DIAGONALIZING

CONICS & QUADRICS

CONICS

Question 1

$$5x^2 - 4xy + 8y^2 = 36 .$$

The conic with the above Cartesian equation is rotated out of its standard position.

By suitably removing the xy term, give a Cartesian equation for the conic in the rotated system.

You are expected to sketch and name the conic.

B , $\boxed{\text{ellipse} : \dfrac{X^2}{4} + \dfrac{Y^2}{9} = 1}$

START BY WRITING THE CONIC IN THE USUAL MATRIX FORM

$\Longrightarrow 5x^2 - 4xy + 8y^2 = 36$

$\Longrightarrow \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 36$

$\Longrightarrow$

THE MATRIX ABOVE BEING SYMMETRIC WILL DIAGONALIZE WITH NORMALIZED EIGENVECTORS — START WITH THE CHARACTERISTIC EQUATION

$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} = 0 \Longrightarrow (5-\lambda)(8-\lambda) - 4 = 0$

$\Longrightarrow (\lambda-5)(\lambda-8) - 4 = 0$

$\Longrightarrow \lambda^2 - 13\lambda + 40 - 4 = 0$

$\Longrightarrow \lambda^2 - 13\lambda + 36 = 0$

$\Longrightarrow (\lambda \quad 4)(\lambda \quad 9) = 0$

$\Longrightarrow \lambda = \genfrac{}{}{0pt}{}{4}{9}$

IF $\lambda = 4$

$\left.\begin{array}{r} 5x - 2y = 4x \\ -2x + 8y = 4y \end{array}\right\}$ BOTH GIVE $x = 2y$

- EIGENVECTOR: $\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- NORMALIZED EIGENVECTOR: $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

IF $\lambda = 9$

$\left.\begin{array}{r} 5x - 2y = 9x \\ -2x + 8y = 9y \end{array}\right\}$ BOTH GIVE $y = -2x$

- EIGENVECTOR: $\beta \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
- NORMALIZED EIGENVECTOR: $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

HENCE THE MATRIX $\underline{P}$ WHICH DIAGONALIZES THE MATRIX INTO $\underline{D}$

$\underline{P} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$ & $\underline{D} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

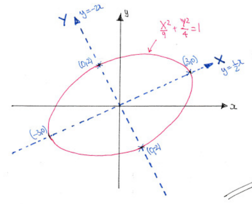
SO WE HAVE IN NEW CO-ORDINATES (X, Y)

$\Longrightarrow (X \; Y) \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 36$

$\Longrightarrow 4X^2 + 9Y^2 = 36$

$\Longrightarrow \dfrac{X^2}{9} + \dfrac{Y^2}{4} = 1$ // IT AN ELLIPSE

FINALLY THE SKETCH



Question 2

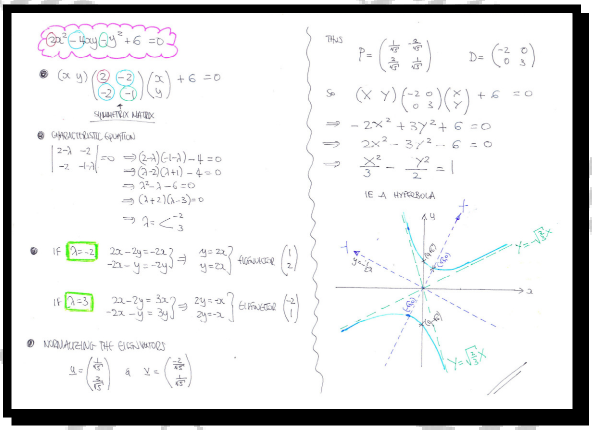
$$2x^2 - 4xy - y^2 + 6 = 0 .$$

The conic with the above Cartesian equation is rotated out of its standard position.

By suitably removing the xy term, give a Cartesian equation for the conic in the rotated system.

You are expected to sketch and name the conic.

$$\boxed{\text{hyperbola} : \frac{X^2}{3} - \frac{Y^2}{2} = 1}$$



Question 3

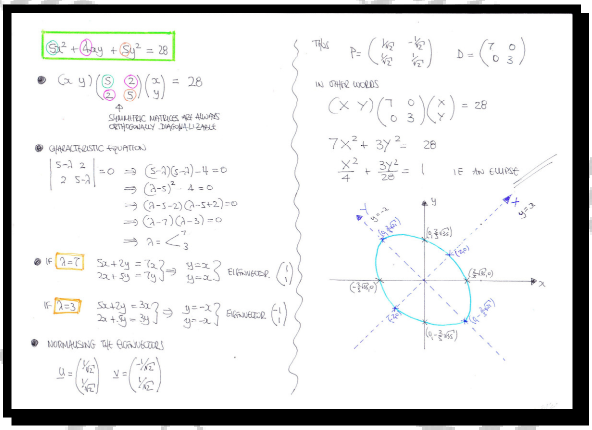
$$5x^2 + 4xy + 5y^2 = 28 .$$

The conic with the above Cartesian equation is rotated out of its standard position.

