IYGB - SYNOPTIC PAPER W - QUESTION 1

a)

$$a_{n+1} = 7a_n - n^3 - 3$$

$a_1 = 1$

$a_2 = 7a_1 - 1^3 - 3 = 7\times1 - 1 - 3 = 3$

$a_3 = 7a_2 - 2^3 - 3 = 7\times3 - 8 - 3 = 10$

$a_4 = 7a_3 - 3^3 - 3 = 7\times10 - 27 - 3 = 40$

$\underbrace{\left[\, a_5 = 7a_4 - 4^3 - 3 \,\right]}_{\text{NEEDED FOR PART (b)}} = 7\times40 - 64 - 3 = 213$

b) ADDING THE FIRST 5 TERMS

$$\sum_{r=1}^{5} a_r = a_1 + a_2 + a_3 + a_4 + a_5$$

$$= 1 + 3 + 10 + 40 + 213$$

$$= 267$$

IYGB - SYNOPTIC PAPER W - QUESTION 2

a) $f(x) = x^4 + 2x^3 + x^2 - 4$

$f(-2) = (-2)^4 + 2(-2)^3 + (-2)^2 - 4$
$= 16 - 16 + 4 - 4$
$= 0$

INDEED A FACTOR

b) LONG DIVISION OR MANIPULATION

$f(x) = x^4 + 2x^3 + x^2 - 4 = x^3(x+2) + (x-2)(x+2)$
$= (x+2)[x^3 + (x-2)]$
$= (x+2)(x^3 + x - 2)$

c) BY INSPECTING THE CUBIC $g(x) = x^3 + x - 2$

$g(1) = 0$

$\therefore$ $(x-1)$ IS ANOTHER FACTOR

d) LONG DIVISION OR MANIPULATION

$x^3 + x - 2 = x^2(x-1) + x(x-1) + 2(x-1)$
$= (x-1)(x^2 + x + 2)$

$\therefore$ $f(x) = (x-1)(x+2)(x^2 + x + 2)$

e) $f(x) = 0$

$(x-1)(x+2)(x^2+x+2) = 0$ EITHER $x = 1$

OR $x = -2$

OR $x^2 + x + 2 = 0$

Φ $b^2 - 4ac = 1^2 - 4 \times 1 \times 2 < 0$

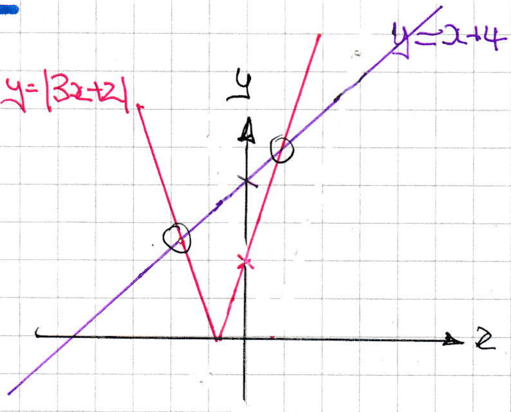
ONLY SOLUTIONS ARE $x=1$ & $x=-2$

IYGB - SYNOPTIC PAPER W - QUESTION 3

a) REWRITE THE EQUATION FIRST

$$\Rightarrow x = |3x+2| - 4$$

$$\Rightarrow x+4 = |3x+2|$$



$$\Rightarrow \begin{cases} x+4 = 3x+2 \\ x+4 = -3x-2 \end{cases}$$

$$\Rightarrow \begin{cases} 2 = 2x \\ 4x = -6 \end{cases}$$

$$\Rightarrow x = \begin{cases} 1 \\ -\frac{3}{2} \end{cases}$$

// BOTH ARE OK (SEE DIAGRAM)

b) IN A SIMILAR FASHION

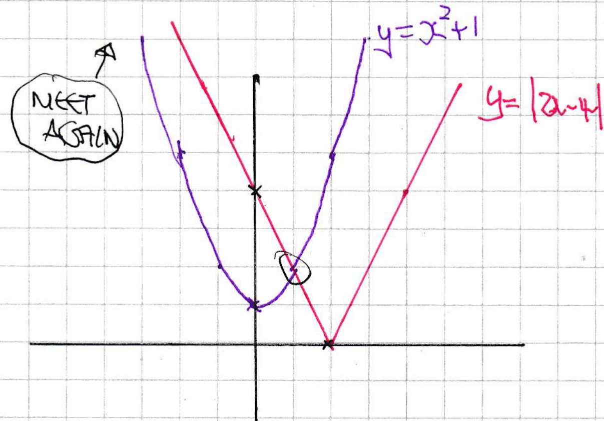
$$\Rightarrow x^2+1 = |2x-4|$$

$$\Rightarrow \begin{cases} x^2+1 = 2x-4 \\ x^2+1 = -2x+4 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 - 2x + 5 = 0 \leftarrow b^2-4ac<0 \text{ SO NO SOLUTIONS} \\ x^2+2x-3 = 0 \end{cases}$$

$$\Rightarrow (x-1)(x+3) = 0$$

$$x = \begin{cases} 1 \\ -3 \end{cases}$$



// BOTH OK

IYGB - SYNOPTIC PAPER W - QUESTION 4

a) BY PYTHAGORAS ON $\hat{OAC}$

$\Rightarrow |OC|^2 + |CA|^2 = |OA|^2$

$\Rightarrow (2x+2)^2 + (2x-1)^2 = (3x)^2$

$\Rightarrow \left(\begin{array}{l} 4x^2+8x+4 \\ 4x^2-4x+1 \end{array}\right. = 9x^2$

$\Rightarrow 8x^2+4x+5 = 9x^2$

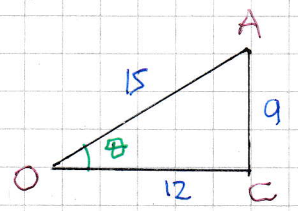
$\Rightarrow 0 = x^2-4x-5$

$\Rightarrow (x+1)(x-5)=0$

$\therefore x = \begin{cases} \cancel{-1} \\ 5 \end{cases}$

b) $|OA| = 3x = 15$ & $2x+2 = 12$, $2x-1 = 9$

"r"



- $\tan\theta = \frac{9}{12} = \frac{3}{4}$

  $\theta = 0.6435^c$

- AREA OF TRIANGLE

  $\frac{1}{2} \times 12 \times 9 = 54$

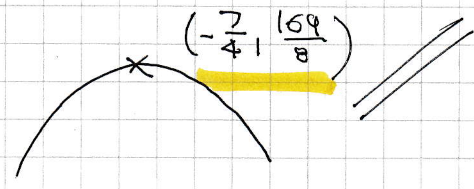
- AREA OF SECTOR

  $\frac{1}{2}r^2\theta^c = \frac{1}{2} \times 15^2 \times 0.6435... \simeq 72.393...$

