COMPLEX VARIABLES

CALCULATIONS OF RESIDUES

Question 1

$$f(z) \equiv \frac{\sin z}{z^2}, \; z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=1}$

- $f(z)$ HAS A SIMPLE POLE AT $z=0$, WHICH IS VERY EASY TO FIND FROM ITS LAURENT EXPANSION

$$f(z) = \frac{\sin z}{z^2} = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)\right]$$

$$= \frac{1}{z} - \frac{1}{6}z + \frac{1}{120}z^3 + O(z^5)$$

$\therefore$ RESIDUE IS 1

- ALTERNATIVE IS TO USE THE STANDARD METHOD FOR A SIMPLE POLE AT $z=a$

$$\lim_{z \to a}\left[\frac{d}{dz}\left[(z-a)^2 f(z)\right]\right] = \lim_{z \to 0}\left[\frac{d}{dz}\left[z^2 \frac{\sin z}{z^2}\right]\right]$$

$$= \lim_{z \to 0}\left[\frac{d}{dz}\left(\frac{\sin z}{z}\right)\right]$$

BY L'HOSPITAL RULE

$$= \lim_{z \to 0}\left(\frac{\cos z}{1}\right)$$

$$= 1$$

AS BEFORE

Question 2

$$f(z) \equiv e^z z^{-5}, \; z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=\frac{1}{24}}$

$f(z)$ HAS A SINGLE POLE OF ORDER 5 AT THE ORIGIN, WHICH IS EASY TO FIND DIRECTLY FROM ITS LAURENT EXPANSION

$$f(z) = e^z z^{-5} = \frac{e^z}{z^5} = \frac{1}{z^5}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots\right]$$

$$= \frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{2z^3} + \frac{1}{6z^2} + \frac{1}{24z} + \frac{1}{120} + \dots$$

$\therefore$ REQUIRED RESIDUE $= \frac{1}{24}$

ALTERNATIVE IS TO USE THE STANDARD FORMULA FOR FINDING A POLE OF ORDER n, AT $z=a$

$$\text{Res}(z=a) = \frac{1}{(n-1)!}\lim_{z \to a}\left[\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right]$$

$$\text{Res}(z=0) = \frac{1}{4!}\lim_{z \to 0}\left[\frac{d^4}{dz^4}\left[z^5 \times \frac{e^z}{z^5}\right]\right]$$

$$\text{Res}(z=0) = \frac{1}{24}\lim_{z \to 0}\left[\frac{d^4}{dz^4}\left(e^z\right)\right]$$

$$\text{Res}(z=0) = \frac{1}{24}\lim_{z \to 0}\left[e^z\right]$$

$$\text{Res}(z=0) = \frac{1}{24}$$

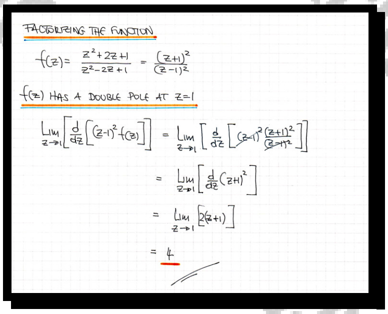
AS BEFORE

Question 3

$$f(z) \equiv \frac{z^2 + 2z + 1}{z^2 - 2z + 1}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=1) = 4}$


FACTORIZING THE FUNCTION

$$f(z) = \frac{z^2+2z+1}{z^2-2z+1} = \frac{(z+1)^2}{(z-1)^2}$$

$f(z)$ HAS A DOUBLE POLE AT $z=1$

$$\lim_{z \to 1}\left[\frac{d}{dz}\left[(z-1)^2 f(z)\right]\right] = \lim_{z \to 1}\left[\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{(z+1)^2}{\cancel{(z-1)^2}}\right]\right]$$

$$= \lim_{z \to 1}\left[\frac{d}{dz}(z+1)^2\right]$$

$$= \lim_{z \to 1}\left[2(z+1)\right]$$

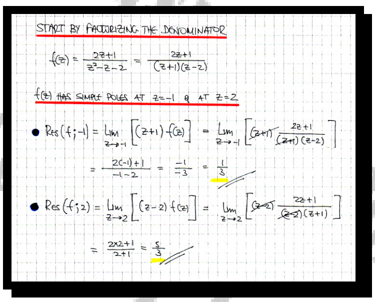
$$= 4$$

Question 4

$$f(z) \equiv \frac{2z + 1}{z^2 - z - 2}, \ z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\boxed{}$, $\boxed{res(z=2) = \frac{5}{3}}$, $\boxed{res(z=-1) = \frac{1}{3}}$


START BY FACTORIZING THE DENOMINATOR

$$f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z+1)(z-2)}$$

$f(z)$ HAS SIMPLE POLES AT $z=-1$ & AT $z=2$

- $\text{Res}(f;-1) = \lim_{z \to -1}\left[(z+1)f(z)\right] = \lim_{z \to -1}\left[\cancel{(z+1)}\frac{2z+1}{\cancel{(z+1)}(z-2)}\right]$

  $= \frac{2(-1)+1}{-1-2} = \frac{-1}{-3} = \frac{1}{3}$

- $\text{Res}(f;2) = \lim_{z \to 2}\left[(z-2)f(z)\right] = \lim_{z \to 2}\left[\cancel{(z-2)}\frac{2z+1}{\cancel{(z-2)}(z+1)}\right]$

  $= \frac{2 \times 2+1}{2+1} = \frac{5}{3}$

Question 5

$$f(z) \equiv \frac{z}{2z^2 - 5z + 2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\boxed{}$, $\boxed{res\left(z = \frac{1}{2}\right) = -\frac{1}{6}}$, $\boxed{res(z = 2) = \frac{2}{3}}$

