$

• The function is even in a symmetrical domain, $T=1$, $L=\frac{1}{2}$

$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \pi t}$ where $C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-i n \pi t} dt$

• Calculate the coefficients C_n (Note C_0 is part of this)

$\Rightarrow C_n = \frac{1}{2 \times \frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \pi t e^{-i n \pi t} dt$

$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \pi t [\cos(2n\pi t) - i \sin(2n\pi t)] dt$

$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2 \cos \pi t \cos 2n\pi t dt$

$\cos(\pi t + 2n\pi t) = \cos \pi t \cos 2n\pi t - \sin \pi t \sin 2n\pi t$
 $\cos(\pi t - 2n\pi t) = \cos \pi t \cos 2n\pi t + \sin \pi t \sin 2n\pi t$
Adding: $\cos(2n+1)\pi t + \cos(1-2n)\pi t = 2 \cos \pi t \cos 2n\pi t$

$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos[(1+2n)\pi t] + \cos[(1-2n)\pi t] dt$

$\Rightarrow C_n = \left[\frac{1}{(1+2n)\pi} \sin[(1+2n)\pi t] + \frac{1}{(1-2n)\pi} \sin[(1-2n)\pi t] \right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$\Rightarrow C_n = \frac{1}{(1+2n)\pi} \sin\left[(1+2n)\pi \frac{1}{2}\right] + \frac{1}{(1-2n)\pi} \sin\left[(1-2n)\pi \frac{1}{2}\right]$

$\Rightarrow C_n = \frac{1}{(2n+1)\pi} \sin\left[(2n+1)\frac{\pi}{2}\right] + \frac{1}{(2n-1)\pi} \sin\left[(2n-1)\frac{\pi}{2}\right]$

• To try or note that

$\sin[(2n+1)\frac{\pi}{2}] = (-1)^n$
 $\sin[(2n-1)\frac{\pi}{2}] = (-1)^{n+1}$

$\Rightarrow C_n = \frac{1}{(2n+1)\pi} (-1)^n + \frac{1}{(2n-1)\pi} (-1)^{n+1}$

$\Rightarrow C_n = \frac{(-1)^n}{\pi} \left[\frac{1}{2n+1} - \frac{1}{2n-1} \right]$

$\Rightarrow C_n = \frac{(-1)^n}{\pi} \left[\frac{2n-1 - (2n+1)}{(2n+1)(2n-1)} \right]$

$\Rightarrow C_n = \frac{(-1)^n}{\pi} \frac{-2}{4n^2 - 1}$

$\therefore f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \pi t}$

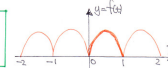
$f(t) = \sum_{n=-\infty}^{\infty} \frac{-2(-1)^n}{\pi(4n^2 - 1)} e^{i n \pi t}$

Question 7

$$f(t) = \sin(\pi t) \quad , \quad 0 \leq t < 1, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{e^{2n\pi i}}{4n^2 - 1} \right]$$

$f(t) = \begin{cases} \sin \pi t & 0 \leq t < 1 \\ f(t+1) = f(t) \end{cases}$


THE FUNCTION IS PERIODIC (PERIOD $T=1$) AND EVEN (ABOUT $T/2=0.5$).
 THE FUNCTION IS NOT EVEN (ABOUT $T/2=0.5$).

$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \pi t}$, where $C_n = \frac{1}{T} \int_0^1 f(t) e^{-i n \pi t} dt$ (see note 1 to Q1 & Q2)

CALCULATE THE COEFFICIENTS C_n

$\Rightarrow C_n = \frac{1}{1} \int_0^1 \sin \pi t e^{-i n \pi t} dt$
 $\Rightarrow C_n = \int_0^1 \sin \pi t [\cos 2n\pi t - i \sin 2n\pi t] dt$
 $\Rightarrow C_n = \int_0^1 \sin \pi t \cos 2n\pi t - i \sin \pi t \sin 2n\pi t dt$

$\bullet \cos(\pi t + 2n\pi t) = \cos \pi t \cos 2n\pi t - \sin \pi t \sin 2n\pi t$
 $\cos(\pi t - 2n\pi t) = \cos \pi t \cos 2n\pi t + \sin \pi t \sin 2n\pi t$
 $\therefore -\sin \pi t \sin 2n\pi t = \cos(\pi t + 2n\pi t) - \cos(\pi t - 2n\pi t)$
 $= \frac{1}{2} [\cos(\pi t + 2n\pi t) - \cos(\pi t - 2n\pi t)]$

$\bullet \sin(\pi t + 2n\pi t) = \sin \pi t \cos 2n\pi t + \cos \pi t \sin 2n\pi t$
 $\sin(\pi t - 2n\pi t) = \sin \pi t \cos 2n\pi t - \cos \pi t \sin 2n\pi t$
 $\therefore 2 \sin \pi t \cos 2n\pi t = \sin(\pi t + 2n\pi t) + \sin(\pi t - 2n\pi t)$
 $\sin \pi t \cos 2n\pi t = \frac{1}{2} [\sin(\pi t + 2n\pi t) + \sin(\pi t - 2n\pi t)]$