By suitably removing the xy term, give a Cartesian equation for the conic in the rotated system.

You are expected to sketch and name the conic.

$$\boxed{\text{ellipse} : \frac{X^2}{4} + \frac{3Y^2}{28} = 1}$$



Question 4

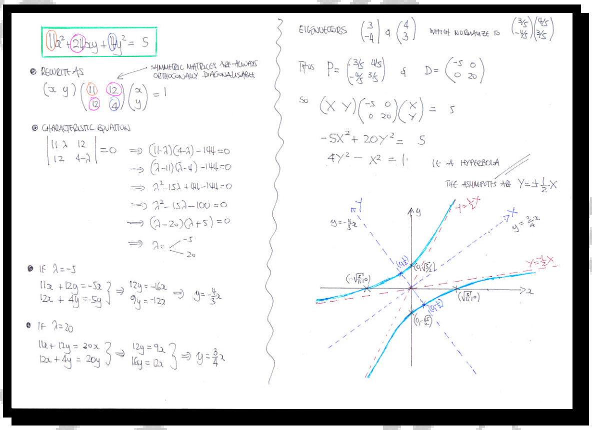
$$11x^2 + 24xy + 4y^2 = 5.$$

The conic with the above Cartesian equation is rotated out of its standard position.

By suitably removing the xy term, give a Cartesian equation for the conic in the rotated system.

You are expected to sketch and name the conic.

$$\boxed{\text{hyperbola}: \ 4Y^2 - X^2 = 1}$$



Question 5

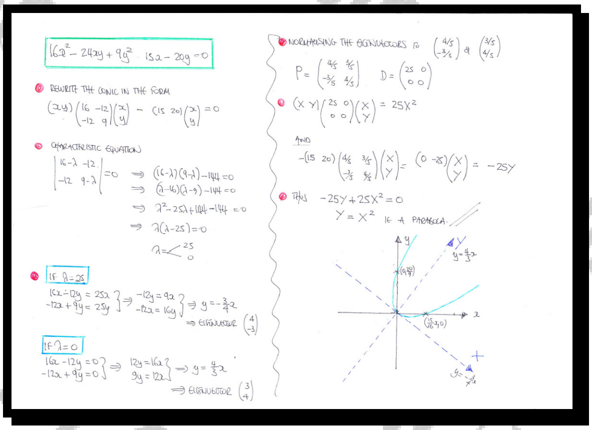
$$16x^2 - 24xy + 9y^2 - 15x - 20y = 0.$$

The conic with the above Cartesian equation is rotated out of its standard position.

By suitably removing the xy term, give a Cartesian equation of the conic in the rotated system.

You are expected to sketch and name the conic.

$\boxed{\text{parabola}: \ Y = X^2}$



Question 6

$$9x^2-4xy+6y^2-10x-20y=100.$$

The conic with the above Cartesian equation has been rotated and translated out of its standard position.

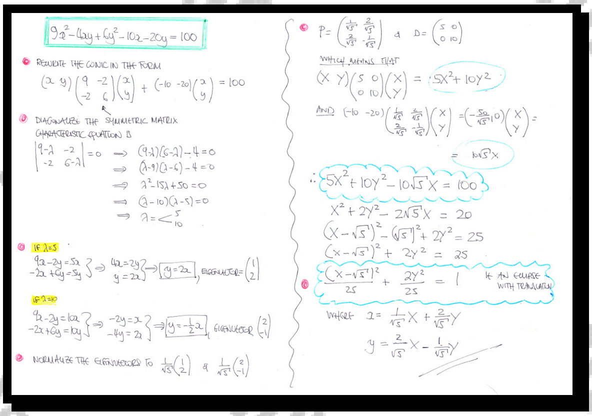
Show that a Cartesian equation in a suitable coordinate system is

$$\frac{(X-k)^2}{25}+\frac{2Y^2}{25}=1,$$

where k is an exact constant to be found.

You are not expected to sketch the conic.

$$\boxed{k=\sqrt{5}}$$



Question 7

$$x^2 + 2xy + y^2 + 8x + y = 0.$$

The conic with the above Cartesian equation has been rotated and translated out of its standard position.

Show that a Cartesian equation of the conic in a suitable coordinate system is

$$4X^2 + 9\sqrt{2}\,X + 7\sqrt{2}\,Y = 0.$$

You are not expected to sketch the conic.

