

Created by T. Madas

# DIFFERENCE EQUATIONS

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**Question 1** (\*\*)

A sequence of numbers  $T_1, T_2, T_3, T_4, T_5, \dots, T_n, \dots$  is generated by the recurrence relation

$$T_{n+1} = 2T_n - 5, \quad T_1 = 6.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$\boxed{\phantom{000}}, \quad T_n = 2^{n-1} + 5$$

Handwritten solution for the recurrence relation  $T_{n+1} = 2T_n - 5, T_1 = 6$ .

**Auxiliary Equation:**

$$T_{n+1} - 2T_n = -5$$

$$\Rightarrow \lambda - 2 = 0$$

$$\Rightarrow \lambda = 2$$

**Particular Solution:**

$$T_n = A \times 2^n$$

**General Solution:**

$$T_n = A \times 2^n + 5$$

**Apply the boundary condition,  $T_1 = 6$**

$$\Rightarrow 6 = A \times 2^1 + 5$$

$$\Rightarrow 1 = 2A$$

$$\Rightarrow A = \frac{1}{2}$$

**Final Answer:**

$$T_n = \frac{1}{2} \times 2^n + 5$$

$$\Rightarrow T_n = 2^{n-1} + 5$$

## Question 2 (\*\*)

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the recurrence relation

$$u_{n+1} = 2u_n + n, \quad u_1 = 2.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$\boxed{\phantom{000}}, \quad \boxed{u_n = 2^{n+1} - n - 1}$$

$u_{n+1} = 2u_n + n, \quad u_1 = 2$

• "HOMOGENEOUS EQUATION"  
 $u_{n+1} - 2u_n = 0$   
 $\Rightarrow \lambda - 2 = 0$   
 $\Rightarrow \lambda = 2$

• "COMPLEMENTARY FUNCTION"  
 $u_n = A \times 2^n$

• "PARTICULAR SOLUTION"  
 $u_n = Pn + Q$   
 $u_{n+1} = P(n+1) + Q$   
 SUB INTO THE RECURRENCE  
 $P(n+1) + Q = 2[Pn + Q] + n$   
 $Pn + (P+Q) = (2P)n + 2Q + n$   
 $\bullet 2P+1 = 0 \quad \bullet 2Q = P+Q$   
 $P = -1 \quad \bullet 2Q = -1+Q$   
 $Q = -1$

$\therefore$  GENERAL SOLUTION:  $u_n = A \times 2^n - n - 1$

APPLY BOUNDARY CONDITION,  $u_1 = 2$   
 $\Rightarrow 2 = A \times 2^1 - 1 - 1$   
 $\Rightarrow 2 = 2A - 2$   
 $\Rightarrow 4 = 2A$   
 $\Rightarrow A = 2$   
 $\Rightarrow u_n = 2 \times 2^n - n - 1$   
 $\Rightarrow u_n = 2^{n+1} - n - 1$

## Question 3 (\*\*+)

A sequence of numbers  $b_1, b_2, b_3, b_4, b_5 \dots, b_n, \dots$  is generated by

$$b_{n+2} = 2b_n + n, \quad b_1 = 1, \quad b_2 = 5.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$b_n = 2^n + (-1)^n$$

$b_{n+2} = b_{n+1} + 2b_n$   
 $b_{n+2} - b_{n+1} - 2b_n = 0$   
 "characteristic equation"  
 $x^2 - x - 2 = 0$   
 $(x+1)(x-2) = 0$   
 $x = -1$   
 $\therefore b_n = A \times 2^n + B \times (-1)^n$   
 $b_1 = 1 \Rightarrow 1 = 2A - B$   
 $b_2 = 5 \Rightarrow 5 = 4A + B$   
 $\therefore b_n = 2^n + (-1)^n$

## Question 4 (\*\*\*)

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the recurrence relation

$$u_{n+1} = 2u_n - n^2 + 3, \quad u_1 = 2.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$u_n = n^2 + 2n - 2^{n-1}$$

$u_{n+1} = 2u_n - n^2 + 3, \quad u_1 = 2$   
 $u_{n+1} - 2u_n = -n^2 + 3$   
 • "SOLVING EQUATION"  
 $\lambda - 2 = 0$   
 $\lambda = 2$   
 $\therefore u_n = A \cdot 2^n$   
 • "PARTICULAR INTEGRAL"  
 LET  $u_n = Pn^2 + Qn + R$   
 $u_{n+1} = P(n+1)^2 + Q(n+1) + R = P(n^2 + 2n + 1) + Q(n+1) + R$   
 $= Pn^2 + 2Pn + P + Qn + Q + R$   
 SUB INTO THE EQUATION  
 $(Pn^2 + 2Pn + P + Qn + Q + R) - 2(Pn^2 + Qn + R) = -n^2 + 3$   
 $-Pn^2 + (2P - Q)n + (P + Q - R) = -n^2 + 3$   
 $\therefore P = 1 \quad 2P - Q = 0 \quad P + Q - R = 3$   
 $2 - Q = 0 \quad 1 + 2 - R = 3$   
 $Q = 2 \quad R = 0$   
 $\therefore u_n = n^2 + 2n$   
 $\therefore$  GENERAL SOLUTION  $u_n = A \cdot 2^n + n^2 + 2n$  WITH  $n=1 \quad u_1 = 2$   
 $2 = A \cdot 2 + 1 + 2$   
 $A = -\frac{1}{2}$   
 $\therefore u_n = -\frac{1}{2} \cdot 2^n + n^2 + 2n$   
 $u_n = n^2 + 2n - 2^{n-1}$

**Question 5 (\*\*\*)**

A sequence of numbers  $t_1, t_2, t_3, t_4, t_5 \dots, t_n, \dots$  is generated by the recurrence relation

$$t_{n+1} = (A+1)t_n - M, \quad t_1 = M,$$

where  $A$  and  $M$  are positive constants.

Determine an expression for the  $n^{\text{th}}$  term of this sequence is given by

$$t_n = \frac{M}{A} \left[ (A+1)^n - 1 \right].$$

proof

Handwritten proof of the  $n^{\text{th}}$  term formula for the recurrence relation  $t_{n+1} = t_n(A+1) - M$ ,  $t_1 = M$ .