START BY FACTORIZING THE DENOMINATOR

$$f(z) = \frac{z}{2z^2 - 5z + 2} = \frac{z}{(2z-1)(z-2)}$$

$f(z)$ HAS SIMPLE POLES AT $z = \frac{1}{2}$ & $z = 2$

- $\text{Res}\left(f; \frac{1}{2}\right) = \lim_{z \to \frac{1}{2}} \left[\left(z - \frac{1}{2}\right) f(z)\right] = \lim_{z \to \frac{1}{2}} \left[\left(z - \frac{1}{2}\right) \times \frac{z}{(2z-1)(z-2)}\right]$

$$= \lim_{z \to \frac{1}{2}} \left[\cancel{\left(z - \frac{1}{2}\right)} \times \frac{z}{2\cancel{\left(z - \frac{1}{2}\right)}(z-2)}\right] = \frac{\frac{1}{2}}{2\left(-\frac{3}{2}\right)} = \frac{\frac{1}{2}}{-3}$$

$$= -\frac{1}{6}$$

- $\text{Res}(f; 2) = \lim_{z \to 2} \left[(z-2) f(z)\right] = \lim_{z \to 2} \left[\cancel{(z-2)} \times \frac{z}{(2z-1)\cancel{(z-2)}}\right]$

$$= \frac{2}{3}$$

Question 6

$$f(z) \equiv \frac{1-\mathrm{e}^{\mathrm{i}z}}{z^3}, \quad z \in \mathbb{C}.$$

a) Find the first four terms in the Laurent expansion of $f(z)$ and hence state the residue of the pole of $f(z)$.

b) Determine the residue of the pole of $f(z)$ by an alternative method

$\boxed{\text{MM3}}$, $\boxed{res(z=0)=\frac{1}{2}}$

a) FIRSTLY OBTAIN THE RESIDUE FROM THE LAURENT EXPANSION

$$f(z) = \frac{1-e^{iz}}{z^3} = \frac{1}{z^3}\left[1-\left[1+(iz)+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\frac{(iz)^4}{4!}+\dots\right]\right]$$

$$= \frac{1}{z^3}\left[-iz-\frac{1}{2}(iz)^2-\frac{1}{6}(iz)^3-\frac{1}{24}(iz)^4+\dots\right]$$

$$= \frac{1}{z^3}\left[-iz+\frac{1}{2}z^2+\frac{1}{6}iz^3-\frac{1}{24}z^4-\dots\right]$$

$$= -\frac{i}{z^2}+\frac{1}{2z}+\frac{i}{6}-\frac{1}{24}z \dots$$

∴ RESIDUE OF THE DOUBLE POLE AT $z=0$ IS $\frac{1}{2}$

b) NEXT USING THE FORMULA FOR THE RESIDUE OF POLE OF ORDER n

$$\text{Res}(f;c) = \frac{1}{(n-1)!}\lim_{z\to c}\left[\frac{d^{n-1}}{dz^{n-1}}\left[(z-c)^n f(z)\right]\right]$$

$$\text{Res}(f;0) = \frac{1}{1!}\lim_{z\to 0}\left[\frac{d}{dz}\left[z^2 \times \frac{1-e^{iz}}{z^3}\right]\right]$$

NOTE THAT IT IS A DOUBLE POLE / NOT TRIPLE

$$= \lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{1-e^{iz}}{z}\right]\right]$$

$$= \lim_{z\to 0}\left[\frac{z(-ie^{iz})-(1-e^{iz})}{z^2}\right] = \lim_{z\to 0}\left[\frac{-ize^{iz}+e^{iz}-1}{z^2}\right]$$

THIS IS "ZERO OVER ZERO", SO BY L'HOSPITAL RULE WE OBTAIN

$$= \lim_{z\to 0}\left[\frac{(\cancel{-ie^{iz}}+ze^{iz})+\cancel{ie^{iz}}}{2z}\right] = \lim_{z\to 0}\left[\frac{ze^{iz}}{2z}\right]$$

$$= \lim_{z\to 0}\left[\frac{e^{iz}}{2}\right] = \frac{1}{2}$$

AS BEFORE

Question 7

$$f(z) \equiv \frac{z^2+4}{z^3+2z^2+2z}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$\boxed{res(z=0)=2}$, $\boxed{res(z=-1+\mathrm{i})=\frac{1}{2}(-1+3\mathrm{i})}$, $\boxed{res(z=-1-\mathrm{i})=-\frac{1}{2}(1+3\mathrm{i})}$

$f(z) = \frac{z^2+4}{z^3+2z^2+2z}$

FIRSTLY $z^3+2z^2+2z = z(z^2+2z+2)$
$= z[(z+1)^2-1+2]$
$= z[(z+1)^2+1]$
$= z[(z+1)-\mathrm{i}^2]$
$= z[z+1-\mathrm{i}][z+1+\mathrm{i}]$

$\therefore$ SIMPLE POLES AT $z=0$, $z=-1+\mathrm{i}$, $z=-1-\mathrm{i}$

- $\lim_{z\to 0}[z f(z)] = \lim_{z\to 0}\left[\cancel{z}\frac{z^2+4}{\cancel{z}(z^2+2z+2)}\right] = \frac{4}{2} = 2$
- $\lim_{z\to -1+\mathrm{i}}\left[\cancel{(z+1-\mathrm{i})}\frac{z^2+4}{z\cancel{(z+1-\mathrm{i})}(z+1+\mathrm{i})}\right] = \lim_{z\to -1+\mathrm{i}}\left[\frac{z^2+4}{z(z+1+\mathrm{i})}\right]$

$= \frac{(-1+\mathrm{i})^2+4}{(-1+\mathrm{i})(-1+\mathrm{i}+1+\mathrm{i})} = \frac{1-2\mathrm{i}-1+4}{2\mathrm{i}(-1+\mathrm{i})} = \frac{4-2\mathrm{i}}{-2-2\mathrm{i}} = \frac{2-\mathrm{i}}{-1-\mathrm{i}}$

$= \frac{(2-\mathrm{i})(-1+\mathrm{i})}{(-1-\mathrm{i})(-1+\mathrm{i})} = \frac{-2+2\mathrm{i}+\mathrm{i}+1}{2} = \frac{-1+3\mathrm{i}}{2} = \frac{1}{2}(-1+3\mathrm{i})$

- $\lim_{z\to -1-\mathrm{i}}\left[\cancel{(z+1+\mathrm{i})}\frac{z^2+4}{z(z+1-\mathrm{i})\cancel{(z+1+\mathrm{i})}}\right] = \lim_{z\to -1-\mathrm{i}}\left[\frac{z^2+4}{z(z+1-\mathrm{i})}\right]$

$= \frac{(-1-\mathrm{i})^2+4}{(-1-\mathrm{i})(-1-\mathrm{i}+1-\mathrm{i})} = \frac{(1+\mathrm{i})^2+4}{-(1+\mathrm{i})(-2\mathrm{i})} = \frac{(1+\mathrm{i})^2+4}{2\mathrm{i}(1+\mathrm{i})} = \frac{1+2\mathrm{i}-1+4}{-2+2\mathrm{i}} = \frac{4+2\mathrm{i}}{-2+2\mathrm{i}}$