$\therefore$ REQUIRED AREA $= 72.39... - 54 \approx 22.4 \text{ cm}^2$

IYGB - SYNOPTIC PAPER W - QUESTION 5

a) LOOKING AT THE "COMPLETED" THE SQUARE FORM



b) MULTIPLY OUT & TIDY

$$\Rightarrow f(x) = \frac{169}{8} - 2\left(x + \frac{7}{4}\right)^2 = \frac{169}{8} - 2\left(x^2 + 2x \times \frac{7}{4} + \frac{49}{16}\right)$$

$$\Rightarrow f(x) = \frac{169}{8} - 2\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) = \frac{169}{8} - 2x^2 - 7x - \frac{49}{8}$$

$$\Rightarrow f(x) = -2x^2 - 7x + 15$$

c) $f(x) = 0$

$$\Rightarrow \frac{169}{8} - 2\left(x + \frac{7}{4}\right)^2 = 0$$

$$\Rightarrow \frac{169}{8} = 2\left(x + \frac{7}{4}\right)^2$$

$$\Rightarrow \frac{169}{16} = \left(x + \frac{7}{4}\right)^2$$

$$\Rightarrow x + \frac{7}{4} = \begin{cases} 13/4 \\ -13/4 \end{cases}$$

$$\Rightarrow x = \begin{cases} 3/2 \\ -5 \end{cases}$$

ALTERNATIVE (BETTER)!

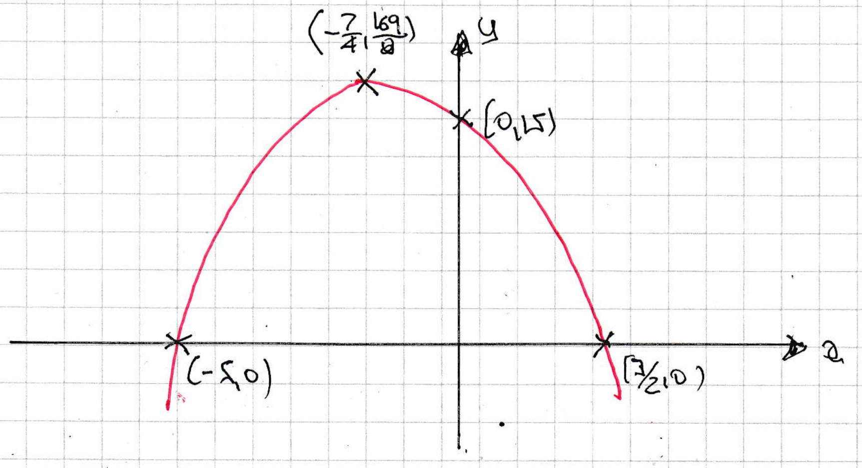
$$\Rightarrow -2x^2 - 7x + 15 = 0$$

$$\Rightarrow 2x^2 + 7x - 15 = 0$$

$$\Rightarrow (2x - 3)(x + 5) = 0$$

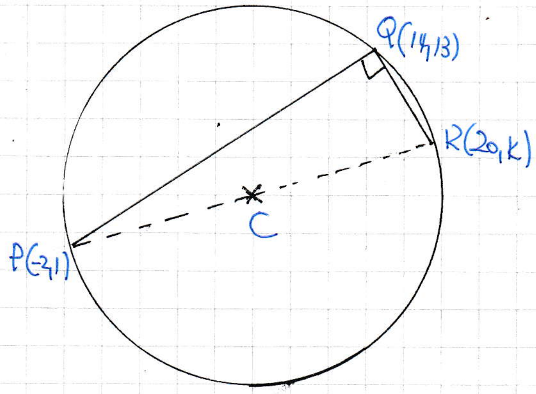
$$x = \begin{cases} 3/2 \\ -5 \end{cases}$$

d) USING PREVIOUS PARTS



IYGB – SYNOPTIC PAPER W – QUESTION 6

AS $\hat{PQR} = 90$, BY CIRCLE THEOREMS PR IS A DIAMETER – START BY FINDING THE VALUE OF k



$$\text{GRAD } PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{13-1}{14+2} = \frac{12}{16} = \frac{3}{4}$$

$$\text{GRAD } QR = \frac{y_2-y_1}{x_2-x_1} = \frac{k-13}{20-14} = \frac{k-13}{6}$$

THESE GRADIENTS MUST MULTIPLY TO -1

$$\frac{k-13}{6} \times \frac{3}{4} = -1 \implies \frac{3(k-13)}{24} = -1$$
$$\implies 3(k-13) = -24$$
$$\implies k-13 = -8$$
$$\implies k = 5$$

NEXT THE MIDPOINT OF $R(20,5)$ & $P(-2,1)$ IS C

$$C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = C\left(\frac{20-2}{2}, \frac{5+1}{2}\right) = C(9,3)$$

CENTRE

THEN THE DISTANCE PC

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(9+2)^2+(3-1)^2} = \sqrt{121+4} = \sqrt{125}$$

RADIUS

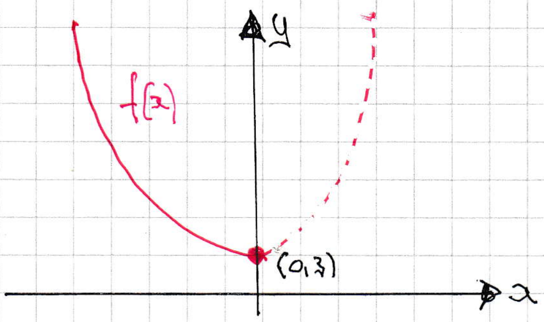
FINALLY WE HAVE

$$(x-a)^2+(y-b)^2 = r^2$$

$$(x-9)^2+(y-3)^2 = 125$$

IYGB - SYNOPTIC PAPER W - QUESTION 7

a)