MM, proof

- REWRITE THE CONIC IN THE STANDARD FORM

$$(x \;\; y)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (8 \;\; 1)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

- DIAGONALIZE THE SYMMETRIC MATRIX ABOVE

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \implies (1-\lambda)^2 - 1 = 0$$
$$\implies (1-\lambda-1)(1-\lambda+1) = 0$$
$$\implies -\lambda(2-\lambda) = 0$$
$$\implies \lambda = \begin{cases} 0 \\ 2 \end{cases}$$

- FIND EIGENVECTORS FOR EACH EIGENVALUE

IF $\lambda = 0$ $\quad \left.\begin{array}{l} x+y=0 \\ x+y=0 \end{array}\right\} \; y=-x \longrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

IF $\lambda = 2$ $\quad \left.\begin{array}{l} x+y=2x \\ x+y=2y \end{array}\right\} \; y=x \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- NORMALIZE THE EIGENVECTORS TO $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ & $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

- THUS IN THE USUAL NOTATION OF DIAGONALIZATION WE HAVE

$$\underline{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \& \quad \underline{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

- AND USING THE TRANSFORMATION

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\implies \boxed{x^2 + 2xy + 8x + y = 0}$$

$$\implies (x \;\; y)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (8 \;\; 1)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\implies (X \;\; Y)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + (8 \;\; 1)\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\implies 2X^2 + \frac{\sqrt{2}}{2}(9 \;\; 7)\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\implies 2X^2 + \frac{\sqrt{2}}{2}(9X + 7Y) = 0$$

$$\implies 4X^2 + \sqrt{2}(9X + 7Y) = 0$$

$$\implies 4X^2 + 9\sqrt{2}X + 7\sqrt{2}Y = 0$$

Question 8

Find, showing a detailed method, the area enclosed by the ellipse with the following Cartesian equation.

$$6x^2+4xy+9y^2-12x-4y=4\,.$$

$\boxed{}$, $\boxed{\text{area}=\pi\sqrt{2}}$

WE SHALL FIND THE AREA BY DIAGONALIZING THE CONIC, INTO A STANDARD CONIC

$$\Rightarrow 6x^2+4xy+9y^2-12x-4y=4$$

$$\Rightarrow (x\ \ y)\begin{pmatrix}6&2\\2&9\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+(-12\ \ -4)\begin{pmatrix}x\\y\end{pmatrix}=4$$

DIAGONIZE THE SYMMETRIC MATRIX – START BY THE CHARACTERISTIC EQUATION

$$\begin{vmatrix}6-\lambda&2\\2&9-\lambda\end{vmatrix}=0 \Rightarrow (6-\lambda)(9-\lambda)-4=0$$
$$\Rightarrow (\lambda-6)(\lambda-9)-4=0$$
$$\Rightarrow \lambda^2-15\lambda+50=0$$
$$\Rightarrow (\lambda-10)(\lambda-5)=0$$
$$\Rightarrow \lambda = \left\langle \begin{matrix}5\\10\end{matrix}\right.$$

FIND THE TWO (NORMALIZED) EIGENVECTORS FOR THE ABOVE EIGENVALUES

- IF $\lambda=5$

$$\left\{\begin{matrix}6x+2y=5x\\2x+9y=5y\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}x=-2y\\2x=-4y\end{matrix}\right\}\Rightarrow y=-\tfrac{1}{2}x,\quad \alpha\begin{pmatrix}2\\-1\end{pmatrix},\quad \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix}$$

NORMALIZED EIGENVECTORS

- IF $\lambda=10$

$$\left\{\begin{matrix}6x+2y=10x\\2x+9y=10y\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}2x=4x\\2x=y\end{matrix}\right\}\Rightarrow y=2x,\quad \beta\begin{pmatrix}1\\2\end{pmatrix},\quad \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$$

$$\therefore\ \underline{P}=\begin{bmatrix}2/\sqrt{5}&1/\sqrt{5}\\-1/\sqrt{5}&2/\sqrt{5}\end{bmatrix} \quad \& \quad \underline{D}=\begin{bmatrix}5&0\\0&10\end{bmatrix}$$

$\lambda=5 \quad \lambda=10$

SO THE CONIC CAN NOW BE WRITTEN AS

$$\Rightarrow (X\ \ Y)\begin{pmatrix}5&0\\0&10\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}+(-12\ \ -4)\begin{pmatrix}\frac{2}{\sqrt{5}}&\frac{1}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}&\frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=4$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2/\sqrt{5}&1/\sqrt{5}\\-1/\sqrt{5}&2/\sqrt{5}\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}$$

$$\Rightarrow 5X^2+10Y^2+(-12\ \ -4)\begin{pmatrix}\frac{2}{\sqrt{5}}X+\frac{1}{\sqrt{5}}Y\\-\frac{1}{\sqrt{5}}X+\frac{2}{\sqrt{5}}Y\end{pmatrix}=4$$