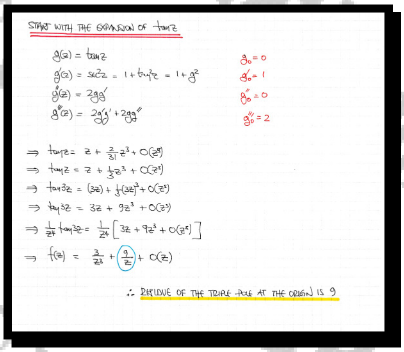
$= \frac{2+\mathrm{i}}{-1+\mathrm{i}} = \frac{(2+\mathrm{i})(-1-\mathrm{i})}{(-1+\mathrm{i})(-1-\mathrm{i})} = \frac{-2-2\mathrm{i}-\mathrm{i}+1}{2} = \frac{-1-3\mathrm{i}}{2} = -\frac{1}{2}(1+3\mathrm{i})$

Question 8

$$f(z) \equiv \frac{\tan 3z}{z^4}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{res(z=0)=9}$



START WITH THE EXPANSION OF $\tan z$

$g(z) = \tan z$ $\qquad g_0 = 0$
$g'(z) = \sec^2 z = 1+\tan^2 z = 1+g^2$ $\qquad g'_0 = 1$
$g''(z) = 2gg'$ $\qquad g''_0 = 0$
$g'''(z) = 2g'g'+2gg''$ $\qquad g'''_0 = 2$

$\Rightarrow \tan z = z + \frac{2}{3!}z^3 + O(z^5)$
$\Rightarrow \tan z = z + \frac{1}{3}z^3 + O(z^5)$
$\Rightarrow \tan 3z = (3z) + \frac{1}{3}(3z)^3 + O(z^5)$
$\Rightarrow \tan 3z = 3z + 9z^3 + O(z^5)$
$\Rightarrow \frac{1}{z^4}\tan 3z = \frac{1}{z^4}\left[3z+9z^3+O(z^5)\right]$
$\Rightarrow f(z) = \frac{3}{z^3} + \frac{9}{z} + O(z)$

$\therefore$ RESIDUE OF THE TRIPLE POLE AT THE ORIGIN IS 9

Question 9

$$f(z) \equiv \frac{z^2-2z}{(z^2+4)(z+1)^2}, \ z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res(z=2\mathrm{i}) = \frac{1}{25}(7+\mathrm{i})}, \ \boxed{res(z=-2\mathrm{i}) = \frac{1}{25}(7-\mathrm{i})}, \ \boxed{res(z=-1) = -\frac{14}{25}}$$

$$f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)} = \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

HAS SIMPLE POLES AT $z = \pm 2i$
HAS A DOUBLE POLE AT $z = -1$

- RESIDUE AT $2i$

$$\lim_{z \to 2i}\left[\cancel{(z-2i)}\frac{z^2-2z}{(z+1)^2\cancel{(z-2i)}(z+2i)}\right] = \lim_{z \to 2i}\left[\frac{z^2-2z}{(z+2i)(z+1)^2}\right] = \frac{-4-4i}{4i(1+2i)^2}$$

$$= \frac{-1-i}{i(1+2i)^2} = \frac{(-1-i)(1-2i)^2}{i \times 5 \times 5} = \frac{-(1+i)(1-4i-4)}{25i} = \frac{-(1+i)(-3-4i)}{25i}$$

$$= \frac{(1+i)(3+4i)}{25i} = \frac{3+4i+3i-4}{25i} = \frac{-1+7i}{25i} = \frac{-i-7}{-25} = \frac{7+i}{25}$$

- RESIDUE AT $-2i$

$$\lim_{z \to -2i}\left[\cancel{(z+2i)}\frac{z^2-2z}{(z+1)^2(z-2i)\cancel{(z+2i)}}\right] = \lim_{z \to -2i}\left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right] = \frac{-4+4i}{(1-2i)^2(-4i)}$$

$$= \frac{1-i}{i(1-2i)^2} = \frac{(1-i)(1+2i)^2}{i \times 5 \times 5} = \frac{(1-i)(1+4i-4)}{25i} = \frac{(1-i)(-3+4i)}{25i} = \frac{-3+4i+3i+4}{25i}$$

$$= \frac{1+7i}{25i} = \frac{i-7}{-25} = \frac{7-i}{25}$$

- RESIDUE AT -1

$$\lim_{z \to -1}\left[\frac{d}{dz}\left[(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+4)}\right]\right] = \lim_{z \to -1}\left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)\right] = \lim_{z \to -1}\left[\frac{(z^2+4)(2z-2)-2z(z^2-2z)}{(z^2+4)^2}\right]$$

$$= \frac{5(-4)+2(3)}{5^2} = \frac{-20+6}{25} = -\frac{14}{25}$$

Question 10

$$f(z) \equiv \frac{1}{e^z-1}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$, at the origin.

$$\boxed{res(z=0)=1}$$

$$f(z) = \frac{1}{e^z-1} = \frac{1}{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots-1} = \frac{1}{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}$$

$$= \frac{1}{z\left[1+\frac{z}{2!}+O(z^2)\right]} = \frac{1}{z}\left(1+\frac{1}{2}z+O(z^2)\right)^{-1}$$

$$= \frac{1}{z}\left[1-\frac{1}{2}z+O(z^2)\right] = \frac{1}{z}-\frac{1}{2}+O(z)$$

$\therefore$ RESIDUE 1

Question 11

$$f(z) \equiv \frac{z}{\left(3z^2 - 10\mathrm{i}\,z - 3\right)^2}, \; z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res\left(z = 3\mathrm{i}\right) = \frac{5}{256}}, \quad \boxed{res\left(z = \tfrac{1}{3}\mathrm{i}\right) = -\frac{5}{256}}$$

$$f(z) = \frac{z}{(3z^2-10iz-3)^2}$$

$$3z^2-10iz-3 = 3\left[z^2-\tfrac{10}{3}iz-1\right] = 3\left[\left(z-\tfrac{5}{3}i\right)^2+\tfrac{25}{9}-1\right]$$
$$= 3\left[\left(z-\tfrac{5}{3}i\right)^2+\tfrac{16}{9}\right] = 3\left[\left(z-\tfrac{5}{3}i\right)^2-\left(\tfrac{4}{3}i\right)^2\right]$$
$$= 3\left[\left(z-\tfrac{5}{3}i-\tfrac{4}{3}i\right)\left(z-\tfrac{5}{3}i+\tfrac{4}{3}i\right)\right]$$
$$= 3(z-3i)\left(z-\tfrac{1}{3}i\right)$$
$$= (z-3i)(3z-i)$$

