IYGB GCE

Mathematics SYNF

Advanced Level

Synoptic Paper C

Difficulty Rating: 3.4325/0.5453

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Further Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

$$I = \int_0^2 3^x \; dx$$

Use the mid-ordinate rule with 4 strips of equal width to obtain an estimate for I.

All steps in the calculation must be recorded and the final answer must be correct to three significant figures. **(4)**

Question 2

The table below shows experimental data connecting two variables x and y.

x	4	6	10	12	14
y	66	36	22	20	17

It is assumed that x and y are related by an equation of the form

$$y = \frac{A}{x^2} + B,$$

where A and B are non zero constants.

By plotting accurately the equation of a suitable straight line, estimate correct to 2 significant figures the value of A and the value of B. **(8)**

Question 3

The complex number z satisfies

$$|z-4| = |z+6\mathrm{i}|.$$

Determine, as an exact simplified surd, the minimum value of $|z|$. **(8)**

Question 4

Show by using standard summation results that …

a) … $\displaystyle\sum_{r=1}^{n}(r+1)(r+5)=\frac{1}{6}n(n+7)(2n+7)$. **(5)**

b) … $\displaystyle\sum_{r=11}^{40}(r+1)(r+5)=26495$. **(4)**

Question 5

The complex number z is defined as

$$z=\mathrm{i}(1+\mathrm{i})(1-2\mathrm{i})^{2}.$$

It is further given that

$$\overline{z-3\mathrm{i}} + P(z-3\mathrm{i}) = Q\,\overline{z}$$

where P and Q are **real** constants.

Find the value of P and the value of Q. **(8)**

Question 6

Show that if x is measured in radians, the general solution of

$$\sin 2x = 1+\cos 2x,$$

is given by

$$x=\tfrac{1}{8}\pi f(n),$$

where $f(n)$ is an integer function to be found. **(8)**

Question 7

The 2×2 matrix $\mathbf{B}$ maps the points with coordinates $(-1,2)$ and $(1,4)$ onto the points with coordinates $(0,1)$ and $(6,-1)$, respectively.

a) Find the elements of $\mathbf{B}$. **(4)**

b) Determine whether $\mathbf{B}$ has an invariant line, or a line of invariant points, or both. **(7)**

c) Describe geometrically the transformation represented by $\mathbf{B}$. **(2)**

Question 8

The curve with equation $y = f(x)$ satisfies $y > 0$, for $x \in [a,b]$.

- The area of the region bounded by the curve with equation $y = f(x)$ and the x axis, for $a \le x \le b$, is denoted by A.
- The length along the curve from the point $P[a, f(a)]$ to the point $Q[b, f(b)]$, is denoted by L.

If A is **numerically equal** to L, determine the equation of the curve. **(10)**

Question 9

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 4 & 5 & 7 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}$$

Show that the system of equations $\mathbf{Ax} = \mathbf{b}$ does not have a unique solution, but for a certain value of b is consistent and its general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+4\lambda \\ \lambda \\ -1-3\lambda \end{pmatrix}$$

where λ is a parameter **(10)**

Question 10

A hyperbola is given parametrically by

$$x = \frac{3}{2}\left(t + \frac{1}{t}\right), \quad y = \frac{5}{2}\left(t - \frac{1}{t}\right), \quad t \neq 0.$$

a) Show that the Cartesian equation of the hyperbola can be written as

$$\frac{x^2}{9} - \frac{y^2}{25} = 1.$$ (6)

b) Find …

i. … the equations of its asymptotes. (2)

ii. … the coordinates of its foci. (3)

iii. … the equations of its directrices. (2)

c) Sketch the hyperbola indicating any intersections with the coordinate axes, as well as the information stated in part **(b)**. (4)

Question 11

The roots of the quadratic equation

$$x^2 + 3x + 3 = 0$$

are denoted by α and β.