Given:  $t_{n+1} = t_n(A+1) - M$ ,  $t_1 = M$

Let  $t_n - t_n(A+1) = -M$

• AUXILIARY EQUATION:  
 $\lambda - (A+1) = 0$   
 $\lambda = A+1$   
 ∴ COMPLEMENTARY FUNCTION:  $t_c = (A+1)^n \times P$  (ARBITRARY CONSTANT)

• "PARTICULAR INTEGRAL"  
 TRY  $t_p = C$  (CONSTANT)  
 $C - (A+1)C = -M$   
 $-AC = -M$   
 $C = \frac{M}{A}$

∴ GENERAL SOLUTION:  $t_n = P(A+1)^n + \frac{M}{A}$

• APPLY CONDITIONS TO FIND P  
 $t_1 = M \Rightarrow M = P(A+1)^1 + \frac{M}{A}$   
 $\Rightarrow M + \frac{M}{A} = P(A+1)$   
 $\Rightarrow \frac{MA + M}{A} = P(A+1)$   
 $\Rightarrow \frac{M(A+1)}{A} = P(A+1)$   
 $\Rightarrow P = \frac{M}{A}$

∴  $t_n = \frac{M}{A}(A+1)^n + \frac{M}{A}$   
 $t_n = \frac{M}{A}[(A+1)^n - 1]$

NOTE:  $A+1 \neq 0$  OTHERWISE NO TERM IN SOLUTION

## Question 6 (\*\*\*)

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the following recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n, \quad u_1 = 1, \quad u_2 = 1.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$\boxed{\phantom{00000}}, \quad u_n = \frac{1}{5} \left[ 3^n - (-2)^n \right]$$

Handwritten solution for Question 6:

$u_{n+2} = u_{n+1} + 6u_n$  with  $u_1 = 1, u_2 = 1$

REARRANGE THE EQUATION  
 $u_{n+2} - u_{n+1} - 6u_n = 0$

AUXILIARY EQUATION  
 $\lambda^2 - \lambda - 6 = 0$   
 $(\lambda + 2)(\lambda - 3) = 0$   
 $\lambda = -2, 3$

GENERAL SOLUTION  
 $u_n = A(-2)^n + B(3)^n$

APPLY CONDITIONS  
 $u_1 = 1 \Rightarrow 1 = -2A + 3B \quad (n=1)$   
 $u_2 = 1 \Rightarrow 1 = 4A + 9B \quad (n=2)$

SOLVING  
 $2 = -4A + 3B \Rightarrow 15B = 3$   
 $1 = 4A + 9B \Rightarrow B = \frac{1}{5}$   
 $\Rightarrow 1 = -2A + \frac{3}{5}$   
 $\Rightarrow 2A = -\frac{2}{5}$   
 $\Rightarrow A = -\frac{1}{5}$

FINALLY WE OBTAIN  
 $u_n = -\frac{1}{5}(-2)^n + \frac{1}{5}(3)^n$   
 $u_n = \frac{1}{5} [ 3^n - (-2)^n ]$

**Question 7 (\*\*\*)**

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n + 4n, \quad u_1 = 1, u_2 = 3.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$\boxed{\phantom{000}}, \quad \boxed{u_n = 4 \times 3^{n-1} - 2^{n+2} + 2n + 3}$$

The image shows two pages of handwritten work on graph paper. The left page details the process of finding a particular solution by assuming a form  $u_n = Pn + Q$  and solving for  $P$  and  $Q$ . The right page shows the process of finding the complementary function by solving the characteristic equation  $\lambda^2 - 5\lambda + 6 = 0$ , which gives roots  $\lambda = 2$  and  $\lambda = 3$ , leading to the complementary function  $u_n = A(2^n) + B(3^n)$ . The final general solution is then given as  $u_n = A(2^n) + B(3^n) + 2n + 3$ .

**Left Page:**

Given:  $u_{n+2} = 5u_{n+1} - 6u_n + 4n, \quad u_1 = 1, u_2 = 3$

REWRITE THE EQUATION IN ORDER TO DERIVE AN AUXILIARY EQUATION

$\Rightarrow u_{n+2} - 5u_{n+1} + 6u_n = 4n$

AUXILIARY EQUATION

$\lambda^2 - 5\lambda + 6 = 0$   
 $(\lambda - 2)(\lambda - 3) = 0$   
 $\lambda = 2, 3$

"COMPLEMENTARY SOLUTION"

$u_n = A(2^n) + B(3^n)$

FOR "PARTICULAR SOLUTION" TRY

$u_n = Pn + Q$   
 $u_{n+1} = P(n+1) + Q$   
 $u_{n+2} = P(n+2) + Q$

SUBSTITUTE INTO THE RELATION

$P(n+2) + Q - 5[P(n+1) + Q] + 6[Pn + Q] \equiv 4n$   
 $Pn + 2P + Q - 5Pn - 5P - 5Q + 6Pn + 6P + 6Q \equiv 4n$   
 $2Pn + (2Q - 3P) \equiv 4n$

$\therefore P = 2, \quad 2Q - 3P = 0$   
 $2Q = 6$   
 $Q = 3$

GENERAL SOLUTION

$u_n = A(2^n) + B(3^n) + 2n + 3$

**Right Page:**

FINDING THE CONSTANTS (GIVE)

$u_1 = 1 \Rightarrow 1 = A + 3B + 5$   
 $u_2 = 3 \Rightarrow 3 = 4A + 9B + 7$

$2A + 3B = -4$   
 $4A + 9B = -4$

$-4A - 6B = 8$   
 $4A + 9B = -4$

$\Rightarrow 3B = 4$   
 $B = \frac{4}{3}$

$\Rightarrow 2A + 3(\frac{4}{3}) = -4$   
 $\Rightarrow 2A + 4 = -4$   
 $\Rightarrow 2A = -8$   
 $\Rightarrow A = -4$

FINDING THE GENERAL

$u_n = -4(2^n) + \frac{4}{3}(3^n) + 2n + 3$   
 $u_n = -2^{n+2} + 4(3^{n-1}) + 2n + 3$   
 $u_n = 4(3^{n-1}) - 2^{n+2} + 2n + 3$



## Question 8 (\*\*\*)

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the recurrence relation

$$u_{n+1} = u_n + 3n - 2, \quad u_1 = -1.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

Give the answer in the form  $u_n = \frac{1}{2}(an+b)(n+c)$ , where  $a, b$  and  $c$  are integers.