$f(z)$ HAS DOUBLE POLES AT $z = 3i$ & $\frac{1}{3}i$

- $$\lim_{z \to 3i}\left[\frac{d}{dz}\left[\cancel{(z-3i)^2}\frac{z}{(3z-i)^2\cancel{(z-3i)^2}}\right]\right] = \lim_{z\to 3i}\left[\frac{d}{dz}\left[\frac{z}{(3z-i)^2}\right]\right]$$
$$= \lim_{z\to 3i}\left[\frac{(3z-i)^2\times 1 - 6(3z-i)z}{(3z-i)^4}\right] = \lim_{z\to 3i}\left[\frac{(3z-i)-6z}{(3z-i)^3}\right]$$
$$= \lim_{z\to 3i}\left[\frac{-i-3z}{(3z-i)^3}\right] = \frac{-i-3(3i)}{(3(3i)-i)^3} = \frac{-10i}{(8i)^3} = \frac{-10i}{-512i} = \frac{5}{256}$$

- $$\lim_{z\to \frac{1}{3}i}\left[\frac{d}{dz}\left[\cancel{\left(z-\tfrac{1}{3}i\right)^2}\frac{z}{3^2(z-3i)^2\cancel{\left(z-\tfrac{1}{3}i\right)^2}}\right]\right] = \lim_{z\to\frac{1}{3}i}\left[\frac{d}{dz}\left[\frac{z}{9(z-3i)^2}\right]\right]$$
$$= \frac{1}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{(z-3i)^2\times 1 - 2(z-3i)z}{(z-3i)^4}\right] = \frac{1}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{(z-3i)-2z}{(z-3i)^3}\right]$$
$$= \frac{1}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{-z-3i}{(z-3i)^3}\right] = -\frac{1}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{z+3i}{(z-3i)^3}\right] = -\frac{1}{9}\left[\frac{\frac{1}{3}i+3i}{\left(-\frac{8}{3}i\right)^3}\right]$$
$$= -\frac{1}{9}\,\frac{\frac{10}{3}i}{+\frac{512}{27}i} = -\frac{\frac{10}{27}}{\frac{512}{27}} = -\frac{10}{512} = -\frac{5}{256}$$

Question 12

$$f(z) \equiv \frac{\cot z \coth z}{z^3}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$ at $z = 0$.

$\boxed{}$, $\boxed{res(z=0) = -\frac{7}{45}}$

IT BEST TO FIND THE RESIDUE BY EXPANSION FOR $f(z) = \frac{\cot z \coth z}{z^3}$

$$f(z) = \frac{1}{z^3} \times \frac{\cos z}{\sin z} \times \frac{\cosh z}{\sinh z}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{z + \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)}$$

$$= \frac{1}{z^5} \times \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^5} \times \frac{1 + \cancel{\frac{z^2}{2}} + \frac{z^4}{24} - \cancel{\frac{z^2}{2}} - \frac{z^4}{4} + \frac{z^4}{24} + O(z^6)}{1 + \cancel{\frac{z^2}{6}} + \frac{z^4}{120} - \cancel{\frac{z^2}{6}} - \frac{z^4}{36} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^5} \times \frac{1 - \frac{1}{6}z^4 + O(z^6)}{1 - \frac{1}{90}z^4 + O(z^6)}$$

REWRITE IN ORDER TO COMPLETE THE EXPANSION

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + O(z^6)\right]\left[1 - \frac{1}{90}z^4 + O(z^6)\right]^{-1}$$

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + O(z^6)\right]\left[1 + \frac{1}{90}z^4 + O(z^6)\right]$$

$$= \frac{1}{z^5}\left[1 + \frac{1}{90}z^4 - \frac{1}{6}z^4 - \frac{1}{540}z^8 + O(z^{10})\right]$$

$$= \frac{1}{z^5}\left[1 - \frac{7}{45}z^4 + O(z^8)\right]$$

$$= \frac{1}{z^5} - \frac{7}{45z} + O(z^3)$$

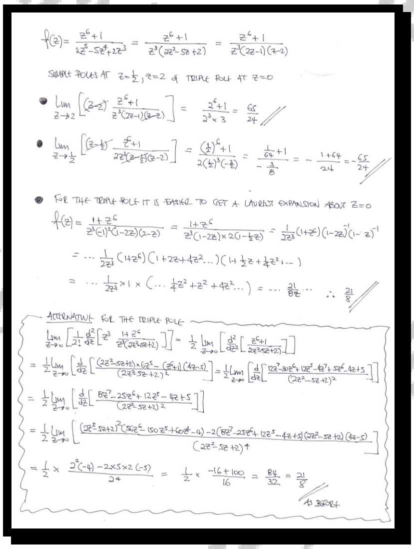
$\therefore$ RESIDUE IS $-\frac{7}{45}$

Question 13

$$f(z) \equiv \frac{z^6+1}{2z^5-5z^4+2z^3}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res\left(z=\frac{1}{2}\right) = -\frac{65}{24}}, \quad \boxed{res(z=2) = \frac{65}{24}}, \quad \boxed{res(z=0) = \frac{21}{8}}$$



$$f(z) = \frac{z^6+1}{2z^5-5z^4+2z^3} = \frac{z^6+1}{z^3(2z^2-5z+2)} = \frac{z^6+1}{z^3(2z-1)(z-2)}$$