b) WRITE $y = f(x)$ FOR SIMPLICITY

$\Rightarrow y = 2x^2 + 3$

$\Rightarrow y - 3 = 2x^2$

$\Rightarrow x^2 = \frac{y-3}{2}$

$\Rightarrow x = -\sqrt{\frac{y-3}{2}}$

$\therefore f^{-1}(x) = -\sqrt{\frac{x-3}{2}}$

c) USING A TWO WAY TABLE

	$f(x)$	$f^{-1}(x)$
D	$x \leq 0$	$x \geq 3$
R	$f(x) \geq 3$	$f^{-1}(x) \leq 0$

DOMAIN $x \geq 3$

RANGE $f^{-1}(x) \leq 0$

d) FINALLY WE HAVE

$\Rightarrow f^{-1}(x) = -3$

$\Rightarrow -\sqrt{\frac{x-3}{2}} = -3$

$\Rightarrow \sqrt{\frac{x-3}{2}} = 3$

$\Rightarrow \frac{x-3}{2} = 9$

$\Rightarrow x - 3 = 18$

$\therefore x = 21$

IYGB - SYNOPTIC PAPER W - QUESTION 8

a) USING A STANDARD METHOD

$$\frac{4x(9x-10)}{(2-x)(2-3x)^2} \equiv \frac{A}{2-x} + \frac{B}{2-3x} + \frac{C}{(2-3x)^2}$$

$$\boxed{4x(9x-10) \equiv A(2-3x)^2 + B(2-3x)(2-x) + C(2-x)}$$

- IF $x=2$

$8 \times 8 = 16A$

$A = 4$

- IF $x = \frac{2}{3}$

$\frac{8}{3}(6-10) = (2-\frac{2}{3})C$

$-\frac{32}{3} = \frac{4}{3}C$

$4C = -32$

$C = -8$

- IF $x=0$

$0 = 4A + 4B + 2C$

$0 = 16 + 4B - 16$

$0 = 4B$

$B = 0$

b) $f(x) = \frac{4}{2-x} - \frac{8}{(2-3x)^2}$

- $\frac{4}{2-x} = 4(2-x)^{-1} = 4 \times 2^{-1}(1-\frac{1}{2}x)^{-1} = 2(1-\frac{1}{2}x)^{-1}$

$= 2\left[1 + \frac{-1}{1}(-\frac{1}{2}x)^1 + \frac{-1(-2)}{1\times 2}(-\frac{1}{2}x)^2 + \dots\right]$

$= 2\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots\right]$

$= 2 + x + \frac{1}{2}x^2 + \dots$

- $-\frac{8}{(2-3x)^2} = -8(2-3x)^{-2} = -8 \times 2^{-2} \times (1-\frac{3}{2}x)^{-2} = -2(1-\frac{3}{2}x)^{-2}$

$= -2\left[1 + \frac{-2}{1}(-\frac{3}{2}x)^1 + \frac{-2(-3)}{1\times 2}(-\frac{3}{2}x)^2 + \dots\right]$

$= -2\left[1 + 3x + \frac{27}{4}x^2 + \dots\right]$

$= -2 - 6x - \frac{27}{2}x^2 - \dots$

ADDING EXPANSIONS

$$f(x) = \begin{pmatrix} 2 + x + \frac{1}{2}x^2 + \dots \\ -2 - 6x - \frac{27}{2}x^2 - \dots \end{pmatrix} = -5x - 13x^2 + O(x^3)$$

IYGB - SYNOPTIC PAPER W - QUESTION 8

c) SOLVING $f(x) = -0.63$ USING THE APPROXIMATION

$$\Rightarrow -5x - 13x^2 = -0.63$$

$$\Rightarrow 13x^2 + 5x = 0.63$$

$$\Rightarrow 13x^2 + 5x - 0.63 = 0$$

$$\Rightarrow 1300x^2 + 500x - 63 = 0$$

QUADRATIC FORMULA OR FACTORIZATION

$$\Rightarrow (130x + 63)(10x - 1) = 0$$

$$\Rightarrow x = \begin{cases} 0.1 \\ \cancel{-\frac{63}{130} \approx -0.4846\ldots} \end{cases}$$

IYGB - SYNOPTIC PAPER W - QUESTION 9

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y = k(4x-17) \\ y = 13-8x-x^2 \end{array}\right\} \Rightarrow k(4x-17) = 13-8x-x^2$$

$$\Rightarrow x^2+8x-13+4kx-17k = 0$$

$$\Rightarrow x^2+(8+4k)x+(-13-17k) = 0$$

NO INTERSECTIONS, NO REAL ROOTS

$$\Rightarrow b^2-4ac < 0$$

$$\Rightarrow (8+4k)^2 - 4\times1\times(-13-17k) < 0$$

$$\Rightarrow 16(2+k)^2 + 4(13+17k) < 0$$

$$\div 4$$

$$\Rightarrow 4(k+2)^2 + (13+17k) < 0$$

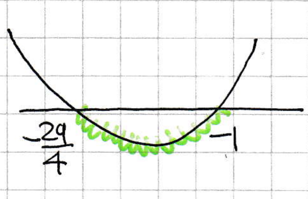
$$\Rightarrow 4(k^2+4k+4) + 17k+13 < 0$$

$$\Rightarrow 4k^2+16k+16+17k+13 < 0$$

$$\Rightarrow 4k^2+33k+29 < 0$$

FACTORIZE TO OBTAIN CRITICAL VALUES

$$\Rightarrow (4k+29)(k+1) < 0$$



$$-\frac{29}{4} < k < -1$$

IYGB - SYNOPTIC PAPER W - QUESTION 10

CHANGE EQUATIONS INTO POWERS OF 2 & 3

$$\Rightarrow 8^{y} = 4^{2x+1}$$
$$\Rightarrow (2^{3})^{y} = (2^{2})^{2x+1}$$
$$\Rightarrow 2^{3y} = 2^{4x+2}$$
$$\therefore 3y = 4x+2$$
$$\therefore 6y = 8x+4$$

$$\Rightarrow 27^{2y} = 9^{x-3}$$
$$\Rightarrow (3^{3})^{2y} = (3^{2})^{x-3}$$
$$\Rightarrow 3^{6y} = 3^{2x-6}$$
$$\therefore 6y = 2x-6$$

$$\Rightarrow 8x+4 = 2x-6$$
$$\Rightarrow 6x = -10$$
$$\Rightarrow x = -\frac{5}{3}$$

FINALLY WE HAVE

$$\Rightarrow 6y = 8x+4$$
$$\Rightarrow 6y = -\frac{40}{3}+4$$
$$\Rightarrow 18y = -40+12$$
$$\Rightarrow 18y = -28$$
$$\Rightarrow 9y = -14$$
$$\Rightarrow y = -\frac{14}{9}$$