$$\boxed{\phantom{000}}, \quad \boxed{u_n = \frac{1}{2}(3n-1)(n-2)}$$

The image shows two pages of handwritten work. The left page is titled 'REWRITE THE DIFFERENCE EQUATION IN THE USUAL FORM' and shows the recurrence relation  $u_{n+1} = u_n + 3n - 2$  and  $u_{n+1} - u_n = 3n - 2$ . It then identifies the 'ANALOGY EQUATION' as  $\Delta = 0$  and  $\Delta = 1$ , leading to  $u_n = A(n)^2 + B$ . It then states 'FOR PARTICULAR INTEGERS WE NEED TO TRY  $Pn^2 + Qn$  AS THE GUESSED IS FIRST OF THE USUAL SEQUENTIAL ALGEBRA'. It shows  $u_n = Pn^2 + Qn$  and  $u_{n+1} = P(n+1)^2 + Q(n+1) = Pn^2 + 2Pn + P + Qn + Q$ . It then sets up the difference equation  $u_{n+1} - u_n = 3n - 2$  and equates coefficients to find  $2P = 3$  and  $P + Q = -2$ , leading to  $P = \frac{3}{2}$  and  $Q = -\frac{7}{2}$ . It then states 'CHECK THE GENERAL EQUATION IS' and gives  $u_n = \frac{3}{2}n^2 - \frac{7}{2}n + A$ . The right page is titled 'FINALLY USING THE INITIAL TERM OF -1' and shows  $u_1 = -1 \Rightarrow -1 = \frac{3}{2}(1)^2 - \frac{7}{2}(1) + A$ , leading to  $-1 = \frac{3}{2} - \frac{7}{2} + A$  and  $A = 1$ . It then gives  $u_n = \frac{3}{2}n^2 - \frac{7}{2}n + 1$  and  $u_n = \frac{1}{2}(3n^2 - 7n + 2)$ , and finally  $u_n = \frac{1}{2}(3n-1)(n-2)$  with a checkmark and 'A REBORN'.

**Question 9** (\*\*\*)

A sequence of numbers,  $u_1, u_2, u_3, u_4, \dots$ , is defined by

$$u_{n+1} = 3u_n - 1, \quad u_1 = 2.$$

Determine, in terms of  $n$ , a simplified expression for

$$\sum_{r=1}^n u_r.$$

$$\boxed{\phantom{000}}, \quad \boxed{S_n = \frac{1}{4} [3^{n+1} + 2n - 3]}$$

**USING STANDARD TECHNIQUE**

$u_{n+1} = 3u_n - 1, \quad u_1 = 2$

"AUXILIARY EQUATION"

$$\Rightarrow u_{n+1} - 3u_n = -1$$

$$\Rightarrow \lambda - 3\lambda = -1$$

$$\Rightarrow -2\lambda = -1$$

$$\Rightarrow \lambda = \frac{1}{2}$$

GENERAL SOLUTION OF  $u_{n+1} = 3u_n - 1$  IS GIVEN BY

$$u_n = A \times 3^n$$

"PARTICULAR WHERE" - TRY  $u_n = C$  (CONSTANT)

$$\Rightarrow C - 3C = -1$$

$$\Rightarrow -2C = -1$$

$$\Rightarrow C = \frac{1}{2}$$

GENERAL SOLUTION OF  $u_{n+1} = 3u_n - 1$  IS GIVEN BY

$$\Rightarrow u_n = \frac{1}{2} + A \times 3^n$$

**USING THE CONDITION,  $u_1 = 2$ , i.e.  $n=1, u_1=2$**

$$\Rightarrow 2 = \frac{1}{2} + A \times 3^1$$

$$\Rightarrow 2 = \frac{1}{2} + 3A$$

$$\Rightarrow 4 = 1 + 6A$$

$$\Rightarrow A = \frac{1}{2}$$

$\therefore u_n = \frac{1}{2} + \frac{1}{2} \times 3^n$

**SUMMING UP, BEGINNING THAT FOR GEOMETRIC PROGRESSIONS**

THE SUM OF THE FIRST  $n$  TERMS IS GIVEN BY  $\frac{n(n+1)}{2}$

$$\Rightarrow S_n = \sum_{r=1}^n \left( \frac{1}{2} + \frac{1}{2} \times 3^r \right) = \frac{1}{2} \sum_{r=1}^n 1 + \frac{1}{2} \sum_{r=1}^n 3^r$$

$$\Rightarrow S_n = \frac{1}{2} \times n + \frac{1}{2} \left( 3 + 3^2 + 3^3 + \dots + 3^n \right)$$

$$\Rightarrow S_n = \frac{n}{2} + \frac{1}{2} \left( \frac{3(3^n - 1)}{3 - 1} \right) = \frac{n}{2} + \frac{1}{4} (3^{n+1} - 3)$$

$$\Rightarrow S_n = \frac{n}{2} + \frac{3^{n+1} - 3}{4}$$

$$\Rightarrow S_n = \frac{2n + 3^{n+1} - 3}{4}$$

**Question 10 (\*\*\*)**

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the recurrence relation

$$u_{n+1} = u_n + n + 1, \quad u_1 = 0.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence is given by

$$u_n = \frac{1}{2}(n + A)(n + B),$$

where  $A$  and  $B$  are integers to be found.

$$u_n = \frac{1}{2}(n + 2)(n - 1)$$

Handwritten solution for Question 10:

Given:  $u_{n+1} = u_n + n + 1$ ,  $u_1 = 0$

Step 1:  $u_{n+1} - u_n = n + 1$

Step 2: Assume a quadratic form:  $u_n = Pn^2 + Qn + R$

Step 3: Substitute into the recurrence relation:

$$P(n+1)^2 + Q(n+1) + R - (Pn^2 + Qn + R) = n + 1$$

$$P(n^2 + 2n + 1) + Qn + Q + R - Pn^2 - Qn - R = n + 1$$

$$2Pn + P + Q = n + 1$$

Step 4: Equate coefficients:

$$2P = 1 \Rightarrow P = \frac{1}{2}$$

$$P + Q = 1 \Rightarrow \frac{1}{2} + Q = 1 \Rightarrow Q = \frac{1}{2}$$

Step 5: Use the initial condition  $u_1 = 0$  to find  $R$ :

$$u_1 = \frac{1}{2}(1)^2 + \frac{1}{2}(1) + R = 0$$

$$\frac{1}{2} + \frac{1}{2} + R = 0 \Rightarrow R = -1$$

Step 6: Final formula:

$$u_n = \frac{1}{2}n^2 + \frac{1}{2}n - 1$$

$$u_n = \frac{1}{2}(n^2 + n - 2)$$

$$u_n = \frac{1}{2}(n+2)(n-1)$$

## Question 11 (\*\*\*)

A sequence of numbers  $u_1, u_2, u_3, u_4, u_5 \dots, u_n, \dots$  is generated by the recurrence relation

$$u_{n+1} = 4u_n - (n+1)^2, \quad u_1 = 2.$$

Determine an expression for the  $n^{\text{th}}$  term of this sequence.

$$u_n = \frac{107}{108} \left( 4^n \right) - \frac{1}{3} n^2 - \frac{8}{9} n - \frac{20}{27} = \frac{1}{108} \left[ 107 \left( 4^n \right) - \left( 36n^2 + 96n + 80 \right) \right]$$

Handwritten solution for Question 11:

$u_{n+1} = 4u_n - (n+1)^2, \quad u_1 = 2.$

- Recurrence AS:  
 $u_{n+1} - 4u_n = -n^2 - 2n - 1$
- Auxiliary Equation:  
 $\lambda - 4 = 0$   
 $\lambda = 4$
- Particular Integral:  
 Try  $u_n = Pn^2 + Qn + R$   
 $u_{n+1} = P(n+1)^2 + Q(n+1) + R$   
 $[P(n+1)^2 + Q(n+1) + R] - 4[Pn^2 + Qn + R] = -n^2 - 2n - 1$   
 $\left. \begin{aligned} Pn^2 + 2Pn + P \\ + Qn + Q \\ + R \end{aligned} \right\} = -n^2 - 2n - 1$   
 $-3Pn^2 + (2P - 3Q)n + [P + Q - 3R] = -n^2 - 2n - 1$   

$-3P = -1$	$2P - 3Q = -2$	$P + Q - 3R = -1$
$P = \frac{1}{3}$	$\frac{2}{3} - 3Q = -2$	$\frac{1}{3} + Q - 3R = -1$
	$\frac{2}{3} = 3Q$	$\frac{2Q}{3} = 3R$
	$Q = \frac{8}{9}$	$R = \frac{20}{27}$
- $u_n = A(4^n) - \left( \frac{1}{3}n^2 + \frac{8}{9}n + \frac{20}{27} \right)$   
 Apply  $u_1 = 2 \Rightarrow 2 = 4A - \frac{1}{3} - \frac{8}{9} - \frac{20}{27}$   
 $A = \frac{107}{108}$   
 Hence  $u_n = \frac{107}{108} \left( 4^n \right) - \frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$

Question 12 (\*\*\*)

A recurrence relation is defined as

$$u_{n+1} = (\sin x)u_n + \cos 2x, \quad u_1 = k, \quad 0 < x < \frac{\pi}{2}.$$

- a) Explain why this recurrence relation will converge to a limit  $L$ , for all real values of the constant  $k$ .
- b) Given that  $L = \frac{1}{2} \sin x$ , write the recurrence relation in the form

$$u_{n+1} = Au_n + B,$$

where  $A$  and  $B$  are rational constants to be found.

- c) Given further that  $k = \frac{305}{24}$ , show that  $u_4 = 4$ .

$$u_{n+1} = \frac{1}{3}u_n + \frac{1}{9}$$

a)  $u_{n+1} = (\sin x)u_n + \cos 2x, \quad u_1 = k, \quad 0 < x < \frac{\pi}{2}$

- RECURRENCE RELATIONS OF THE FORM  $u_{n+1} = Au_n + B$  WILL CONVERGE IF  $-1 < A < 1$
- AS  $\sin x$ , WITH  $0 < x < \frac{\pi}{2}$ , SATISFIES THIS CONDITION, THE SEQUENCE CONVERGES. (GIVE ALL STARTING VALUES)

b) NEXT, LET THE LIMIT BE  $L$  (AS  $n \rightarrow \infty$ )

$$\Rightarrow L = L \sin x + \cos 2x$$

$$\Rightarrow \frac{1}{2} \sin 2x = \frac{1}{2} \sin 2x + \cos 2x$$

$$\Rightarrow \sin 2x = \sin 2x + 2 \cos 2x$$

$$\Rightarrow \sin 2x = \sin 2x + 2(1 - 2\sin^2 x)$$

$$\Rightarrow \sin 2x = \sin 2x + 2 - 4\sin^2 x$$

$$\Rightarrow 3\sin 2x + \sin 2x - 2 = 0$$

$$\Rightarrow (3\sin x - 2)(\sin x + 1) = 0$$

$$\Rightarrow \sin x = \frac{2}{3} \quad \text{or} \quad \sin x = -1$$

$$\therefore \cos 2x = 1 - 2\sin^2 x = 1 - 2 \times \left(\frac{2}{3}\right)^2 = 1 - \frac{8}{9} = \frac{1}{9}$$

$$\therefore u_{n+1} = \frac{1}{3}u_n + \frac{1}{9}$$

c) NOW  $u_{n+1} = \frac{1}{3}u_n + \frac{1}{9}, \quad u_1 = \frac{305}{24}$

$$u_2 = \frac{1}{3}u_1 + \frac{1}{9} = \frac{1}{3} \times \frac{305}{24} + \frac{1}{9} = \frac{305}{72} + \frac{1}{9} = \frac{305}{72} + \frac{8}{72} = \frac{313}{72}$$

$$u_3 = \frac{1}{3}u_2 + \frac{1}{9} = \frac{1}{3} \times \frac{313}{72} + \frac{1}{9} = \frac{313}{216} + \frac{1}{9} = \frac{313}{216} + \frac{24}{216} = \frac{337}{216}$$

$$u_4 = \frac{1}{3}u_3 + \frac{1}{9} = \frac{1}{3} \times \frac{337}{216} + \frac{1}{9} = \frac{337}{648} + \frac{1}{9} = \frac{337}{648} + \frac{72}{648} = \frac{409}{648}$$

As Required

## Question 13 (\*\*\*\*)

The Fibonacci sequence of numbers is generated by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n, \quad n \in \mathbb{N},$$

with  $u_1 = 1, u_2 = 1$ .

Solve the above difference equation to show that the  $n^{\text{th}}$  term of Fibonacci sequence is given by

$$u_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n.$$

proof

$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 1, u_2 = 1$

$u_{n+2} - u_{n+1} - u_n = 0$   
 $\lambda^2 - \lambda - 1 = 0$   
 $(\lambda - \frac{1}{2})^2 - \frac{5}{4} = 0$   
 $(\lambda - \frac{1}{2})^2 = \frac{5}{4}$   
 $\lambda - \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$   
 $\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$\therefore$  General solution  
 $u_n = A \left( \frac{1+\sqrt{5}}{2} \right)^n + B \left( \frac{1-\sqrt{5}}{2} \right)^n$

Apply conditions  
 $1 = A \left( \frac{1+\sqrt{5}}{2} \right) + B \left( \frac{1-\sqrt{5}}{2} \right)$   
 $2 = (1+\sqrt{5})A + (1-\sqrt{5})B$   
 $1 = A \left( \frac{1+\sqrt{5}}{2} \right)^2 + B \left( \frac{1-\sqrt{5}}{2} \right)^2$   
 $4 = A(6+2\sqrt{5}) + B(6-2\sqrt{5})$   
 $2 = A(3+\sqrt{5}) + B(3-\sqrt{5})$