SIMPLE POLES AT $z=\frac{1}{2}$, $z=2$ & TRIPLE POLE AT $z=0$

- $\lim_{z\to 2}\left[\cancel{(z-2)}\frac{z^6+1}{z^3(2z-1)\cancel{(z-2)}}\right] = \frac{2^6+1}{2^3\times 3} = \frac{65}{24}$
- $\lim_{z\to \frac{1}{2}}\left[\cancel{(z-\tfrac{1}{2})}\frac{z^6+1}{2z^3\cancel{(z-\tfrac{1}{2})}(z-2)}\right] = \frac{(\frac{1}{2})^6+1}{2(\frac{1}{2})^3(-\frac{3}{2})} = \frac{\frac{1}{64}+1}{-\frac{3}{8}} = -\frac{1+64}{24} = -\frac{65}{24}$
- FOR THE TRIPLE POLE IT IS EASIER TO GET A LAURENT EXPANSION ABOUT $z=0$

$$f(z) = \frac{1+z^6}{z^3(-1)^2(1-2z)(2-z)} = \frac{1+z^6}{z^3(1-2z)\times 2(1-\frac{1}{2}z)} = \frac{1}{2z^3}(1+z^6)(1-2z)^{-1}(1-\tfrac{1}{2}z)^{-1}$$

$$= \dots \frac{1}{2z^3}(1+z^6)(1+2z+4z^2\dots)(1+\tfrac{1}{2}z+\tfrac{1}{4}z^2+\dots)$$

$$= \dots \frac{1}{2z^3}\times 1\times(\dots \tfrac{1}{4}z^2+z^2+4z^2\dots) = \dots \frac{21}{8z}\dots \quad \therefore \frac{21}{8}$$

ALTERNATIVE FOR THE TRIPLE POLE

$$\lim_{z\to 0}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left[z^3\frac{1+z^6}{z^3(2z^2-5z+2)}\right]\right] = \frac{1}{2}\lim_{z\to 0}\left[\frac{d^2}{dz^2}\left[\frac{z^6+1}{2z^2-5z+2}\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{(2z^2-5z+2)\times 6z^5-(z^6+1)(4z-5)}{(2z^2-5z+2)^2}\right]\right] = \frac{1}{2}\lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{12z^7-30z^6+12z^5-4z^7+5z^6-4z+5}{(2z^2-5z+2)^2}\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{(2z^2-5z+2)^2}\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-2(8z^7-25z^6+12z^5-4z+5)(2z^2-5z+2)(4z-5)}{(2z^2-5z+2)^4}\right]$$

$$= \frac{1}{2}\times\frac{2^2(-4)-2\times 5\times 2(-5)}{2^4} = \frac{1}{2}\times\frac{-16+100}{16} = \frac{84}{32} = \frac{21}{8}$$

AS BEFORE

Question 14

$$f(z) \equiv \frac{4}{z^2(1-2\mathrm{i})+6z\mathrm{i}-(1+2\mathrm{i})}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res(z=2-\mathrm{i})=\mathrm{i}}, \quad \boxed{res\left(z=\tfrac{1}{5}(2-\mathrm{i})\right)=-\mathrm{i}}$$

$$f(z) = \frac{4}{z^2(1-2i)+6iz-(1+2i)}$$

BY THE QUADRATIC FORMULA

$$z = \frac{-6i \pm \sqrt{(6i)^2+4(1-2i)(1+2i)}}{2(1-2i)} = \frac{-6i \pm \sqrt{-36+4\times5}}{2(1-2i)} = \frac{-6i \pm \sqrt{-16}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)} = \frac{(-3\pm2)i}{1-2i} = \begin{cases} \dfrac{-i}{1-2i} = \dfrac{-i(1+2i)}{1+4} = \dfrac{1}{5}(2-i) \\[2ex] \dfrac{-5i}{1-2i} \quad \dfrac{-5i(1+2i)}{1+4} = (2-i) \end{cases}$$

$f(z)$ HAS SIMPLE POLES AT $z = 2-i$ & $\frac{1}{5}(2-i)$

- $$\lim_{z\to 2-i}\left[(z-2+i)\times\frac{4}{z^2(1-2i)+6iz-(1+2i)}\right] = \frac{0}{0} = \ldots \text{BY L'HOSPITAL} \ldots$$

$$= \lim_{z\to 2-i}\left[\frac{4}{2z(1-2i)+6i}\right] = \frac{4}{2(2-i)(1-2i)+6i} = \frac{2}{(2-i)(1-2i)+3i}$$

$$= \frac{2}{2-4i-i-2+3i} = \frac{2}{-2i} = -\frac{1}{i} = i$$

$$\lim_{z\to\frac{1}{5}(2-i)}\left[\left[z-\tfrac{1}{5}(2-i)\right]\times\frac{4}{z^2(1-2i)+6iz-(1+2i)}\right] = \frac{0}{0} = \ldots \text{BY L'HOSPITAL}$$

$$= \lim_{z\to\frac{1}{5}(2-i)}\left[\frac{4}{2z(1-2i)+6i}\right] = \frac{4}{2\times\frac{1}{5}(2-i)(1-2i)+6i} = \frac{10}{(2-i)(1-2i)+15i}$$

$$= \frac{10}{\cancel{2}-4i-i-\cancel{2}+15i} = \frac{10}{10i} = \frac{1}{i} = -i$$

Question 15

$$f(z) \equiv \frac{z\,\mathrm{e}^{kz}}{z^4+1}, \quad z \in \mathbb{C}, \; k \in \mathbb{R}, \; k > 0.$$

Show that the sum of the residues of the four poles of $f(z)$, is

$$\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right).$$

[ ], [proof]

IT IS BEST TO WORK WITH EXPONENTIALS IN THIS QUESTION

$f(z) = \dfrac{ze^{kz}}{z^4+1}$ HAS SIMPLE POLES AT:

$z^4 = -1 = e^{i(\pi+2n\pi)} = e^{i\pi(2n+1)}, \quad n=0,1,2,3$

$z = e^{\frac{i\pi}{4}(2n+1)}, \quad n=0,1,2,3 \quad (\text{OR } -2,-1,0,1)$

$z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{-i\frac{\pi}{4}}, e^{-i\frac{3\pi}{4}}$

CALCULATE THESE RESIDUES USING A GENERAL METHOD — LET A POLE BE AT $z=z_0$

$$\text{Res}(f;z_0) = \lim_{z \to z_0}\left[\frac{(z-z_0)ze^{kz}}{z^4+1}\right]$$