IYGB - SYNOPTIC PAPER W - QUESTION 11

START WITH AN OBVIOUS SUBSTITUTION

- $u = \ln x$
- $\frac{du}{dx} = \frac{1}{x}$
- $dx = x\,du$
- $e^u = x$
- $dx = e^u\,du$

$$\int 3^{\ln x}\,dx = \int 3^u e^u\,du = \int (3e)^u\,du = \int a^u\,du$$

WHERE $a = 3e$

NOW WE KNOW THAT

$$\frac{d}{dx}(a^x) = a^x \ln a \implies a^x = \int a^x \ln a\,dx$$

$$\implies \frac{1}{\ln a}a^x = \int a^x\,dx$$

RETURNING TO OUR INTEGRAL IN u

$$\int a^u\,du = \frac{1}{\ln a}a^u + C = \frac{1}{\ln(3e)}(3e)^u + C$$

$$= \frac{3^u e^u}{\ln 3 + \ln e} + C = \frac{3^{\ln x} \times e^{\ln x}}{\ln 3 + 1} + C$$

$$= \frac{3^{\ln x} \times x}{1 + \ln 3} + C = \frac{x(3^{\ln x})}{1 + \ln 3} + C$$

IYGB – SYNOPTIC PAPER W – QUESTION 12

START BY FINDING THE VALUE OF A

$$\left(\tfrac{\pi}{3}, 2\right) \Rightarrow 2 \times \sec^2\tfrac{\pi}{3} = A + 2\ln\left(\sec\tfrac{\pi}{3}\right)$$

$$\Rightarrow \frac{2}{\left(\cos\frac{\pi}{3}\right)^2} = A + 2\ln\left(\frac{1}{\cos\frac{\pi}{3}}\right)$$

$$\Rightarrow \frac{2}{\frac{1}{4}} = A + 2\ln\left(\frac{1}{\frac{1}{2}}\right)$$

$$\Rightarrow 8 = A + 2\ln 2$$

$$\Rightarrow \underline{\underline{A = 8 - 2\ln 2}}$$

REWRITE WITH THE VALUE OF A FOUND & $x = \pi/6$

$$\Rightarrow \frac{y}{\cos^2 x} = 8 - 2\ln 2 + 2\ln\left(\frac{1}{\cos x}\right)$$

$$\Rightarrow \frac{y}{\left(\cos\frac{\pi}{6}\right)^2} = 8 - 2\ln 2 + 2\ln\left(\frac{1}{\cos\frac{\pi}{6}}\right)$$

$$\Rightarrow \frac{y}{3/4} = 8 - 2\ln 2 + 2\ln\left(\frac{1}{\sqrt{3}/2}\right)$$

$$\Rightarrow \frac{4}{3}y = 8 - 2\ln 2 + 2\ln\frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{4}{3}y = 8 - 2\ln 2 + 2\left[\ln 2 - \ln\sqrt{3}\right]$$

$$\Rightarrow \frac{4}{3}y = 8 - \cancel{2\ln 2} + \cancel{2\ln 2} - 2\ln\sqrt{3}$$

$$\Rightarrow \frac{4}{3}y = 8 - 2\ln 3^{\frac{1}{2}}$$

$$\Rightarrow \frac{4}{3}y = 8 - \ln 3$$

$$\Rightarrow y = \frac{3}{4}(8 - \ln 3)$$

AS REQUIRED

IYGB - SYNOPTIC PAPER W - QUESTION 13

a) REWRITE THE L.H.S. AND "EXPAND"

$$
\begin{aligned}
L.H.S &= \left[\sin\left(\theta+\tfrac{\pi}{4}\right)\right]^2 - \left[\sin\left(\theta-\tfrac{\pi}{4}\right)\right]^2 \\
&= \left(\sin\theta\cos\tfrac{\pi}{4}+\cos\theta\sin\tfrac{\pi}{4}\right)^2 - \left(\sin\theta\cos\tfrac{\pi}{4}-\cos\theta\sin\tfrac{\pi}{4}\right)^2 \\
&= \left(\tfrac{\sqrt{2}}{2}\sin\theta+\tfrac{\sqrt{2}}{2}\cos\theta\right)^2 - \left(\tfrac{\sqrt{2}}{2}\sin\theta-\tfrac{\sqrt{2}}{2}\cos\theta\right)^2 \\
&= \left(\cancel{\tfrac{1}{2}\sin^2\theta}+\sin\theta\cos\theta+\cancel{\tfrac{1}{2}\cos^2\theta}\right) - \left(\cancel{\tfrac{1}{2}\sin^2\theta}-\sin\theta\cos\theta+\cancel{\tfrac{1}{2}\cos^2\theta}\right) \\
&= 2\sin\theta\cos\theta \\
&= \sin 2\theta \\
&= RHS
\end{aligned}
$$

ALTERNATIVE AS DIFFERENCE OF SQUARES

$$
\begin{aligned}
LHS &= \left[\sin\left(\theta+\tfrac{\pi}{4}\right)\right]^2 - \left[\sin\left(\theta-\tfrac{\pi}{4}\right)\right]^2 \\
&= \left[\sin\left(\theta+\tfrac{\pi}{4}\right)+\sin\left(\theta-\tfrac{\pi}{4}\right)\right]\left[\sin\left(\theta+\tfrac{\pi}{4}\right)-\sin\left(\theta-\tfrac{\pi}{4}\right)\right] \\
&= \left[\sin\theta\cos\tfrac{\pi}{4}+\cancel{\cos\theta\sin\tfrac{\pi}{4}}+\sin\theta\cos\tfrac{\pi}{4}-\cancel{\cos\theta\sin\tfrac{\pi}{4}}\right] \\
&\quad\times \\
&\quad\left[\cancel{\sin\theta\cos\tfrac{\pi}{4}}+\cos\theta\sin\tfrac{\pi}{4}-\cancel{\sin\theta\cos\tfrac{\pi}{4}}+\cos\theta\sin\tfrac{\pi}{4}\right] \\
&= 2\sin\theta\cos\tfrac{\pi}{4} \times 2\cos\theta\sin\tfrac{\pi}{4} \\
&= \sqrt{2}\sin\theta \times \sqrt{2}\cos\theta \\
&= 2\sin\theta\cos\theta \\
&= \sin 2\theta \\
&= RHS
\end{aligned}
$$

IYGB - SYNOPTIC PAPER W - QUESTION 13

b) I) USE PART (a) WITH DIFFERENTIATION

$$\Rightarrow \sin 2\theta \equiv \sin^2\left(\theta+\frac{\pi}{4}\right) - \sin^2\left(\theta-\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{d}{d\theta}\left[\sin 2\theta\right] \equiv \frac{d}{d\theta}\left[\sin^2\left(\theta+\frac{\pi}{4}\right) - \sin^2\left(\theta-\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \cancel{2}\cos 2\theta \equiv \cancel{2}\sin\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta+\frac{\pi}{4}\right) - \cancel{2}\sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)$$

$$\Rightarrow \cos 2\theta \equiv \sin\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta+\frac{\pi}{4}\right) - \sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)$$