INTEGRATING THE NEW EXPRESSIONS

$\Rightarrow C_n = \int_0^1 \frac{1}{2} \sin(\pi t + 2n\pi t) + \frac{1}{2} \sin(\pi t - 2n\pi t) + \frac{1}{2} \cos(\pi t + 2n\pi t) - \frac{1}{2} \cos(\pi t - 2n\pi t) dt$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{-\cos(\pi t + 2n\pi t)}{(1+2n)\pi} - \frac{\cos(\pi t - 2n\pi t)}{(1-2n)\pi} + \frac{\sin(\pi t + 2n\pi t)}{(1+2n)\pi} - \frac{\sin(\pi t - 2n\pi t)}{(1-2n)\pi} \right]_0^1$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{-\cos(\pi + 2n\pi)}{(1+2n)\pi} - \frac{\cos(\pi - 2n\pi)}{(1-2n)\pi} + \frac{\sin(\pi + 2n\pi)}{(1+2n)\pi} - \frac{\sin(\pi - 2n\pi)}{(1-2n)\pi} \right]$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{-\cos(\pi)}{(1+2n)\pi} - \frac{\cos(\pi)}{(1-2n)\pi} + \frac{\sin(\pi)}{(1+2n)\pi} - \frac{\sin(\pi)}{(1-2n)\pi} \right]$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{-(-1)}{(1+2n)\pi} - \frac{-(-1)}{(1-2n)\pi} + 0 - 0 \right]$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{1}{(1+2n)\pi} + \frac{1}{(1-2n)\pi} \right]$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{1-2n + 1+2n}{(1+2n)(1-2n)} \right]$
 $\Rightarrow C_n = \frac{1}{2} \left[\frac{2}{1-4n^2} \right] = \frac{1}{1-4n^2}$

FINALLY WE CAN OBTAIN A COMPLEX FOURIER SERIES

$f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{i n \pi t}$ (here C_0 is 0)

$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \pi t}$
 $f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1-4n^2} e^{i n \pi t}$
 $f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{2n\pi i}}{4n^2 - 1}$

Question 8

The function f is defined as

$$f(t) = V \cos\left(\frac{\pi t}{T}\right), \quad -\frac{1}{2}T \leq t < \frac{1}{2}T, \quad f(t) = f(t+T),$$

where V and T are positive constants.

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{2V}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{e^{2n\pi i}}{(1-4n^2)} \right]$$

Handwritten solution for the complex Fourier series expansion of $f(t)$.

Left page:

- Definition of $f(t)$: $f(t) = V \cos\left(\frac{\pi t}{T}\right)$ for $-\frac{T}{2} < t < \frac{T}{2}$. A graph shows a cosine wave over one period.
- Periodicity: $f(t) = f(t+T)$.
- Complex Fourier series formula: $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \frac{2\pi}{T} t}$ where $C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i n \frac{2\pi}{T} t} dt$.
- Calculation of C_n : $C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} V \cos\left(\frac{\pi t}{T}\right) e^{-i n \frac{2\pi}{T} t} dt$.
- Using trigonometric identities: $\cos\left(\frac{\pi t}{T}\right) = \frac{e^{i \frac{\pi t}{T}} + e^{-i \frac{\pi t}{T}}}{2}$.
- Integration: $C_n = \frac{V}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{i \frac{\pi t}{T}} + e^{-i \frac{\pi t}{T}}}{2} e^{-i n \frac{2\pi}{T} t} dt$.
- Result: $C_n = \frac{V}{2T} \left[\frac{e^{i \frac{\pi t}{T} - i n \frac{2\pi}{T} t}}{i \left(\frac{\pi}{T} - 2n\pi\right)} + \frac{e^{-i \frac{\pi t}{T} - i n \frac{2\pi}{T} t}}{-i \left(\frac{\pi}{T} + 2n\pi\right)} \right]_{-\frac{T}{2}}^{\frac{T}{2}}$.
- Simplification: $C_n = \frac{V}{2T} \left[\frac{e^{i \frac{\pi}{2} (1-2n)} - e^{-i \frac{\pi}{2} (1-2n)}}{i \left(\frac{\pi}{T} - 2n\pi\right)} + \frac{e^{-i \frac{\pi}{2} (1+2n)} - e^{i \frac{\pi}{2} (1+2n)}}{-i \left(\frac{\pi}{T} + 2n\pi\right)} \right]$.
- Final result: $C_n = \frac{V}{2T} \left[\frac{2i \cos\left(\frac{\pi}{2} (1-2n)\right)}{i \left(\frac{\pi}{T} - 2n\pi\right)} + \frac{2i \cos\left(\frac{\pi}{2} (1+2n)\right)}{-i \left(\frac{\pi}{T} + 2n\pi\right)} \right]$.
- Using $\cos\left(\frac{\pi}{2} (1-2n)\right) = \sin(n\pi)$ and $\cos\left(\frac{\pi}{2} (1+2n)\right) = \sin(n\pi)$.
- Final result: $C_n = \frac{V}{2T} \left[\frac{2 \sin(n\pi)}{\left(\frac{\pi}{T} - 2n\pi\right)} - \frac{2 \sin(n\pi)}{\left(\frac{\pi}{T} + 2n\pi\right)} \right]$.
- Using $\sin(n\pi) = 0$ for integer n , the result is $C_n = 0$ for $n \neq 0$.
- For $n=0$, $C_0 = \frac{V}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{\pi t}{T}\right) dt = \frac{V}{T} \left[\frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi}{T}} \right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{V}{\pi} \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) \right] = \frac{2V}{\pi}$.

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- Final result: $C_n = \frac{2V}{\pi} \frac{1}{1-4n^2}$ for $n \neq 0$ and $C_0 = \frac{2V}{\pi}$.
- Complex Fourier series expansion: $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \frac{2\pi}{T} t} = \frac{2V}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i n \frac{2\pi}{T} t}}{1-4n^2}$.