$$\Rightarrow 5X^2+10Y^2-\frac{20}{\sqrt{5}}X-\frac{20}{\sqrt{5}}Y=4$$

$$\Rightarrow 5X^2+10Y^2-4\sqrt{5}X-4\sqrt{5}Y=4$$

MANIPULATE INTO "STANDARD" FORM

$$\Rightarrow \tfrac{5}{4}X^2+\tfrac{5}{2}Y^2-\sqrt{5}Y-\sqrt{5}X=1$$

$$\Rightarrow \tfrac{5}{4}\left(X^2-\tfrac{4\sqrt{5}}{5}X\right)+\tfrac{5}{2}\left(Y^2-\tfrac{2\sqrt{5}}{5}Y\right)=1$$

$$\Rightarrow \tfrac{5}{4}\left[\left(X-\tfrac{2}{5}\sqrt{5}\right)^2-\tfrac{4}{5}\right]+\tfrac{5}{2}\left[\left(Y-\tfrac{\sqrt{5}}{5}\right)^2-\tfrac{1}{5}\right]=1$$

$$\Rightarrow \tfrac{5}{4}\left(X-\tfrac{2}{5}\sqrt{5}\right)^2-1+\tfrac{5}{2}\left(Y-\tfrac{\sqrt{5}}{5}\right)^2-\tfrac{1}{2}=1$$

$$\Rightarrow \tfrac{5}{4}\left(X-\tfrac{2}{5}\sqrt{5}\right)^2+\tfrac{5}{2}\left(Y-\tfrac{\sqrt{5}}{5}\right)^2=\tfrac{5}{2}$$

$$\Rightarrow \tfrac{1}{2}\left(X-\tfrac{2}{5}\sqrt{5}\right)^2+\left(Y-\tfrac{\sqrt{5}}{5}\right)^2=1$$

$$\Rightarrow \frac{\left(X-\frac{2}{5}\sqrt{5}\right)^2}{\underbrace{2}_{a^2}}+\frac{\left(Y-\frac{\sqrt{5}}{5}\right)^2}{\underbrace{1}_{b^2}}=1$$

FINALLY AS TRANSLATIONS DO NOT AFFECT AREA

$$\text{AREA} = \text{"}\pi ab\text{"} = \pi\times\sqrt{2}\times 1 = \pi\sqrt{2}$$

Question 9

$$3x^2 - 8xy + 12y^2 - 30x - 64y = 90\,.$$

The conic with the above Cartesian equation has been rotated and translated out of its standard position.

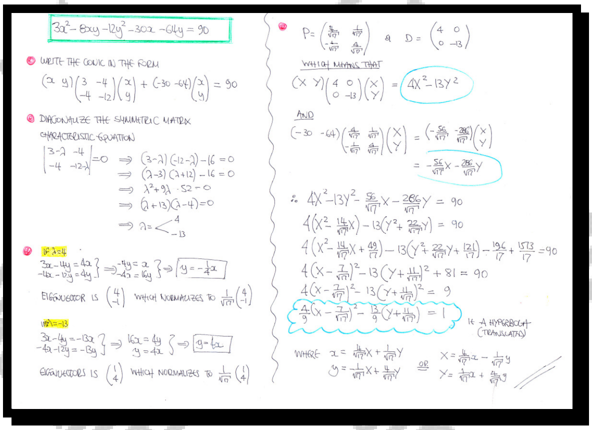
Show that a Cartesian equation of the conic in a suitable coordinate system is

$$\frac{4}{9}(X-a)^2 - \frac{13}{9}(Y+b)^2 = 1\,,$$

where a and b are exact constants to be found.

You are not expected to sketch the conic.

$$\boxed{a = \frac{7}{\sqrt{17}}\,,\quad b = \frac{11}{\sqrt{17}}}$$



$\boxed{3x^2 - 8xy - 12y^2 - 30x - 64y = 90}$

WRITE THE CONIC IN THE FORM

$(x\ \ y)\begin{pmatrix}3 & -4\\ -4 & -12\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + (-30\ \ -64)\begin{pmatrix}x\\ y\end{pmatrix} = 90$

DIAGONALIZE THE SYMMETRIC MATRIX

CHARACTERISTIC EQUATION

$\begin{vmatrix}3-\lambda & -4\\ -4 & -12-\lambda\end{vmatrix} = 0 \Rightarrow (3-\lambda)(-12-\lambda) - 16 = 0$

$\Rightarrow (\lambda-3)(\lambda+12) - 16 = 0$

$\Rightarrow \lambda^2 + 9\lambda - 52 = 0$

$\Rightarrow (\lambda+13)(\lambda-4) = 0$

$\Rightarrow \lambda = 4,\ -13$

IF $\lambda = 4$

$\left.\begin{array}{l}3x - 4y = 4x\\ -4x - 12y = 4y\end{array}\right\} \Rightarrow \left.\begin{array}{l}-4y = x\\ -4x = 16y\end{array}\right\} \Rightarrow \boxed{y = -\tfrac{1}{4}x}$