$A = \frac{2 - (3-\sqrt{5})B}{1+\sqrt{5}}$   
 $2 = \frac{2 - (3-\sqrt{5})B}{1+\sqrt{5}} \times (3+\sqrt{5}) + B(3-\sqrt{5})$   
 $2 + 2\sqrt{5} = 6 + 2\sqrt{5} - (3-\sqrt{5})(3+\sqrt{5})B + (3-\sqrt{5})(3+\sqrt{5})B$   
 $2 = 6 - (3+\sqrt{5})(3-\sqrt{5})B + (3+\sqrt{5})(3-\sqrt{5})B$   
 $2 = 6 - (-2+2\sqrt{5})B + (-2+2\sqrt{5})B$   
 $-4 = [2+2\sqrt{5}-2+2\sqrt{5}]B$   
 $B = -\frac{4}{4\sqrt{5}} = -\frac{1}{\sqrt{5}}$   
 $A = \frac{2 - (3-\sqrt{5})(-\frac{1}{\sqrt{5}})}{1+\sqrt{5}} = \frac{2\sqrt{5} + (3-\sqrt{5})}{\sqrt{5}+5} = \frac{\sqrt{5}+1}{\sqrt{5}+5} \times \frac{(\sqrt{5}+1)(\sqrt{5}-1)}{(\sqrt{5}+5)(\sqrt{5}-1)}$   
 $A = \frac{5-5\sqrt{5}+5-5}{5-25} = \frac{-5\sqrt{5}}{-20} = \frac{\sqrt{5}}{4} \therefore A = \frac{1}{\sqrt{5}}$

$\therefore u_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$   $\checkmark$  Q.E.D.

**Question 14** (\*\*\*\*)

A sequence of numbers is generated by the recurrence relation

$$u_{n+2} = 2u_{n+1} - 2u_n, \quad n \in \mathbb{N},$$

with  $u_1 = 1$ ,  $u_2 = 6$ .

Determine a simplified expression for the  $n^{\text{th}}$  term of this sequence.

*The final answer may not contain complex numbers*

$$u_n = \frac{1}{2}(-2+3i)(1-i)^n - \frac{1}{2}(2+3i)(1+i)^n = 2^{\frac{1}{2}n} \left[ 3 \sin\left(\frac{1}{4}n\pi\right) - 2 \cos\left(\frac{1}{4}n\pi\right) \right],$$

START WITH THE AUXILIARY EQUATION OF THE DIFFERENCE EQUATION

$$r^2 - 2r + 2 = 0$$

$$\Rightarrow r^2 - 2r = -2$$

$$\Rightarrow r^2 - 2r + 1 = -1$$

$$\Rightarrow (r-1)^2 = -1$$

$$\Rightarrow r-1 = \pm i$$

$$\Rightarrow r = 1 \pm i$$

THE GENERAL SOLUTION IS

$$u_n = A(1+i)^n + B(1-i)^n$$

WHERE A & B MAY BE COMPLEX

APPLYING CONDITIONS

$$\begin{aligned} u_1 = 1 &\Rightarrow A(1+i) + B(1-i) = 1 \\ u_2 = 6 &\Rightarrow A(1+i)^2 + B(1-i)^2 = 6 \end{aligned}$$

$$\begin{cases} A(1+i) + B(1-i) = 1 \\ A(2i) + B(-2i) = 6 \end{cases} \Rightarrow \begin{cases} A(1+i) + B(1-i) = 1 \\ A - B = 3i \end{cases}$$

REARRANGE THE SECOND EQUATION & SUB INTO THE FIRST

$$\begin{aligned} A - B &= 3i \\ A &= B + 3i \end{aligned}$$

$$\begin{aligned} (B+3i)(1+i) + B(1-i) &= 1 \\ B(1+i) + 3i(1+i) + B(1-i) &= 1 \\ 2B + 1 + 3i(1+i) &= 1 \\ 2B + 1 + 3i + 3i^2 &= 1 \\ 2B + 1 + 3i - 3 &= 1 \\ 2B - 2 + 3i &= 1 \\ 2B &= 3 - 3i \\ B &= \frac{3}{2} - \frac{3}{2}i \end{aligned}$$

AND THEREFORE

$$A = -1 + \frac{3}{2}i - 3i = -1 - \frac{3}{2}i$$

$$A = -1 - \frac{3}{2}i \quad \text{or} \quad \frac{1}{2}(-2+3i)$$

THE SOLUTION CAN NOW BE WRITTEN AS

$$u_n = \frac{1}{2}(-2+3i)(1-i)^n - \frac{1}{2}(2+3i)(1+i)^n$$

PROCEED TO ELIMINATE THE COMPLEX NUMBERS AS REQUIRED

$$\begin{aligned} u_n &= \frac{1}{2} \left[ (-2+3i)(\sqrt{2}e^{-i\frac{\pi}{4}})^n - (2+3i)(\sqrt{2}e^{i\frac{\pi}{4}})^n \right] \\ u_n &= \frac{1}{2} \left[ (-2+3i)(2^{\frac{n}{2}}e^{-i\frac{n\pi}{4}}) - (2+3i)(2^{\frac{n}{2}}e^{i\frac{n\pi}{4}}) \right] \\ u_n &= \frac{1}{2} \times 2^{\frac{n}{2}} \left[ -2e^{-i\frac{n\pi}{4}} + 3ie^{-i\frac{n\pi}{4}} - 2e^{i\frac{n\pi}{4}} - 3ie^{i\frac{n\pi}{4}} \right] \end{aligned}$$

ALTERNATIVE SOLUTION - QUICKER & MORE DIRECT

STARTING WITH THE GENERAL SOLUTION FROM BEFORE

$$u_n = A(1+i)^n + B(1-i)^n$$

MINIMALLY USE EXPANSION HERE USING THE CONDITIONS

$$\begin{aligned} u_1 &= A(\sqrt{2}e^{i\frac{\pi}{4}}) + B(\sqrt{2}e^{-i\frac{\pi}{4}}) \\ u_2 &= A(2^{\frac{1}{2}}e^{i\frac{\pi}{2}}) + B(2^{\frac{1}{2}}e^{-i\frac{\pi}{2}}) \\ u_1 &= 2^{\frac{1}{2}} \left[ A \cos \frac{\pi}{4} + A i \sin \frac{\pi}{4} + B \cos \frac{\pi}{4} - B i \sin \frac{\pi}{4} \right] \\ u_1 &= 2^{\frac{1}{2}} \left[ (A+B) \cos \frac{\pi}{4} + i(A-B) \sin \frac{\pi}{4} \right] \\ u_1 &= 2^{\frac{1}{2}} \left[ P \cos \frac{\pi}{4} + Q \sin \frac{\pi}{4} \right] \end{aligned}$$