AS REQUIRED

ALTERNATIVE

$$\text{L.H.S.} = \sin\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta+\frac{\pi}{4}\right) - \sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left[2\sin\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta+\frac{\pi}{4}\right) - 2\sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\sin\left(2\theta+\frac{\pi}{2}\right) - \sin\left(2\theta-\frac{\pi}{2}\right)\right]$$

USED $2\sin A\cos A \equiv \sin 2A$
WITH $A = \theta+\frac{\pi}{4}$ & $A = \theta-\frac{\pi}{4}$

$$= \frac{1}{2}\left[\cancel{\sin 2\theta\cos\frac{\pi}{2}} + \cos 2\theta\sin\frac{\pi}{2} - \cancel{\sin 2\theta\cos\frac{\pi}{2}} + \cos 2\theta\sin\frac{\pi}{2}\right]$$

$$= \frac{1}{2} \times 2\cos 2\theta\sin\frac{\pi}{2}$$

$$= \cos 2\theta$$

$$= \text{R.H.S}$$

b) II) LET $\theta = \frac{\pi}{6}$ IN PART b) I)

$$\Rightarrow \sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}-\frac{\pi}{4}\right) = \cos\left(2\times\frac{\pi}{6}\right)$$

$$\Rightarrow \sin\frac{5\pi}{6}\cos\frac{5\pi}{6} - \sin\left(-\frac{\pi}{12}\right)\cos\left(-\frac{\pi}{12}\right) = \cos\frac{\pi}{3}$$

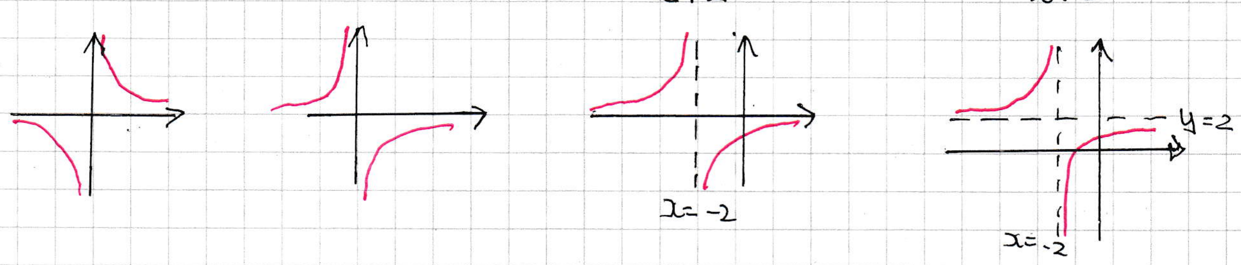
$\sin(-A) \equiv \sin A$ & $\cos(-A) \equiv \cos A$

$$\therefore \sin\frac{5\pi}{6}\cos\frac{5\pi}{6} + \sin\frac{\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}$$

IYGB – SYNOPTIC PAPER N – QUESTION 14

a) WORKING AS FOLLOWS

$$\frac{1}{x} \xmapsto{T_1} -\frac{1}{x} \xmapsto{T_2} -\frac{1}{x+2} \xmapsto{T_3} -\frac{1}{x+2}+2$$

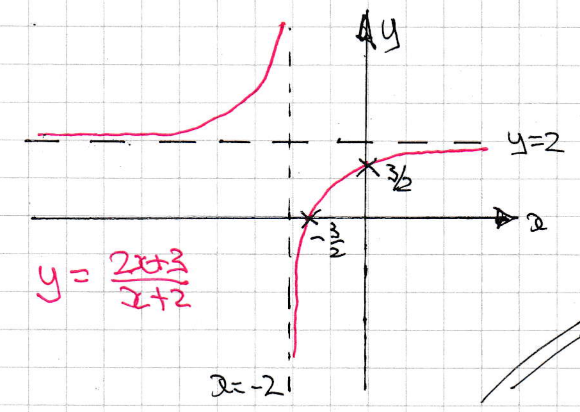


THUS WE HAVE

$$y_2 = 2 - \frac{1}{x+2} = \frac{2(x+2)-1}{x+2} = \frac{2x+4-1}{x+2} = \frac{2x+3}{x+2}$$

AS REQUIRED

b) LOOKING AT THE LAST GRAPH



- $x=0 \quad y=\frac{3}{2}$
- $y=0 \quad x=-\frac{3}{2}$

c) USE THE EXPRESSION FROM PART (a)

$$\frac{2x+3}{x+2} = 2 + \frac{2}{x-1}$$

$$\Rightarrow \cancel{2} - \frac{1}{x+2} = \cancel{2} + \frac{2}{x-1}$$

$$\Rightarrow -(x-1) = 2(x+2)$$

$$\Rightarrow -x+1 = 2x+4$$

$$\Rightarrow -3 = 3x$$

$$\therefore x = -1$$

IYGB – SYNOPTIC PAPER W – QUESTION 15

PREPARE THE DERIVATIVES – PRODUCT RULE

- $y = (x+2)^2 e^{1-x}$

- $\frac{dy}{dx} = 2(x+2)e^{1-x} + (x+2)^2 e^{1-x}(-1)$

$$= 2(x+2)e^{1-x} - (x+2)^2 e^{1-x}$$

$$= e^{1-x}\left(2(x+2) - (x+2)^2\right)$$

← AVOIDING FACTORIZING $(x+2)$ TO AVOID TRIPLE PRODUCT

$$= e^{1-x}\left(2x + \cancel{4} - x^2 - 4x - \cancel{4}\right)$$

$$= e^{1-x}\left(-x^2 - 2x\right)$$

$$\therefore \frac{dy}{dx} = -(x^2+2x)e^{1-x}$$

- $\frac{d^2y}{dx^2} = -(2x+2)e^{1-x} - (x^2+2x)e^{1-x}(-1)$

$$= -(2x+2)e^{1-x} + (x^2+2x)e^{1-x}$$

$$= e^{1-x}\left(\cancel{-2x} - 2 + x^2 + \cancel{2x}\right)$$

$$\therefore \frac{d^2y}{dx^2} = (x^2-2)e^{1-x}$$

SUBSTITUTE AND VERIFY

$$(x+2)^2\frac{d^2y}{dx^2} + x(x+2)\frac{dy}{dx} + 2y$$

$$= (x-2)^2(x^2-2)e^{1-x} + (x^2+2x)\left[-(x^2+2x)\right]e^{1-x} + 2(x+2)^2e^{1-x}$$

$$= (x-2)^2(x^2-2)e^{1-x} - x^2(x+2)^2e^{1-x} + 2(x+2)^2e^{1-x}$$

$$= (x+2)^2e^{1-x}\left[(x^2-2) - x^2 + 2\right]$$

$$= (x+2)^2e^{1-x}\left[\cancel{x^2 - 2 - x^2 + 2}\right]$$

$$= 0$$

// AS REQUIRED

IYGB – SYNOPTIC PAPER W – QUESTION 16

a) PRODUCT RULE

$$y = (9x)(\sin 3x)$$

$$\frac{dy}{dx} = 9\sin 3x + 9x(3\cos 3x)$$

$$\frac{dy}{dx} = 9(\sin 3x + 3x\cos 3x)$$

WHEN $x = \frac{\pi}{6}$

$$y = \left(9 \times \frac{\pi}{6}\right)\sin\frac{\pi}{2}$$

$$y = \frac{3\pi}{2}$$

$$P\left(\frac{\pi}{6}, \frac{3\pi}{2}\right)$$

$\frac{dy}{dx}$ AT $x = \frac{\pi}{6}$ (TANGENT GRAD)