EIGENVECTOR IS $\begin{pmatrix}4\\ -1\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{17}}\begin{pmatrix}4\\ -1\end{pmatrix}$

IF $\lambda = -13$

$\left.\begin{array}{l}3x - 4y = -13x\\ -4x - 12y = -13y\end{array}\right\} \Rightarrow \left.\begin{array}{l}16x = 4y\\ y = 4x\end{array}\right\} \Rightarrow \boxed{y = 4x}$

EIGENVECTORS IS $\begin{pmatrix}1\\ 4\end{pmatrix}$ WHICH NORMALIZES TO $\frac{1}{\sqrt{17}}\begin{pmatrix}1\\ 4\end{pmatrix}$

$P = \begin{pmatrix}\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}}\\ -\frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}}\end{pmatrix}$ & $D = \begin{pmatrix}4 & 0\\ 0 & -13\end{pmatrix}$

WHICH MEANS THAT

$(X\ \ Y)\begin{pmatrix}4 & 0\\ 0 & -13\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix} = 4X^2 - 13Y^2$

AND

$(-30\ \ -64)\begin{pmatrix}\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}}\\ -\frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}}\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix} = \left(-\frac{56}{\sqrt{17}}\ \ -\frac{286}{\sqrt{17}}\right)\begin{pmatrix}X\\ Y\end{pmatrix}$

$= -\frac{56}{\sqrt{17}}X - \frac{286}{\sqrt{17}}Y$

$\therefore\ 4X^2 - 13Y^2 - \frac{56}{\sqrt{17}}X - \frac{286}{\sqrt{17}}Y = 90$

$4\left(X^2 - \frac{14}{\sqrt{17}}X\right) - 13\left(Y^2 + \frac{22}{\sqrt{17}}Y\right) = 90$

$4\left(X^2 - \frac{14}{\sqrt{17}}X + \frac{49}{17}\right) - 13\left(Y^2 + \frac{22}{\sqrt{17}}Y + \frac{121}{17}\right) - \frac{196}{17} + \frac{1573}{17} = 90$

$4\left(X - \frac{7}{\sqrt{17}}\right)^2 - 13\left(Y + \frac{11}{\sqrt{17}}\right)^2 + 81 = 90$

$4\left(X - \frac{7}{\sqrt{17}}\right)^2 - 13\left(Y + \frac{11}{\sqrt{17}}\right)^2 = 9$

$\frac{4}{9}\left(X - \frac{7}{\sqrt{17}}\right)^2 - \frac{13}{9}\left(Y + \frac{11}{\sqrt{17}}\right) = 1$ IE A HYPERBOLA (TRANSLATED)

WHERE $x = \frac{4}{\sqrt{17}}X + \frac{1}{\sqrt{17}}Y$, $y = -\frac{1}{\sqrt{17}}X + \frac{4}{\sqrt{17}}Y$ OR $X = \frac{4}{\sqrt{17}}x - \frac{1}{\sqrt{17}}y$, $Y = \frac{1}{\sqrt{17}}x + \frac{4}{\sqrt{17}}y$

QUADRICS

Question 1

$$x^2 + 2y^2 + z^2 + 2xy + 2yz = 9.$$

The quadric surface, with the above Cartesian equation, is rotated out of its standard position.

By suitably removing the cross terms, give the Cartesian equation of the quadric surface in the rotated system.

You are expected to name the conic and give the direction of any axes of symmetry in the original coordinate system.

$\boxed{}$, $\boxed{\text{elliptic cylinder}: Y^2 + 3Z^2 = 9 \quad \text{or} \quad \frac{Y^2}{9} + \frac{Z^2}{3} = 1}$

START BY REWRITING THE QUADRIC SURFACE IN MATRIX FORM

$\Rightarrow x^2 + 2y^2 + z^2 + 2xy + 2yz = 9$

$$\Rightarrow (x \ \ y \ \ z) \begin{array}{c} x \\ y \\ z \\ \\ \end{array} \!\!\begin{array}{c} \begin{bmatrix} 1_{x^2} & 1_{xy} & 0_{xz} \\ 1_{yx} & 2_{y^2} & 1_{yz} \\ 0_{zx} & 1_{zy} & 1_{z^2} \end{bmatrix} \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9$$

FINDING THE EIGENVALUES OF THE ABOVE SYMMETRIC MATRIX

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(\lambda-2)(\lambda-1)-1\right] - (1-\lambda) = 0$