APPLY CONDITIONS

$$\begin{aligned} u_1 = 1 &\Rightarrow 2^{\frac{1}{2}} \left[ P \cos \frac{\pi}{4} + Q \sin \frac{\pi}{4} \right] = 1 \\ u_2 = 6 &\Rightarrow 2 \left[ P \cos \frac{\pi}{2} + Q \sin \frac{\pi}{2} \right] = 6 \end{aligned}$$

$\Rightarrow u_1 = \frac{1}{2} \times 2^{\frac{1}{2}} \left[ -3i \left( \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} \right) - 2 \left( \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \right) \right]$

$\Rightarrow u_1 = \frac{1}{2} \times 2^{\frac{1}{2}} \left[ -3i \times 2 \sin \left( \frac{\pi}{4} \right) - 2 \times 2 \cos \left( \frac{\pi}{4} \right) \right]$

$\Rightarrow u_1 = \frac{1}{2} \times 2^{\frac{1}{2}} \left[ -3i \times 2 \sin \left( \frac{\pi}{4} \right) - 2 \times 2 \cos \frac{\pi}{4} \right]$

$\Rightarrow u_1 = 2^{\frac{1}{2}} \left[ -3 \sin \frac{\pi}{4} - 2 \cos \frac{\pi}{4} \right]$

## Question 15 (\*\*\*\*)

A sequence of numbers is generated by the recurrence relation

$$u_n = 5u_{n-1} - 4u_{n-2} - 12n + 31, \quad n \in \mathbb{N}, \quad n \geq 2$$

with  $u_0 = 7$ ,  $u_1 = 9$ .

Determine a simplified expression for the  $n^{\text{th}}$  term of this sequence.

$$u_n = 4^n + 2n^2 - 3n + 6$$

REWRITE THE QUESTION

$$u_n = 5u_{n-1} - 4u_{n-2} - 12n + 31$$

$$u_n - 5u_{n-1} + 4u_{n-2} = -12n + 31$$

"HOMOGENEOUS" EQUATION      "CONSTANT" EQUATION

$$r^2 - 5r + 4 = 0 \quad u_n = A \times 1^n + B \times 4^n$$

$$(r-1)(r-4) = 0 \quad u_n = A + B \times 4^n$$

$$r = 1, 4$$

NOT LOSE FOR "THE PARTICULAR INTEGRAL" — TRY A POLYNOMIAL

$$u_n = Pn + Qr^2 \quad (\text{As the constant } 'A' \text{ is already there})$$

$$u_n = P(n-1) + Q(n-1)^2$$

$$u_{n-1} = P(n-2) + Q(n-2)^2$$

$$u_{n-2} = P(n-3) + Q(n-3)^2$$

SUBSTITUTE BACK INTO THE LHS OF THE DIFFERENCE EQUATION

$$\Rightarrow Pn + Qn^2 - 5[P(n-1) + Q(n-1)^2] + 4[P(n-2) + Q(n-2)^2] = -12n + 31$$

$$\Rightarrow Pn + Qn^2 - 5[Pn - P + Qn^2 - 2Qn + Q] + 4[Pn - 2P + Qn^2 - 4Qn + 4P] = -12n + 31$$

$$\Rightarrow Pn^2 - 5Pn + 5P - 5Qn^2 + 10Qn - 5Q + 4Pn^2 - 8Pn + 4Qn^2 - 16Qn + 16P = -12n + 31$$

$$\Rightarrow -3P - 6Qn + 11Q = -12n + 31$$

$$\Rightarrow -6Qn + (11Q - 3P) = -12n + 31$$

COMPARING COEFFICIENTS

$$\bullet -6Q = -12 \quad \bullet 11Q - 3P = 31$$

$$Q = 2 \quad 22 - 3P = 31$$

$$-3P = 9 \quad P = -3$$

$\therefore u_n = A + B \times 4^n - 3n + 2n^2$

FINALLY APPLY CONDITIONS

$$\bullet u_0 = 7 \quad \bullet u_1 = 9$$

$$A + B = 7 \quad A + 4B - 3 + 2 = 9$$

$$A + 4B = 10$$

$$\begin{array}{r} \downarrow \\ 4 + B = 7 \\ 4 + 4B = 10 \\ \hline 3B = 3 \\ B = 1 \end{array}$$

$$\downarrow \quad A = 6$$

$\therefore u_n = 6 + 4^n - 3n + 2n^2$



**Question 16** (\*\*\*)

A sequence of numbers is generated by the recurrence relation

$$a_{n+2} = a_{n+1} - a_n + 21(2^n), \quad n \in \mathbb{N},$$

with  $a_0 = 10$ ,  $a_1 = \frac{1}{2}[31 + 5\sqrt{3}]$ .

Determine a simplified expression for the  $n^{\text{th}}$  term of this sequence.

$$a_n = 7 \times 2^n + 3 \cos\left(\frac{1}{3}\pi n\right) + 5 \sin\left(\frac{1}{3}\pi n\right)$$

The handwritten solution is divided into two main sections: finding the auxiliary equation and finding the particular solution.

**Auxiliary Equation:**

$$\lambda^2 = \lambda - 1$$

$$\lambda^2 - \lambda + 1 = 0$$

$$4\lambda^2 - 4\lambda + 4 = 0$$

$$4\lambda^2 - 4\lambda + 4 = 0$$

$$(2\lambda - 1)^2 = -3$$

$$2\lambda - 1 = \pm i\sqrt{3}$$

$$\lambda = \frac{1 \pm i\sqrt{3}}{2}$$

$\therefore a_n = A \left(\frac{1+i\sqrt{3}}{2}\right)^n + B \left(\frac{1-i\sqrt{3}}{2}\right)^n$

**Particular Solution:**

$$a_n = P \times 2^n$$

$$a_{n+1} = P \times 2^{n+1} = 2P \times 2^n$$

$$a_{n+2} = P \times 2^{n+2} = 4P \times 2^n$$

Substituting into the recurrence relation:

$$4P \times 2^n = 2P \times 2^n - P \times 2^n + 21 \times 2^n$$

$$4P = 2P - P + 21$$

$$3P = 21$$

$$P = 7$$

$\therefore a_n = A \left(\frac{1+i\sqrt{3}}{2}\right)^n + B \left(\frac{1-i\sqrt{3}}{2}\right)^n + 7 \times 2^n$