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = 9\left(\sin\frac{\pi}{2} + \cancel{3 \times \frac{\pi}{6} \times \cos\frac{\pi}{2}}\right)$$

TANG GRAD $= 9$

FINALLY WE HAVE

$$y - y_0 = m(x - x_0)$$

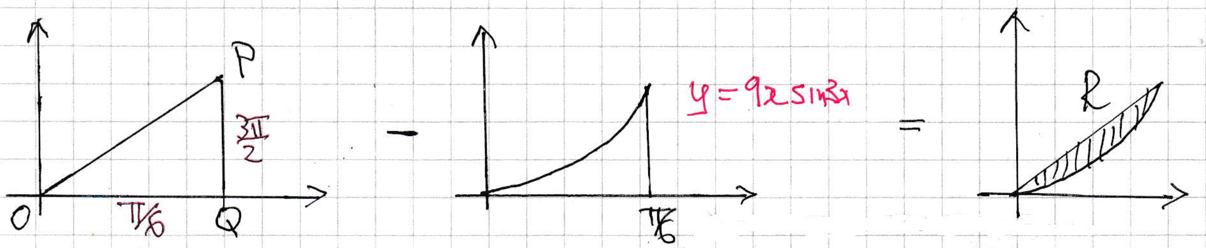
$$y - \frac{3\pi}{2} = 9\left(x - \frac{\pi}{6}\right)$$

$$y - \cancel{\frac{3\pi}{2}} = 9x - \cancel{\frac{3\pi}{2}}$$

$\therefore\ y = 9x$

PASSES THROUGH O

b) LOOKING AT THE PICTORIAL EQUATION



$$\frac{1}{2} \times \frac{\pi}{6} \times \frac{3\pi}{2} = \frac{\pi^2}{8}$$

$$\int_0^{\pi/6} 9x\sin 3x \, dx$$

IYGB - SYNOPTIC PAPER W - QUESTION 16

INTEGRATION BY PARTS

$9x$	9
$-\frac{1}{3}\cos 3x$	$\sin 3x$

$$\int 9x\sin 3x \, dx = -3x\cos 3x - \int -\frac{1}{3}\cos 3x \times 9 \, dx$$

$$= -3x\cos 3x + \int 3\cos 3x \, dx$$

$$= -3x\cos 3x + \sin 3x + C$$

INSERTING LIMITS 0 to $\pi/6$

$$\int_0^{\pi/6} 9x\sin 3x \, dx = \Big[\sin 3x - 3x\cos 3x\Big]_0^{\pi/6}$$

$$= (1 - 3\times 0) - (0 - 0)$$

$$= 1$$

$\therefore$ REQUIRED AREA $= \frac{\pi^2}{8} - 1$

$$= \frac{1}{8}(\pi^2 - 8)$$

AS REQUIRED

IYGB - SYNOPTIC PAPER W - QUESTION 17

USING THE TAN COMPOUND IDENTITY

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \tan(\overset{A}{\arctan x} + \overset{B}{\arctan y}) = \tan(\arctan 8)$$

$$\Rightarrow \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = 8$$

$$\Rightarrow \frac{x+y}{1-xy} = 8$$

$$\Rightarrow \frac{2}{1-xy} = 8 \qquad x+y=2$$

$$\Rightarrow \frac{1}{4} = 1 - xy$$

$$\Rightarrow xy = \frac{3}{4}$$

COMBINE WITH $x+y=2$

$$\Rightarrow xy + y^2 = 2y$$

$$\Rightarrow \frac{3}{4} + y^2 = 2y$$

$$\Rightarrow y^2 - 2y + \frac{3}{4} = 0$$

$$\Rightarrow 4y^2 - 8y + 3 = 0$$

$$\Rightarrow (2y-3)(2y-1)$$

$$\Rightarrow y = \begin{cases} 3/2 \\ 1/2 \end{cases}$$

$\therefore \frac{3}{2}$ & $\frac{1}{2}$

(EITHER ORDER)

IYGB - SYNOPTIC PAPER W - QUESTION 18

a) FIRSTLY USING $y = 5/2$

$\Rightarrow \frac{5}{2} = 5\sin t$

$\Rightarrow \sin t = \frac{1}{2}$

$\Rightarrow t = \begin{cases} \pi/6, \cancel{5\pi/6}, \ldots \\ \cancel{-\frac{11\pi}{6}}, \cancel{-\frac{7\pi}{6}}, \ldots \end{cases}$

GRADIENT NEXT

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-4\sin 2t} = \frac{5\cancel{\cos t}}{-8\sin t\cancel{\cos t}} = -\frac{5}{8\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = -\frac{5}{8 \times \sin\frac{\pi}{6}} = -\frac{5}{8 \times \frac{1}{2}} = -\frac{5}{4}$$

NORMAL EQUATION

$$y - y_0 = m(x - x_0)$$

$$y - \frac{5}{2} = +\frac{4}{5}(x - 1)$$

$$y - \frac{5}{2} = \frac{4}{5}x - \frac{4}{5}$$

$$10y - 25 = 8x - 8$$

$\therefore$ $8x - 10y + 17 = 0$ // AS REQUIRED

b) SOLVING SIMULTANEOUSLY LINE & CURVE

$\Rightarrow 8x - 10y + 17 = 0$

$\Rightarrow 8(2\cos 2t) - 10(5\sin t) + 17 = 0$

$\Rightarrow 16\cos 2t - 50\sin t + 17 = 0$

$\Rightarrow 16(1 - 2\sin^2 t) - 50\sin t + 17 = 0$

$\Rightarrow 16 - 32\sin^2 t - 50\sin t + 17 = 0$

$\Rightarrow 0 = 32\sin^2 t + 50\sin t - 33$

IYGB - SYNOPTIC PAPER W - QUESTION 18

NOW $\sin t = \frac{1}{2}$ MUST BE A SOLUTION (POINT P)

$$\Rightarrow (2\sin t - 1)(16\sin t + 33) = 0$$

$$\Rightarrow \sin t = \begin{cases} \frac{1}{2} & \leftarrow P \\ -\frac{33}{16} & \leftarrow Q \end{cases}$$