$\Rightarrow (1-\lambda)\left[(\lambda-2)(\lambda-1)-1-1\right] = 0$

$\Rightarrow (1-\lambda)(\lambda^2-3\lambda+2-2) = 0$

$\Rightarrow (1-\lambda)(\lambda^2-3\lambda) = 0$

$\Rightarrow \lambda(\lambda-3)(1-\lambda) = 0$

$\Rightarrow \lambda = 0, 1, 3$

NEXT FIND EIGENVECTORS FOR EACH EIGENVALUE

- IF $\lambda = 0$

$$\left.\begin{array}{l} 1x + 1y + 0z = 0x \\ 1x + 2y + 1z = 0y \\ 0x + 1y + 1z = 0z \end{array}\right\} \rightarrow \left.\begin{array}{l} x + y = 0 \\ x + 2y + z = 0 \\ y + z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = -y \\ \text{...UNDETERMINED...} \\ z = -y \end{array}$$

$$\Rightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ OR } \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

- IF $\lambda = 1$

$$\left.\begin{array}{l} 1x + 1y + 0z = 1x \\ 1x + 2y + 1z = 1y \\ 0x + 1y + 1z = 1z \end{array}\right\} \Rightarrow \begin{array}{l} y = 0 \\ x = -z \\ z = z \end{array} \quad \therefore \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ OR } \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

- IF $\lambda = 3$

$$\left.\begin{array}{l} 1x + 1y + 0z = 3x \\ 1x + 2y + 1z = 3y \\ 0x + 1y + 1z = 3z \end{array}\right\} \Rightarrow \begin{array}{l} y = 2x \\ x - y + z = 0 \\ y = 2z \end{array} \quad \therefore \gamma \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

NORMALIZING THE 3 EIGEN-VECTORS

$$\left.\begin{array}{ll} \lambda = 0 & \underline{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \lambda = 1 & \underline{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \lambda = 3 & \underline{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \end{array}\right\} \text{NORMALIZED TO} \left.\begin{array}{l} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \end{array}\right\} \text{REQUIRED DIRECTION VECTORS OF LINES THROUGH THE ORIGIN}$$

- $\underline{P} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$
- $\underline{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\Rightarrow (X \ \ Y \ \ Z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 9$$

$\Rightarrow Y^2 + 3Z^2 = 9$

I.E AN ELLIPTIC CYLINDER

Question 2

$$3x^2 + 3y^2 + 3z^2 + 2xy + 2yz + 2xz = 20 .$$

The quadric surface, with the above Cartesian equation, is rotated out of its standard position.

By suitably removing the cross terms, give the Cartesian equation of the quadric surface in the rotated system.

You are expected to name the conic and give the direction of any axes of symmetry in the original coordinate system.

$$\boxed{\text{ellipsoid}: 5X^2 + 2Y^2 + 2Z^2 = 20 \quad \text{or} \quad \frac{x^2}{4} + \frac{Y^2}{10} + \frac{Z^2}{10} = 1}$$

$\boxed{3x^2+3y^2+3z^2+2xy+2yz+2zx=20}$

- REWRITE IN "STANDARD FORM"

$$(x\ y\ z)\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

- DIAGONALIZE THE ABOVE SYMMETRIC MATRIX
CHARACTERISTIC EQUATION IS

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$\boxed{C_3(-1)}$ $\begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & \lambda-2 & 3-\lambda \end{vmatrix} = 0$

$(\lambda-2)\begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$

$\boxed{r_{23}(1)}$ $(\lambda-2)\begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 4-\lambda \end{vmatrix} = 0$

- EXPAND BY THE MIDDLE COLUMN

$\Rightarrow (\lambda-2)\times\cancel{(-1)}\begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda-2)\left[(3-\lambda)(4-\lambda)-2\right]=0$

$\Rightarrow (\lambda-2)\left[(\lambda-3)(\lambda-4)-2\right]=0$

$\Rightarrow (\lambda-2)(\lambda^2-7\lambda+10)=0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-5)=0$

$\Rightarrow \lambda = \begin{cases} 2 & \text{(REPEATED)} \\ 5 \end{cases}$

- $\boxed{\text{IF } \lambda=5}$

$$\left.\begin{aligned} 3x+y+z&=5x \\ x+3y+z&=5y \\ x+y+3z&=5z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} -2x+y+z&=0 \\ x-2y+z&=0 \\ x+y-2z&=0 \end{aligned}\right\} \Rightarrow \boxed{z=2x-y}$$

$$\left.\begin{aligned} x-2y+(2x-y)&=0 \\ x+y-2(2x-y)&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} y&=x \\ z&=x \end{aligned} \quad \therefore \text{EIGENVECTOR } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- $\boxed{\text{IF } \lambda=2}$

$$\left.\begin{aligned} 3x+y+z&=2x \\ x+3y+z&=2y \\ x+y+3z&=2z \end{aligned}\right\} \Rightarrow \begin{aligned} x+y+z&=0 \\ x+y+z&=0 \\ x+y+z&=0 \end{aligned}$$