**Using Initial Conditions:**

$$a_0 = 10 \Rightarrow 10 = A + B + 7 \Rightarrow A + B = 3$$

$$a_1 = \frac{1}{2}(31 + 5\sqrt{3}) \Rightarrow \frac{1}{2}(31 + 5\sqrt{3}) = A \left(\frac{1+i\sqrt{3}}{2}\right) + B \left(\frac{1-i\sqrt{3}}{2}\right) + 7$$

$$\Rightarrow \frac{1}{2}(31 + 5\sqrt{3}) - 7 = \frac{1}{2}(A + B) + \frac{i\sqrt{3}}{2}(A - B)$$

$$\Rightarrow \frac{1}{2}(17 + 5\sqrt{3}) = \frac{1}{2}(3) + \frac{i\sqrt{3}}{2}(A - B)$$

$$\Rightarrow 14 + 5\sqrt{3} = 3 + i\sqrt{3}(A - B)$$

$$\Rightarrow 11 + 5\sqrt{3} = i\sqrt{3}(A - B)$$

$$\Rightarrow A - B = \frac{11 + 5\sqrt{3}}{i\sqrt{3}}$$

$$\Rightarrow A - B = \frac{(11 + 5\sqrt{3})(-i\sqrt{3})}{-3}$$

$$\Rightarrow A - B = \frac{-i\sqrt{3}(11 + 5\sqrt{3})}{-3}$$

$$\Rightarrow A - B = \frac{i\sqrt{3}(11 + 5\sqrt{3})}{3}$$

$$\Rightarrow A - B = \frac{11i\sqrt{3} + 15}{3}$$

$$\Rightarrow A - B = \frac{15}{3} + \frac{11i\sqrt{3}}{3}$$

$$\Rightarrow A - B = 5 + \frac{11i\sqrt{3}}{3}$$

$\therefore a_n = 7 \times 2^n + 3 \cos\left(\frac{1}{3}\pi n\right) + 5 \sin\left(\frac{1}{3}\pi n\right)$

**Question 17** (\*\*\*\*)

A sequence is defined by the recurrence relation

$$u_n = \frac{2n}{2n+1}u_{n-1}, \quad n \in \mathbb{N} \quad u_0 = 1.$$

Show, by direct manipulation, that

$$u_n = \frac{4^n \times (n!)^2}{(2n+1)!}.$$

[you may not use proof by induction]

proof

$$\begin{aligned}
 U_n &= \frac{2^n}{2n+1} - U_{n-1}, \quad U_0 = 1 \\
 \Rightarrow U_1 &= \frac{2^1}{2 \cdot 1 + 1} - \frac{2^{0+1}}{2 \cdot 0 + 1} \quad U_0 \\
 \Rightarrow U_2 &= \frac{2^2}{2 \cdot 2 + 1} - \frac{2^{1+1}}{2 \cdot 1 + 1} - \frac{2^{0+1}}{2 \cdot 0 + 1} \quad U_{-1} \\
 \Rightarrow U_3 &= \frac{(2^3 - 2^2) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_4 &= \frac{2^4 - (2^3 - 2^2) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_5 &= \frac{2^5 - (2^4 - 2^3) - (2^{3+1} - 2^{2+1}) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 5 + 1)(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_6 &= \frac{2^6 - (2^5 - 2^4) - (2^{4+1} - 2^{3+1}) - (2^{3+1} - 2^{2+1}) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 6 + 1)(2 \cdot 5 + 1)(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_7 &= \frac{2^7 - (2^6 - 2^5) - (2^{5+1} - 2^{4+1}) - (2^{4+1} - 2^{3+1}) - (2^{3+1} - 2^{2+1}) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 7 + 1)(2 \cdot 6 + 1)(2 \cdot 5 + 1)(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_8 &= \frac{2^8 - (2^7 - 2^6) - (2^{6+1} - 2^{5+1}) - (2^{5+1} - 2^{4+1}) - (2^{4+1} - 2^{3+1}) - (2^{3+1} - 2^{2+1}) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 8 + 1)(2 \cdot 7 + 1)(2 \cdot 6 + 1)(2 \cdot 5 + 1)(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_9 &= \frac{2^9 - (2^8 - 2^7) - (2^{7+1} - 2^{6+1}) - (2^{6+1} - 2^{5+1}) - (2^{5+1} - 2^{4+1}) - (2^{4+1} - 2^{3+1}) - (2^{3+1} - 2^{2+1}) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 9 + 1)(2 \cdot 8 + 1)(2 \cdot 7 + 1)(2 \cdot 6 + 1)(2 \cdot 5 + 1)(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0 \\
 \Rightarrow U_{10} &= \frac{2^{10} - (2^9 - 2^8) - (2^{8+1} - 2^{7+1}) - (2^{7+1} - 2^{6+1}) - (2^{6+1} - 2^{5+1}) - (2^{5+1} - 2^{4+1}) - (2^{4+1} - 2^{3+1}) - (2^{3+1} - 2^{2+1}) - (2^{2+1} - 2^{1+1}) - (2^{1+1} - 2^{0+1})}{(2 \cdot 10 + 1)(2 \cdot 9 + 1)(2 \cdot 8 + 1)(2 \cdot 7 + 1)(2 \cdot 6 + 1)(2 \cdot 5 + 1)(2 \cdot 4 + 1)(2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1)} \quad U_0
 \end{aligned}$$

**Question 18** (\*\*\*\*)

A sequence is defined by the recurrence relation

$$u_{n+1} = \frac{n}{2n+1} u_n, \quad n \in \mathbb{N} \quad u_1 = 2.$$

Show, by direct manipulation, that

$$u_n = \frac{2^n \times [(n-1)!]^2}{(2n-1)!}.$$

[you may not use proof by induction]

proof

Handwritten mathematical proof for the sequence problem. The proof starts with the recurrence relation  $u_{n+1} = \frac{n}{2n+1} u_n$  and the initial condition  $u_1 = 2$ . It then shows the calculation of  $u_2$  and  $u_3$  to establish a pattern. The general term  $u_n$  is derived by multiplying the recurrence relation for  $n=1$  to  $n=n-1$ , resulting in  $u_n = \frac{2^n \times [(n-1)!]^2}{(2n-1)!}$ .