THIS MUST PRODUCE "ZERO OVER ZERO" AS $z-z_0$ MUST BE A FACTOR OF THE DENOMINATOR SO WE PROCEED BY L'HOSPITAL RULE

$$\text{Res}(f;z_0) = \lim_{z \to z_0}\left[\frac{\frac{d}{dz}\left[(z-z_0)ze^{kz}\right]}{\frac{d}{dz}(z^4+1)}\right] = \lim_{z \to z_0}\left[\frac{ze^{kz}+(z-z_0)e^{kz}+(z-z_0)kze^{kz}}{4z^3}\right]$$

$$\text{Res}(f;z_0) = \frac{z_0e^{kz_0}}{4z_0^3} = \boxed{\frac{e^{kz_0}}{4z_0^2}}$$

NEXT OBTAIN THE RESIDUE AT EACH OF THE FOUR POLES

$z_0 = e^{i\frac{\pi}{4}}$ GIVES $\dfrac{e^{ke^{i\frac{\pi}{4}}}}{4e^{i\frac{\pi}{2}}} = \dfrac{e^{k\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4i}$

$z_0 = e^{-i\frac{\pi}{4}}$ GIVES $\dfrac{e^{ke^{-i\frac{\pi}{4}}}}{4e^{-i\frac{\pi}{2}}} = \dfrac{e^{k\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{-4i}$

$z_0 = e^{\frac{3\pi}{4}i}$ GIVES $\dfrac{e^{ke^{i\frac{3\pi}{4}}}}{4e^{i\frac{3\pi}{2}}} = \dfrac{e^{k\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{-4i}$

$z_0 = e^{-\frac{3\pi}{4}i}$ GIVES $\dfrac{e^{ke^{-i\frac{3\pi}{4}}}}{4e^{-i\frac{3\pi}{2}}} = \dfrac{e^{k\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{4i}$

ADDING THE 4 RESIDUES — LET $a = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ FOR SIMPLICITY

$$\text{SUM OF 4 RESIDUES} = \frac{e^{k\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{4i} + \frac{e^{k\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{-4i} + \frac{e^{k\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}}{-4i} + \frac{e^{k\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)}}{4i}$$

$$= \frac{1}{4i}\left[e^{k(a+ai)} - e^{k(a-ai)} - e^{k(-a+ai)} + e^{k(-a-ai)}\right]$$

$$= \frac{1}{4i}\left[e^{ak}e^{kai} - e^{ak}e^{-kai} + e^{-ak}e^{-iak} - e^{-ak}e^{aki}\right]$$

$$= \frac{1}{4i}\left[e^{ak}\left(e^{aki}-e^{-aki}\right) - e^{-ak}\left(e^{aki}-e^{-aki}\right)\right]$$

$$= \frac{1}{4i}\left[\left(e^{aki}-e^{-aki}\right)\left(e^{ak}-e^{-ak}\right)\right]$$

$$= \frac{1}{4i} \times 2\sinh(aki) \times 2\sinh(ak)$$

$$= \frac{1}{\cancel{4i}} \times \cancel{2i}\sin(ak) \times \cancel{2}\sinh(ak)$$

$$= \sin(ak)\sinh(ak)$$

$$= \sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right)$$

AS REQUIRED

LAURENT SERIES

Question 1

Determine a Laurent series for

$$f(z)=\frac{1}{z},$$

which is valid in the infinite annulus $|z-1|>1$.

$$\boxed{\frac{1}{z}=\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}}{(z-1)^{r}}\right]}$$

- WE PROCEED AS FOLLOWS

$$\frac{1}{z}=\frac{1}{1+(z-1)}=\frac{1}{z-1}\left(\frac{1}{1+\frac{1}{z-1}}\right)$$

- EXPAND USING $\frac{1}{1+x}=1-x+x^2+x^3-\dots \quad |x|<1$

$$\Rightarrow \frac{1}{z}=\frac{1}{z-1}\left[1-\frac{1}{(z-1)}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\dots\right]$$

$$\Rightarrow \frac{1}{z}=\frac{1}{z-1}-\frac{1}{(z-1)^2}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}\dots$$

VALID FOR $\left|\frac{1}{z-1}\right|<1$

$|z-1|>1$

$$\therefore \frac{1}{z}=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{(z-1)^r} \quad \text{FOR ANNULUS } |z-1|>1$$

Question 2

Determine a Laurent series for

$$f(z) = \frac{e^{2z}}{(z-1)^3},$$

about its singularity, and hence state the residue of $f(z)$ about its singularity.

$$\boxed{}, \quad \boxed{f(z) = \sum_{r=0}^{\infty}\left[\frac{e^2\, 2^r (z-1)^{r-3}}{r!}\right]}, \quad \boxed{res(z=1) = 2e^2}$$

f(z) HAS A TRIPLE POLE AT $z=1$ — LET $w=z-1$

$$\Rightarrow f(z) = \frac{e^{2z}}{(z-1)^3}$$

$$\Rightarrow f(w) = \frac{e^{2(w+1)}}{w^3} = \frac{e^2}{w^3} \times e^{2w}$$

$$\Rightarrow f(w) = \frac{e^2}{w^3}\sum_{r=0}^{\infty}\frac{(2w)^r}{r!}$$

$$\Rightarrow f(w) = \sum_{r=0}^{\infty}\frac{e^2 2^r w^r}{w^3 r!}$$

$$\Rightarrow f(w) = \sum_{r=0}^{\infty}\frac{e^2 2^r w^{r-3}}{r!}$$

$$\Rightarrow f(z) = \sum_{r=0}^{\infty}\frac{e^2 2^r (z-1)^{r-3}}{r!}$$

$$\left[f(z) = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \frac{4e^2}{15}(z-1)^2 + \dots\right]$$

AND THE RESIDUE IS $2e^2$

Question 3

Determine a Laurent series for

$$f(z) = (z+2)\sin\left(\frac{1}{z-2}\right),$$

about $z=2$.

$$\boxed{},$$

$$\boxed{f(z) = \ldots + \frac{4}{5!(z-2)^5} + \frac{1}{5!(z-2)^4} - \frac{4}{3!(z-2)^3} - \frac{1}{3!(z-2)^2} + \frac{4}{z-2} + 1}$$