Find the cubic equation, with integer coefficients, whose roots are

$$\frac{\alpha}{\beta}, \frac{\beta}{\alpha} \text{ and } \alpha\beta.$$ (7)

Question 12

The curve with equation $y = f(x)$, passes through the point $(0,1)$ and satisfies the following differential equation.

$$\frac{dy}{dx} = 3x^2 y + x^5 .$$

a) Use the approximation

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h},$$

with $h = 0.1$, to find, correct to 6 decimal places, the value of y at $x = 0.2$. **(4)**

b) Find the solution of the differential equation, and use it to obtain the value of y at $x = 0.2$. **(10)**

Question 13

$$f(w) \equiv 5\sinh w + 7\cosh w, \ w \in \mathbb{R}$$

a) Express $f(w)$ in the form $R\cosh(w + a)$, where R and a are exact constants with $R > 0$. **(7)**

b) Use the result of part **(a)** to find, in exact logarithmic form, the solutions of the following equation.

$$5\sinh w + 7\cosh w = 5 .$$ **(8)**

Question 14

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = \frac{1}{4}, \ k > 0 .$$ **(7)**

Question 15

The straight lines l_1 and l_2 have the following Cartesian equations

$$l_1: \quad x-a=\frac{y+4}{-4}=\frac{z}{-2}$$

$$l_2: \quad \frac{x-a}{2}=\frac{y+1}{-5}=\frac{z-1}{-3}$$

where a is a scalar constant.

a) Show that l_1 and l_2 intersect at for all values of a.

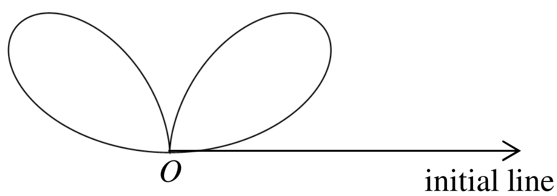
(6)

The intersection point of l_1 and l_2 has coordinates (b,b,b), where b is a scalar constant. **(2)**

b) Find the value of a and the value of b.

c) Calculate the acute angle formed by l_1 and l_2. **(3)**

Question 16



The figure above shows the polar curve C with equation

$$r=2\sin 2\theta\sqrt{\cos\theta}\,,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

Show that the area enclosed by one of the two identical loops of the curve is $\frac{16}{15}$. **(8)**

Question 17

$$z^7 - 1 = 0, \ z \in \mathbb{C}.$$

One of the roots of the above equation is denoted by ω, where $0 < \arg \omega < \frac{\pi}{3}$.

a) Find ω in the form $\omega = r\mathrm{e}^{\mathrm{i}\theta}, \quad r > 0, \quad 0 < \theta \le \frac{\pi}{3}$. **(3)**

b) Show clearly that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0.$$ **(3)**

c) Show further that

$$\omega^2 + \omega^5 = 2\cos\left(\frac{4\pi}{7}\right).$$ **(4)**

d) Hence, using the results from the previous parts, deduce that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$ **(6)**

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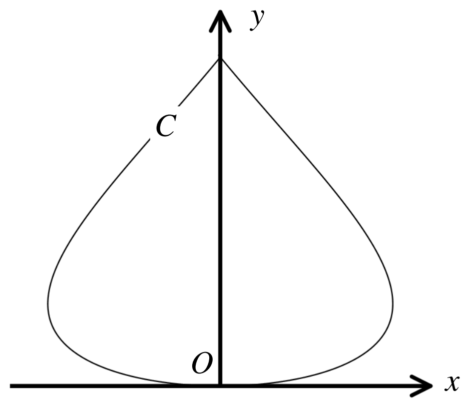
Question 18



The figure above shows the curve C with parametric equations

$$x = \sin t, \quad y = t^2, \quad 0 \le t \le 2\pi .$$

It is given that C is symmetrical about the y axis.

The region bounded by C is to be revolved about the y axis by π radians to form a solid of revolution with volume V.

By considering a suitable integral in parametric, or otherwise, find an exact value for this volume. **(10)**

Question 19

Show, without using proof by induction, that the sum of cubes of any 3 consecutive positive integers is a multiple of 9. (7)

Question 20

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx.$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\ \delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_0^1 x^2\ dx = \tfrac{1}{3}. \qquad \textbf{(10)}$$