$$\begin{aligned}
 u_{n+1} &= \frac{n}{2n+1} u_n \quad u_1 = 2 \\
 u_2 &= \frac{1}{2 \times 1 + 1} u_1 = \frac{1}{3} \times 2 \\
 u_3 &= \frac{2}{2 \times 2 + 1} u_2 = \frac{2}{5} \times \frac{2}{3} \\
 u_4 &= \frac{3}{2 \times 3 + 1} u_3 = \frac{3}{7} \times \frac{2}{5} \times \frac{2}{3} \\
 u_5 &= \frac{4}{2 \times 4 + 1} u_4 = \frac{4}{9} \times \frac{2}{5} \times \frac{2}{3} \times \frac{2}{7} \\
 u_6 &= \frac{5}{2 \times 5 + 1} u_5 = \frac{5}{11} \times \frac{2}{5} \times \frac{2}{3} \times \frac{2}{7} \times \frac{2}{9} \\
 u_7 &= \frac{6}{2 \times 6 + 1} u_6 = \frac{6}{13} \times \frac{2}{5} \times \frac{2}{3} \times \frac{2}{7} \times \frac{2}{9} \times \frac{2}{11} \\
 u_8 &= \frac{7}{2 \times 7 + 1} u_7 = \frac{7}{15} \times \frac{2}{5} \times \frac{2}{3} \times \frac{2}{7} \times \frac{2}{9} \times \frac{2}{11} \times \frac{2}{13} \\
 u_9 &= \frac{8}{2 \times 8 + 1} u_8 = \frac{8}{17} \times \frac{2}{5} \times \frac{2}{3} \times \frac{2}{7} \times \frac{2}{9} \times \frac{2}{11} \times \frac{2}{13} \times \frac{2}{15} \\
 u_{10} &= \frac{9}{2 \times 9 + 1} u_9 = \frac{9}{19} \times \frac{2}{5} \times \frac{2}{3} \times \frac{2}{7} \times \frac{2}{9} \times \frac{2}{11} \times \frac{2}{13} \times \frac{2}{15} \times \frac{2}{17} \\
 u_n &= \frac{2^n \times [(n-1)!]^2}{(2n-1)!}
 \end{aligned}$$

## Question 19 (\*\*\*\*+)

The  $n^{\text{th}}$  term of a series is given recursively by

$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \quad n \in \mathbb{N}, \quad n \geq 1,$$

where  $a$  is a positive constant.

Given further that  $A_0 = 1$ , show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$

proof

$A_n = \frac{a(2n+1)}{2n+4} A_{n-1} = \frac{a(2n+1)}{2(n+2)} A_{n-1}$

● DERIVATE A PATTERN FROM THE RECURSIVE DEFINITION

- $A_1 = \left(\frac{a}{2}\right) \left(\frac{2 \cdot 1 + 1}{2 \cdot 1 + 2}\right) A_0$
- $A_2 = \left(\frac{a}{2}\right) \left(\frac{2 \cdot 2 + 1}{2 \cdot 2 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 1 + 1}{2 \cdot 1 + 2}\right) A_0$
- $A_3 = \left(\frac{a}{2}\right) \left(\frac{2 \cdot 3 + 1}{2 \cdot 3 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 2 + 1}{2 \cdot 2 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 1 + 1}{2 \cdot 1 + 2}\right) A_0$
- $A_4 = \left(\frac{a}{2}\right) \left(\frac{2 \cdot 4 + 1}{2 \cdot 4 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 3 + 1}{2 \cdot 3 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 2 + 1}{2 \cdot 2 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 1 + 1}{2 \cdot 1 + 2}\right) A_0$
- $A_5 = \left(\frac{a}{2}\right) \left(\frac{2 \cdot 5 + 1}{2 \cdot 5 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 4 + 1}{2 \cdot 4 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 3 + 1}{2 \cdot 3 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 2 + 1}{2 \cdot 2 + 2}\right) \times \left(\frac{a}{2}\right) \left(\frac{2 \cdot 1 + 1}{2 \cdot 1 + 2}\right) A_0$

● NOW  $A_0 = 1$  SO WE MAY SUMMARISE THE EXPRESSION AS FOLLOWS

$$\Rightarrow A_1 = \left(\frac{a}{2}\right)^1 \frac{(2 \cdot 1 + 1)(2 \cdot 2 - 1)(2 \cdot 3 - 1) \dots \times 5 \times 3}{(1+2)(2+2)(3+2) \dots \times 4 \times 3}$$

$$\Rightarrow A_2 = \left(\frac{a}{2}\right)^2 \frac{(2 \cdot 2 + 1)(2 \cdot 3 + 1)(2 \cdot 4 + 1)(2 \cdot 5 + 1) \dots \times 6 \times 5 \times 4 \times 3 \times 2}{(2+2)(3+2)(4+2)(5+2) \dots \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_3 = \left(\frac{a}{2}\right)^3 \frac{(2 \cdot 3 + 1)(2 \cdot 4 + 1)(2 \cdot 5 + 1)(2 \cdot 6 + 1)(2 \cdot 7 + 1) \dots \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3+2)(4+2)(5+2)(6+2)(7+2) \dots \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_4 = \left(\frac{a}{2}\right)^4 \frac{(2 \cdot 4 + 1)(2 \cdot 5 + 1)(2 \cdot 6 + 1)(2 \cdot 7 + 1)(2 \cdot 8 + 1)(2 \cdot 9 + 1) \dots \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4+2)(5+2)(6+2)(7+2)(8+2)(9+2) \dots \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_5 = \left(\frac{a}{2}\right)^5 \frac{(2 \cdot 5 + 1)(2 \cdot 6 + 1)(2 \cdot 7 + 1)(2 \cdot 8 + 1)(2 \cdot 9 + 1)(2 \cdot 10 + 1) \dots \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5+2)(6+2)(7+2)(8+2)(9+2)(10+2) \dots \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_6 = \left(\frac{a}{2}\right)^6 \frac{(2 \cdot 6 + 1)(2 \cdot 7 + 1)(2 \cdot 8 + 1)(2 \cdot 9 + 1)(2 \cdot 10 + 1)(2 \cdot 11 + 1) \dots \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(6+2)(7+2)(8+2)(9+2)(10+2)(11+2) \dots \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_7 = \left(\frac{a}{2}\right)^7 \frac{(2 \cdot 7 + 1)(2 \cdot 8 + 1)(2 \cdot 9 + 1)(2 \cdot 10 + 1)(2 \cdot 11 + 1)(2 \cdot 12 + 1) \dots \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(7+2)(8+2)(9+2)(10+2)(11+2)(12+2) \dots \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_8 = \left(\frac{a}{2}\right)^8 \frac{(2 \cdot 8 + 1)(2 \cdot 9 + 1)(2 \cdot 10 + 1)(2 \cdot 11 + 1)(2 \cdot 12 + 1)(2 \cdot 13 + 1) \dots \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(8+2)(9+2)(10+2)(11+2)(12+2)(13+2) \dots \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_9 = \left(\frac{a}{2}\right)^9 \frac{(2 \cdot 9 + 1)(2 \cdot 10 + 1)(2 \cdot 11 + 1)(2 \cdot 12 + 1)(2 \cdot 13 + 1)(2 \cdot 14 + 1) \dots \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(9+2)(10+2)(11+2)(12+2)(13+2)(14+2) \dots \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\Rightarrow A_{10} = \left(\frac{a}{2}\right)^{10} \frac{(2 \cdot 10 + 1)(2 \cdot 11 + 1)(2 \cdot 12 + 1)(2 \cdot 13 + 1)(2 \cdot 14 + 1)(2 \cdot 15 + 1) \dots \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(10+2)(11+2)(12+2)(13+2)(14+2)(15+2) \dots \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

As required