PICK ANY TWO INDEPENDENT EIGENVECTORS, AND IN THIS CASE MUST ALSO BE ORTHOGONAL

SAY $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

- NORMALIZE THE THREE EIGENVECTORS TO $\underset{\lambda=5}{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$ $\underset{\lambda=2}{\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}}$ $\underset{\lambda=2}{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$

- $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\therefore (X\ Y\ Z)\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 20$

IE $5X^2+2Y^2+2Z^2=20$

$\frac{X^2}{4}+\frac{Y^2}{10}+\frac{Z^2}{10}=1$

IE AN ELLIPSOID

WHERE $\left\{\begin{aligned} x &= \tfrac{1}{\sqrt{3}}X+\tfrac{2}{\sqrt{6}}Y \\ y &= \tfrac{1}{\sqrt{3}}X-\tfrac{1}{\sqrt{6}}Y+\tfrac{1}{\sqrt{2}}Z \\ z &= \tfrac{1}{\sqrt{3}}X-\tfrac{1}{\sqrt{6}}Y-\tfrac{1}{\sqrt{2}}Z \end{aligned}\right\}$

Question 3

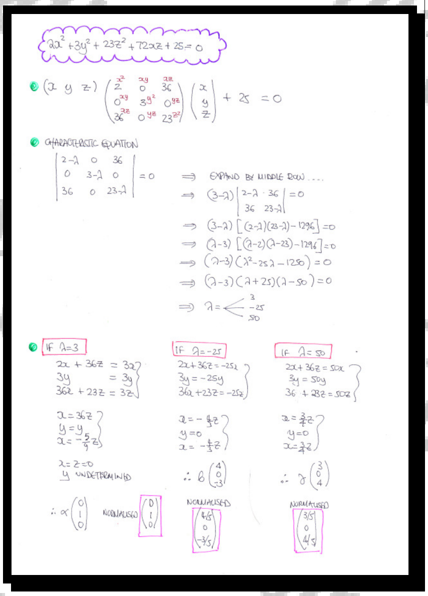
$$2x^2 + 3y^2 + 23z^2 + 72xz + 25 = 0.$$

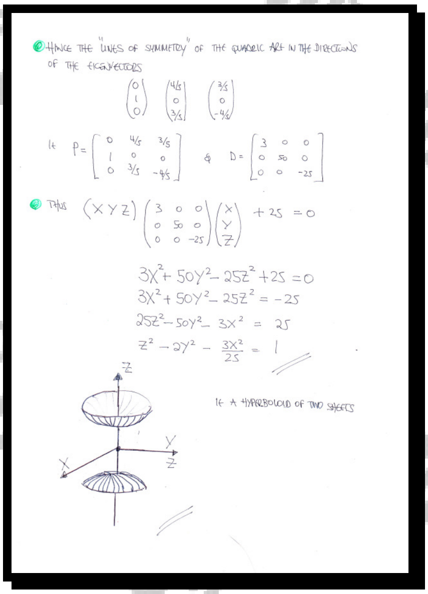
The quadric surface, with the above Cartesian equation, is rotated out of its standard position.

By suitably removing the cross terms, give the Cartesian equation of the quadric surface in the rotated system.

You are expected to name the conic and give the direction of any axes of symmetry in the original coordinate system.

$$\boxed{\text{hyperboloid of 2 sheets}: Z^2 - 2Y^2 - \frac{3X^2}{25} = 1}$$





Question 4

$$x^2 + y^2 + z^2 + xy + yz + xz = 2\,.$$

The quadric surface, with the above Cartesian equation, is rotated out of its standard position.

By suitably removing the cross terms, give the Cartesian equation of the quadric surface in the rotated system.

You are expected to name the conic and give the direction of any axes of symmetry in the original coordinate system.

$\boxed{}$, $\boxed{\text{ellipsoid}: \frac{X^2}{1} + \frac{Y^2}{4} + \frac{Z^2}{4} = 1}$

$\boxed{x^2 + y^2 + z^2 + xy + xz + yz = 2}$

START BY WRITING THE QUADRIC IN THE USUAL NOTATION

$$(x \;\; y \;\; z)\begin{bmatrix} 1_{x^2} & \frac{1}{2}_{xy} & \frac{1}{2}_{xz} \\ \frac{1}{2}_{yx} & 1_{y^2} & \frac{1}{2}_{yz} \\ \frac{1}{2}_{zx} & \frac{1}{2}_{zy} & 1_{z^2} \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2$$