$\therefore$ AT Q $y = 5\sin t = 5 \times \frac{-33}{16} = -\frac{165}{16}$

AS REQUIRED

ALTERNATIVE IN CARTESIAN

$$\left.\begin{array}{l} x = 2\cos 2t \\ y = 5\sin t \end{array}\right\} \Rightarrow \left.\begin{array}{l} 8x = 16(1 - 2\sin^2 t) \\ \sin t = \frac{y}{5} \end{array}\right\} \Rightarrow \begin{array}{l} 8x = 16 - 32\sin^2 t \\ \sin^2 t = \frac{y^2}{25} \end{array}$$

$\therefore$ CURVE HAS EQUATION

$$8x = 16 - 32 \times \frac{y^2}{25}$$

$$8x = 16 - \frac{32}{25}y^2$$

NORMAL HAS EQUATION $8x - 10y + 17 = 0$

$8x = 10y - 17$

$$(10y - 17) = 16 - \frac{32}{25}y^2$$

$$\frac{32}{25}y^2 + 10y - 33 = 0$$

$$32y^2 + 250y - 825 = 0$$

BUT $y = \frac{5}{2}$ IS A SOLUTION, POINT P

$$(2y - 5)(16y + 165) = 0$$

$$\therefore y = -\frac{165}{16}$$

AS BEFORE

IYGB – SYNOPTIC PAPER W – QUESTION 19

SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{dH}{dt} = -ke^{-0.1t}$$

$$\Rightarrow 1\,dH = -ke^{-0.1t}$$

$$\Rightarrow \int 1\,dH = \int -ke^{-0.1t}\,dt$$

$$\Rightarrow H = 10ke^{-0.1t} + C$$

$$\Rightarrow H = Ae^{-0.1t} + B$$

WHEN $t=0$, $H=3$

$$\Rightarrow 3 = A + B$$

WHEN $t=10$, $H=2$

$$\Rightarrow 2 = Ae^{-1} + B$$

SOLVING TO FIND A & B

$$\Rightarrow 1 = A - Ae^{-1}$$

$\times e$

$$\Rightarrow e = Ae - A$$

$$\Rightarrow e = A(e-1)$$

$$\Rightarrow A = \frac{e}{e-1}$$

$$\Rightarrow B = 3 - A = 3 - \frac{e}{e-1}$$

$$\Rightarrow B = \frac{3e - 3 - e}{e-1}$$

$$\Rightarrow B = \frac{2e-3}{e-1}$$

IYGB - SYNOPTIC PAPER W - QUESTION 19

HENCE THE SOLUTION OF THE O.D.E BECOMES

$$H = \frac{e}{e-1} e^{-0.1t} + \frac{2e-3}{e-1}$$

$$H = \frac{e^{1-0.1t} + 2e - 3}{e-1}$$

FINALLY WE HAVE

$$\lim_{t \to \infty} H = \frac{2e-3}{e-1} \qquad \left(\text{AS } e^{-0.1t} \to 0\right)$$

IYGB - SYNOPTIC PAPER W - QUESTION 20

START WITH AN EXPRESSION FOR THE VOLUME OF THE COMPOSITE

$$V = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right) + \pi R^2 H$$

$$V = \frac{2}{3}\pi R^3 + \pi R^2 H$$

NEXT AN EXPRESSION FOR THE VARIABLE SURFACE AREA

$$A = \pi R^2 + 2\pi R H + \frac{1}{2}(4\pi R^2)$$

$$A = \pi R^2 + 2\pi R H + 2\pi R^2$$

$$A = 3\pi R^2 + 2\pi R H$$

MANIPULATE THE VOLUME EXPRESSION AS FOLLOWS

$$V = \frac{2}{3}\pi R^3 + \pi R^2 H \quad \times 2$$

$$2V = \frac{4}{3}\pi R^3 + 2\pi R^2 H \quad \div R$$

$$\frac{2V}{R} = \frac{4}{3}\pi R^2 + 2\pi R H$$

$$2\pi R H = \frac{2V}{R} - \frac{4}{3}\pi R^2$$

FINALLY SUBSTITUTE THE ABOVE INTO THE SURFACE AREA EXPRESSION

$$A = 3\pi R^2 + 2\pi R H$$

$$A = 3\pi R^2 + \frac{2V}{R} - \frac{4}{3}\pi R^2$$

$$A = \frac{5}{3}\pi R^2 + \frac{2V}{R}$$

(NOTE THAT R MAY VARY BUT V IS A CONSTANT)

IYGB – SYNOPTIC PAPER W – QUESTION 20

NEXT DIFFERENTIATE A W.R.T R & SOLVE FOR ZERO

$$\frac{dA}{dR} = \frac{10}{3}\pi R - \frac{2V}{R^2}$$

$$0 = \frac{10}{3}\pi R - \frac{2V}{R^2}$$

$$\frac{10}{3}\pi R = \frac{2V}{R^2}$$

$$10\pi R^3 = 6V$$

$$R^3 = \frac{3V}{5\pi} \quad \text{IE} \quad R = \left(\frac{3V}{5V}\right)^{\frac{1}{3}}$$

JUSTIFY THE NATURE

$$\frac{d^2A}{dR^2} = \frac{10}{3}\pi + \frac{4V}{R^3}$$

$$\left.\frac{d^2A}{dR^2}\right|_{R^3 = \frac{3V}{5\pi}} = \frac{10}{3}\pi + \frac{4V}{\frac{3V}{5\pi}} = \frac{10}{3}\pi + \frac{20}{3}\pi = 10\pi > 0$$