USING A SUBSTITUTION

$f(z) = (z+2)\sin\left(\frac{1}{z-2}\right)$

$f(w) = (w+4)\sin\left(\frac{1}{w}\right)$

$w = z-2$
$z = w+2$
$z+2 = w+4$

USING A SINE STANDARD EXPANSION

$$f(w) = (w+4)\left[\frac{1}{w} - \frac{1}{3!w^3} + \frac{1}{5!w^5} - \frac{1}{7!w^7}\right]$$

$$f(w) = 1 - \frac{1}{3!w^2} + \frac{1}{5!w^4} - \frac{1}{7w^6} + \ldots$$

$$\frac{4}{w} - \frac{4}{3!w^3} + \frac{4}{5!w^5} - \frac{4}{7!w^7} + \ldots$$

WRITE IN ORDER

$$f(w) = 1 + \frac{4}{w} - \frac{1}{3!w^2} - \frac{4}{3!w^3} + \frac{1}{5!w^4} + \frac{4}{5!w^5} - \ldots$$

$$f(w) = \ldots \frac{4}{5!w^5} + \frac{1}{5!w^4} - \frac{4}{3!w^3} - \frac{1}{3!w^2} + \frac{4}{w} + 1$$

$$f(z) = \ldots \frac{4}{5!(z-2)^5} + \frac{1}{5!(z-2)^4} - \frac{4}{3!(z-2)^3} - \frac{1}{3!(z-2)^2} + \frac{4}{z-2} + 1$$

$$f(z) = \ldots \frac{1/30}{(z-2)^5} + \frac{1/120}{(z-2)^2} - \frac{2/3}{(z-2)^3} - \frac{1/6}{(z-2)^2} + \frac{4}{z-2} + 1$$

Question 4

Determine a Laurent series for

$$f(z) = \frac{1}{z^2 - 1},$$

which is valid in the punctured disc $0 < |z-1| < 2$.

$$\boxed{}, \quad \boxed{\frac{1}{z^2-1} = \sum_{r=-1}^{\infty}\left[\frac{(z-1)^r(-1)^{r+1}}{2^{r+2}}\right]}$$

- NEED AN EXPANSION IN POWERS OF $z-1$

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{z-1}\left[\frac{1}{z+1}\right] = \frac{1}{z-1}\left[\frac{1}{(z-1)+2}\right]$$

- CREATING A STANDARD BINOMIAL

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{z-1}\left[\frac{1}{2+(z-1)}\right] = \frac{1}{2(z-1)}\left[\frac{1}{1+\frac{z-1}{2}}\right]$$

- THIS EXPANSION IS VALID FOR

$$0 < \left|\frac{z-1}{2}\right| < 1$$

$$0 < |z-1| < 2 \quad \text{AS REQUIRED}$$

- RETURNING TO THE LAURENT

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{2(z-1)}\left[1 - \frac{z-1}{2} + \frac{(z-1)^2}{2^2} - \frac{(z-1)^3}{2^3} + \frac{(z-1)^4}{2^4} \dots\right]$$

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2^2} + \frac{(z-1)}{2^3} - \frac{(z-1)^2}{2^4} + \frac{(z-1)^4}{2^5} \dots$$

$$\Rightarrow \frac{1}{z^2-1} = \sum_{r=-1}^{\infty}\left[\frac{(z-1)^r(-1)^{r+1}}{2^{r+2}}\right]$$

Question 5

Determine a Laurent series for

$$f(z)=\frac{1}{z+4},$$

which is valid for $|z|>4$.

$\boxed{}$, $\boxed{\frac{1}{z+4}=\sum_{r=0}^{\infty}\left[\frac{(-4)^r}{z^{r+1}}\right]}$

MANIPULATE THE FUNCTION AS FOLLOWS

$$f(z)=\frac{1}{z+4}=\frac{1}{z\left(1+\frac{4}{z}\right)}=\frac{1}{z}\left(1+\frac{4}{z}\right)^{-1}$$

EXPANDING BINOMIALLY NOTING THAT THE RADIUS OF CONVERGENCE MUST BE

$$\left|\frac{4}{z}\right|<1$$

$$|z|>4$$

HENCE WE OBTAIN

$$f(z)=\frac{1}{z}\left[1-\frac{4}{z}+\frac{16}{z^2}-\frac{64}{z^3}+\frac{256}{z^4}-\dots\right]$$

$$f(z)=\frac{1}{z}-\frac{4}{z^2}+\frac{16}{z^3}-\frac{64}{z^4}+\frac{256}{z^5}-\dots$$

Question 6

Determine a Laurent series for

$$f(z) = \frac{5z+3\mathrm{i}}{z(z+\mathrm{i})},$$

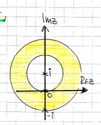
which is valid in the annulus $1 < |z-\mathrm{i}| < 2$.

$$\boxed{\frac{5z+3\mathrm{i}}{z(z+\mathrm{i})} = \sum_{r=0}^{\infty}\left[\frac{3(-\mathrm{i})^r}{(z-\mathrm{i})^{r+1}}\right] - \mathrm{i}\sum_{r=0}^{\infty}\left[\frac{(z-\mathrm{i})^r}{(-2\mathrm{i})^r}\right]}$$

PROCEED BY PARTIAL FRACTIONS (COVER UP)

$$f(z) = \frac{5z+3i}{z(z+i)} = \frac{\frac{3i}{i}}{z} + \frac{\frac{-2i}{-i}}{z+i} = \frac{3}{z} + \frac{2}{z+i}$$

THE SINGULARITIES OF $f(z)$ ARE SHOWN OPPOSITE, AT THE ORIGIN & AT $-i$



EXPANDING $\frac{3}{z}$ FOR $|z-i|>1$

$$\frac{3}{z} = \frac{3}{i+(z-i)} = \frac{3}{z-i}\left(\frac{1}{1+\frac{i}{z-i}}\right)$$

$$= \frac{3}{z-i}\left[1+\frac{i}{z-i}\right]^{-1} \longleftarrow \left|\frac{i}{z-i}\right|<1$$

$$= \frac{3}{z-i}\left[1 - \frac{i}{z-i} + \frac{i^2}{(z-i)^2} - \frac{i^3}{(z-i)^3} + \frac{i^4}{(z-i)^4} - \cdots\right]$$

$$= \frac{3}{z-i}\sum_{r=0}^{\infty}\left[\frac{i^r}{(z-i)^r}(-1)^r\right] = 3\sum_{r=0}\frac{(-i)^r}{(z-i)^{r+1}}$$

EXPANDING $\frac{2}{z+i}$ FOR $|z-i|<2$

$$\frac{2}{z+i} = \frac{2}{2i+(z-i)} = \frac{2}{2i\left(1+\frac{z-i}{2i}\right)} = \frac{1}{i}\left(1+\frac{z-i}{2i}\right)^{-1}$$

$$\left|\frac{z-i}{2i}\right|<1$$

$$= -i\left[1 - \frac{z-i}{2i} + \frac{(z-i)^2}{(2i)^2} - \frac{(z-i)^3}{(2i)^3} + \cdots\right]$$

$$= -i\sum_{r=1}^{\infty}\left(\frac{z-i}{2i}\right)^r(-1)^r$$

ADDING THE EXPANSIONS FOR $1<|z-i|<2$

$$f(z) = \sum_{r=0}^{\infty}\frac{3(-i)^r}{(z-i)^{r+1}} - i\sum_{r=0}^{\infty}\left(\frac{z-i}{-2i}\right)^r$$

NEGATIVE POWERS

POSITIVE POWERS.