SOLVE THE CHARACTERISTIC EQUATION OF THE SYMMETRIC MATRIX

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \Rightarrow \cancel{\left(\frac{1}{2}\right)^3}\begin{vmatrix} 2-2\lambda & 1 & 1 \\ 1 & 2-2\lambda & 1 \\ 1 & 1 & 2-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow C_{32}(-1)\begin{vmatrix} 2-2\lambda & 0 & 1 \\ 1 & 1-2\lambda & 1 \\ 1 & 2\lambda-1 & 2-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2\lambda-1)\begin{vmatrix} 2-2\lambda & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow r_{23}(1)\ (2\lambda-1)\begin{vmatrix} 2-2\lambda & 0 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 3-2\lambda \end{vmatrix} = 0$$

EXPAND BY MIDDLE COLUMN $\Rightarrow (2\lambda-1) \times \left\{ -\begin{vmatrix} 2-2\lambda & 1 \\ 2 & 3-2\lambda \end{vmatrix} \right\} = 0$

$$\Rightarrow (2\lambda-1)\big(2-(2-2\lambda)(3-2\lambda)\big) = 0$$

$$\Rightarrow (2\lambda-1)\big(2-(2\lambda-2)(2\lambda-3)\big) = 0$$

$$\Rightarrow (2\lambda-1)(2-4\lambda^2+10\lambda-6) = 0$$

$$\Rightarrow (2\lambda-1)(-4\lambda^2+10\lambda-4) = 0 \quad \div(-2)$$

$$\Rightarrow (2\lambda-1)(2\lambda^2-5\lambda+2) = 0$$

$$\Rightarrow (2\lambda-1)(2\lambda-1)(\lambda-2) = 0$$

$$\Rightarrow \lambda = \begin{cases} \frac{1}{2} \\ \frac{1}{2} \text{ (REPEATED)} \\ 2 \end{cases}$$

FIND EIGENVECTORS FOR EACH EIGENVALUE

- IF $\lambda = 2$

$$\left.\begin{aligned} x + \tfrac{1}{2}y + \tfrac{1}{2}z &= 2x \\ \tfrac{1}{2}x + y + \tfrac{1}{2}z &= 2y \\ \tfrac{1}{2}x + \tfrac{1}{2}y + z &= 2z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} -x + \tfrac{1}{2}y + \tfrac{1}{2}z &= 0 \\ \tfrac{1}{2}x - y + \tfrac{1}{2}z &= 0 \\ \tfrac{1}{2}x + \tfrac{1}{2}y - z &= 0 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} \text{(I)}\quad -2x + y + z &= 0 \\ \text{(II)}\quad x - 2y + z &= 0 \\ \text{(III)}\quad x + y - 2z &= 0 \end{aligned}\right\} \Rightarrow z = 2x - y \quad \text{— (I)}$$

SUB INTO THE OTHER TWO EQUATIONS

$$\Rightarrow \begin{cases} x - 2y + 2x - y = 0 \\ x + y - 4x + 2y = 0 \end{cases}$$

$$\Rightarrow y = x \quad \& \quad z = x$$

$$\Rightarrow \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ NORMALIZED TO } \boxed{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}}$$

- IF $\lambda = \frac{1}{2}$

$$\left.\begin{aligned} x + \tfrac{1}{2}y + \tfrac{1}{2}z &= \tfrac{1}{2}x \\ \tfrac{1}{2}x + y + \tfrac{1}{2}z &= \tfrac{1}{2}y \\ \tfrac{1}{2}x + \tfrac{1}{2}y + z &= \tfrac{1}{2}z \end{aligned}\right\} \text{ ALL REDUCE TO } \begin{aligned} \tfrac{1}{2}x + \tfrac{1}{2}y + \tfrac{1}{2}z &= 0 \\ x + y + z &= 0 \end{aligned}$$

PICK ANY TWO INDEPENDENT EIGENVECTORS THAT SATISFY ABOVE RELATIONSHIP AND ARE PERPENDICULAR TO ONE ANOTHER (ALL 3)

SAY $\beta\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ & $\gamma\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ NOTE $\alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

NORMALIZED WE HAVE

$$\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

I.E. $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & 0 \end{bmatrix}$ & $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

$\lambda = 2 \quad \lambda = \frac{1}{2} \quad \lambda = \frac{1}{2}$

HENCE THE DIAGONALIZED FORM OF THE QUADRIC IS

$$2X^2 + \tfrac{1}{2}Y^2 + \tfrac{1}{2}Z^2 = 2$$

$$X^2 + \tfrac{1}{4}Y^2 + \tfrac{1}{4}Z^2 = 1$$

$$\frac{X^2}{1} + \frac{Y^2}{4} + \frac{Z^2}{4} = 1$$

$\therefore$ AN ELLIPSOID WITH ITS LINES OF SYMMETRY IN THE ORIGINAL COORDINATE SYSTEM IN THE DIRECTION OF THE EIGENVECTORS

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$