INDEED MINIMIZES

NOW RETURNING TO THE CONSTRAINT

WITH $R^3 = \frac{3V}{5\pi}$

$$2\pi RH = \frac{2V}{R} - \frac{4}{3}\pi R^2$$

$$2\pi R^2 H = 2V - \frac{4}{3}\pi R^3$$

$$6\pi R^2 H = 6V - 4\pi R^3$$

$$3\pi R^2 H = 3V - 2\pi R^3$$

NOW $V = \frac{5\pi R^3}{3}$

$$3\pi R^2 H = 3\left(\frac{5\pi R^3}{3}\right) - 2\pi R^3$$

$$3\pi R^2 H = 5\pi R^3 - 2\pi R^3$$

$$3\pi R^2 H = 3\pi R^3$$

$$H = R$$

AS REQUIRED

MYGB - SYNOPTIC PAPER N - QUESTION 20

FINALLY TO FIND THE MINIMUM SURFACE AREA

$$A = \frac{5}{3}\pi R^2 + \frac{2V}{R} = \frac{1}{R}\left[\frac{5}{3}\pi R^3 + 2V\right]$$

$$A_{MIN} = \frac{1}{R}\left[\frac{5}{3}\pi\left(\frac{3V}{5\pi}\right) + 2V\right]$$

$$A_{MIN} = \frac{1}{R}\left[V + 2V\right]$$

$$A_{MIN} = \frac{3V}{R}$$

$$A_{MIN} = 3V \times R^{-1}$$

$$A_{MIN} = 3V \times \left(\frac{5\pi}{3V}\right)^{\frac{1}{3}}$$

$$A_{MIN} = \left(27V^3\right)^{\frac{1}{3}} \times \left(\frac{5\pi}{3V}\right)^{\frac{1}{3}}$$

$$A_{MIN} = \left(27V^3 \times \frac{5\pi}{3V}\right)^{\frac{1}{3}}$$

$$A_{MIN} = \left(45\pi V^2\right)^{\frac{1}{3}}$$

$$A_{MIN} = \sqrt[3]{45\pi V^2}$$

AS REQUIRED

IYGB - SYNOPTIC PAPER W - QUESTION 21

LOOKING AT THE VOLUME WHICH IS STRICTLY INCREASING FOR $h \geq 6$

$\Rightarrow h = 6 \qquad V = \pi(36 + 30 - 16) = 50\pi$ ← ALREADY IN THE CONTAINER

NOW $\frac{dV}{dt} = 2\pi$ = CONSTANT

$\Rightarrow 2\pi \text{ cm}^3$ EVERY SECOND

$\Rightarrow 2\pi \times 30 \text{ cm}^3$ AFTER 30 SECONDS

$\Rightarrow 60\pi \text{ cm}^3$ EXTRA GONE IN

TOTAL LIQUID AFTER 30 SECONDS MUST BE $60\pi + 50\pi = 110\pi$

USING $V = \pi[h^2 + 5h - 16]$ WITH $V = 110\pi$ TO FIND h

$$\Rightarrow 110\pi = \pi(h^2 + 5h - 16)$$
$$\Rightarrow 110 = h^2 + 5h - 16$$
$$\Rightarrow h^2 + 5h - 126 = 0$$
$$\Rightarrow (h + 14)(h - 9)$$
$$\Rightarrow h = \begin{cases} \cancel{-14} \\ 9 \end{cases}$$

FINALLY WE HAVE

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi(2h+5)} \times 2\pi$$

$$\frac{dh}{dt} = \frac{2}{2h+5}$$

$$V = \pi(h^2 + 5h - 16)$$
$$\frac{dV}{dh} = \pi[2h + 5]$$
$$\frac{dh}{dV} = \frac{1}{\pi(2h+5)}$$

$$\left.\frac{dh}{dt}\right|_{t=30} = \left.\frac{dh}{dt}\right|_{h=9} = \frac{2}{2 \times 9 + 5} = \frac{2}{23} \approx 0.087 \text{ cm s}^{-1}$$

IYGB - SYNOPTIC PAPER W - QUESTION 22

STARTING WITH THE "GIVENS"

$$S_\infty = \frac{a}{1-r} \qquad 0<r<1 \qquad a>0 \qquad u_2 = ar$$

NOW CONSIDER THE "RATIO" BELOW

$$\frac{S_\infty}{u_2} = \frac{\frac{a}{1-r}}{ar} = \frac{a}{ar(1-r)} = \frac{1}{r(1-r)}$$

PARTIAL FRACTIONS (BY COVER UP)

$$\frac{S_\infty}{u_2} = \frac{1}{r} + \frac{1}{1-r} = \frac{1}{r} - \frac{1}{r-1}$$

NOW CONSIDER A NEW FUNCTION & CALCULUS

$$\frac{S_\infty}{u_2} = f(r) = \frac{1}{r} - \frac{1}{r-1}$$

$$f'(r) = -\frac{1}{r^2} + \frac{1}{(r-1)^2}$$

$$f''(r) = \frac{2}{r^3} - \frac{2}{(r-1)^3}$$

LOOK FOR STATIONARY POINTS

$$\Rightarrow -\frac{1}{r^2} + \frac{1}{(r-1)^2} = 0$$

$$\Rightarrow \frac{1}{(r-1)^2} = \frac{1}{r^2}$$

$$\Rightarrow (r-1)^2 = r^2$$

$$\Rightarrow r^2 - 2r + 1 = r^2$$

$$\Rightarrow 1 = 2r$$

$$\Rightarrow r = \frac{1}{2}$$

IYGB - SYNOPTIC PAPER W - QUESTION 22

HENCE WE HAVE

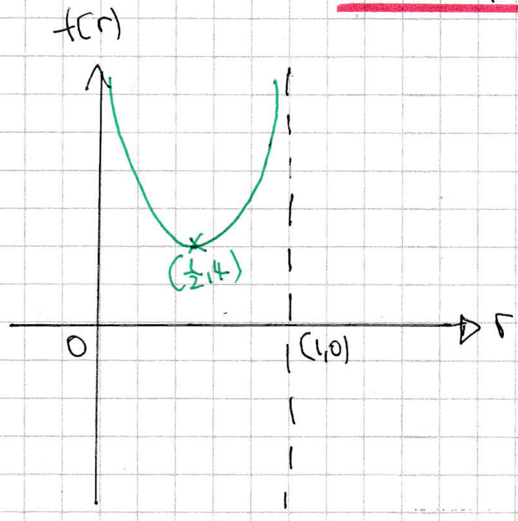
$$f\left(\tfrac{1}{2}\right) = \frac{S_\infty}{u_2} = \frac{1}{\frac{1}{2}\left(1-\frac{1}{2}\right)} = \frac{1}{\frac{1}{2}\times\frac{1}{2}} = 4$$

$$f''\left(\tfrac{1}{2}\right) = \frac{2}{1/8} - \frac{2}{-1/8} = 16+16 = 32 > 0$$

IE LOCAL MINIMUM

NEED A SKETCH OF $f(r)$ FOR $0<r<1$

OR SOME OTHER ARGUMENT



$f(0) = f(1) = +\infty$

$$\therefore \quad f(r) \geqslant 4 \qquad 0<r<1$$

$$\frac{S_\infty}{u_2} \geqslant 4$$

$$S_\infty \geqslant 4u_2$$

AS REQUIRED