Question 7

Determine a Laurent series for

$$f(z)=\frac{1}{(z-1)(z+2)},$$

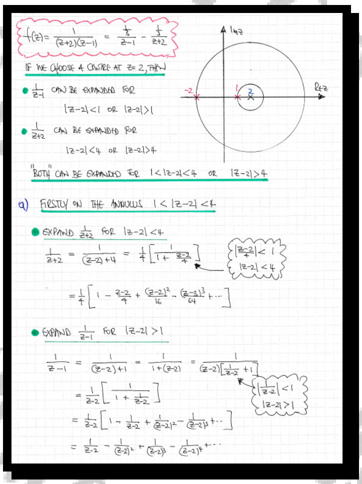
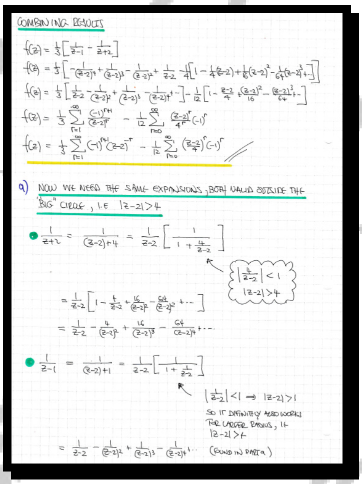
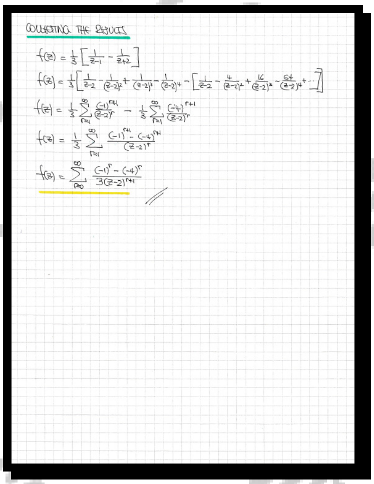
which is valid in ...

a) ... the annulus $1<|z-2|<4$.

b) ... in the region for which $|z-2|>4$.

$$\boxed{f(z) = \frac{1}{3}\sum_{r=1}^{\infty}\left[(-1)^{r+1}(z-2)^{-r}\right] - \frac{1}{12}\sum_{r=0}^{\infty}\left[(-1)^{r}\left(\frac{z-2}{4}\right)^{r}\right]},$$

$$\boxed{f(z) = \sum_{r=0}^{\infty}\frac{(-1)^r-(-4)^r}{3(z-2)^{r+1}}}$$

VARIOUS PROBLEMS

Question 1

The complex number $z = c + a\cos\theta + \mathrm{i}b\sin\theta$, $0 \le \theta < 2\pi$, traces a closed contour C, where a, b and c are positive real numbers with $a > c$.

By considering

$$\oint_C \frac{1}{z}\,dz,$$

show that

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + (b\sin\theta)^2}\,d\theta = \frac{2\pi}{b}.$$

$\boxed{\text{proof}}$

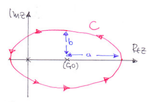
$$\boxed{\int_0^{2\pi} \frac{a + c\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\,d\theta = \frac{2\pi}{b}, \quad \begin{matrix} a,b,c \in \mathbb{R},\ a>c \\ a,b,c>0 \end{matrix}}$$

- LET $z = c + a\cos\theta + ib\sin\theta \qquad \theta \in [0, 2\pi]$

$a > c$

$dz = -a\sin\theta\,d\theta + ib\cos\theta\,d\theta$

$dz = (-a\sin\theta + ib\cos\theta)\,d\theta$



- NOW CONSIDERING THE INTEGRAL GIVEN

$\Longrightarrow \oint_C \frac{1}{z}\,dz = 2\pi i \qquad$ (ORIGIN IS INSIDE C AS AS $a > c$)

$\Longrightarrow \int_{\theta=0}^{2\pi} \frac{1}{[c + a\cos\theta] + i[b\sin\theta]}(-a\sin\theta + ib\cos\theta)\,d\theta = 2\pi i$

$\Longrightarrow \int_0^{2\pi} \frac{[-a\sin\theta + ib\cos\theta][(c + a\cos\theta) - ib\sin\theta]}{[(c + a\cos\theta) + ib\sin\theta][(c + a\cos\theta) - ib\sin\theta]}\,d\theta = 2\pi i$

$\Longrightarrow \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)[(c + a\cos\theta) - ib\sin\theta]}{(c + a\cos\theta)^2 + b^2\sin^2\theta}\,d\theta = 2\pi i$

$\Longrightarrow \int_0^{2\pi} \frac{[-a(c + a\cos\theta)\sin\theta + b^2\cos\theta\sin\theta] + i[bc\cos\theta + ab\cos^2\theta + ab\sin^2\theta]}{(c + a\cos\theta)^2 + b^2\sin^2\theta}\,d\theta$

$= 2\pi i$

- NOW SEPARATING THE INTEGRAL INTO REAL & IMAGINARY

$\int_0^{2\pi} \frac{(b^2 - a^2)\cos\theta\sin\theta - ac\sin\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\,d\theta + i\int_0^{2\pi} \frac{ab + bc\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\,d\theta = 2\pi i$

- THE REAL INTEGRAL YIELDS ZERO

LOOKING AT THE IMAGINARY INTEGRAL

$ib\int_0^{2\pi} \frac{a + c\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\,d\theta = 2\pi i$

$\int_0^{2\pi} \frac{a + c\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\,d\theta = \frac{2\pi}{b}$