IYGB

Special Extension Paper H

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

It is given that a curve with equation $y = f(x)$ passes through the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ and satisfies the differential equation

$$\left(\frac{dy}{dx} - \sqrt{\tan x}\right)\sin 2x = y .$$

Find an equation for the curve in the form $y = f(x)$. **(8)**

Question 2

A hyperbola H has Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 ,$$

where a and b are positive constants.

The straight line T_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$.

T_1 meets the x axis at the point P.

The straight line T_2 is a tangent to the hyperbola at the point $(a, 0)$.

T_1 and T_2 meet each other at the point Q.

Given further that M is the midpoint of PQ, show that as θ varies, the locus of M traces the curve with equation

$$x\left(4y^2 + b^2\right) = ab^2 .$$ **(10)**

Question 3

The function f is defined below.

$$f(x) \equiv \ln\left[\sin x + \sqrt{2-\cos^2 x}\right], \ x \in \mathbb{R}.$$

Prove that f is odd. **(7)**

Question 4

$$z^4 - 2z^3 + z - 20 = 0, \ z \in \mathbb{C}.$$

By using the substitution $w = z^2 - z$, or otherwise, find in exact form the four solutions of the above equation. **(7)**

Question 5

It is given that

$$y = \arcsin\left[\frac{\alpha + \cos x}{1 + \alpha\cos x}\right],$$

where α is a constant.

Show that

$$\frac{dy}{dx} = -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}.$$ **(8)**

Question 6

A quadratic curve has equation

$$f(x) \equiv 9x^2 + 3(1-8a)x + 4a(4a-1), \; x \in \mathbb{R},$$

where a is a constant.

a) Express $f(x)$ as the product of two linear factors. **(4)**

b) Solve the equation $f(x) = 2$, giving the answers in terms of a. **(8)**

Question 7

Show clearly that the general solution of the equation

$$\sin z = 2, \; z \in \mathbb{C},$$

can be written in the form

$$z = \frac{\pi}{2}(4k+1) \pm \mathrm{i}\,\mathrm{arcosh}\,2, \; k \in \mathbb{Z}.$$ **(8)**

Question 8

Prove by induction that for $n \geq 1$, $n \in \mathbb{N}$

$$\prod_{r=1}^{n} \left(\cos\left(2^{r-1}x\right) \right) = \frac{\sin\left(2^n x\right)}{2^n \sin x}.$$ **(7)**

Question 9

Two curves, C_1 and C_2, have polar equations

$$C_1: \; r = 12\cos\theta, \; -\frac{\pi}{2} < \theta \le \frac{\pi}{2}$$

$$C_2: \; r = 4 + 4\cos\theta, \; -\pi < \theta \le \pi.$$

One of the points of intersection between the graphs of C_1 and C_2 is denoted by A. The area of the **smallest** of the two regions bounded by C_1 and the straight line segment OA is

$$6\pi - 9\sqrt{3}.$$

The finite region R represents points which lie inside C_1 but outside C_2.

Show that the area of R is 16π. **(10)**

Question 10

$$\frac{dy}{dx} = \frac{2x+5y+3}{4x+y-3}, \; y(1) = 1.$$

a) Show that the transformation equations

$$x = X + 1$$
$$y = Y - 1$$ **(3)**

transform the above differential equation to

$$\frac{dY}{dX} = \frac{2X+5Y}{4X+Y}.$$

b) Use the substitution $Y = XV$, where $V = f(x)$, to show that the solution of the original differential equation is

$$(y - 2x + 3)^2 = 2(x+y).$$ **(9)**

Question 11

Use the substitution $x = \tan\left(\frac{1}{2}\theta\right)$, to find a simplified expression for

$$\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx. \qquad \mathbf{(10)}$$

Question 12

Sketch the curve with equation

$$y = \frac{x+1}{|x-1|}, \ x \in \mathbb{R}, \ x \neq 1.$$

The sketch must include …

- … the coordinates of all the points where the curve meets the coordinate axes.
- … the equations of the asymptotes of the curve. **(8)**

Question 13

The point M is the midpoint of AB, on a triangle ABC.

Given further that

$$\tan\left[\measuredangle CAM\right] = \frac{2}{5} \quad \text{and} \quad \tan\left[\measuredangle CBM\right] = \frac{2}{3},$$

Use trigonometric identities to find the value of $\tan\left[\measuredangle CMB\right]$. **(10)**

Question 14

It is given that

$$y = 2\ln\left(e^x + 1\right) - \ln\left(e^x - 1\right), \ x \in \mathbb{R}.$$

Express x in terms of y. **(10)**

Question 15

$$f(x,n) = \sum_{r=1}^{n}\left[\frac{1}{(x-1)^r}\right], \quad x \in \mathbb{R}, \ n \in \mathbb{N}.$$

By observing the simplification of

$$\frac{1}{(x-2)(x-1)^r} - \frac{1}{(x-2)(x-1)^{r+1}}$$

find a simplified expression for $f(x,n)$. **(7)**

Question 16

A curve has parametric equations

$$x = 2 + \tanh t, \quad y = \operatorname{sech} t, \ t \in \mathbb{R}$$

The part of the curve for which

$$0 \le t \le \ln\left[\frac{\sqrt{2}+\sqrt{6}}{2}\right],$$

is rotated through 2π radians in the x axis.

Show that the exact area of the surface generated is

$$\frac{1}{6}\pi\left[4 + 3\sqrt{3}\right].$$ **(12)**

Question 17

The radius of curvature ρ at any point on a curve with Cartesian equation $y = f(x)$ is given by

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

a) Given that the curve can be parameterized as $x = g(t)$, $y = h(t)$, for some parameter t, show that

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}},$$

where a dot above a variable denoted differentiation with respect to t. **(8)**

A curve C is given parametrically by

$$x = \cos t + t\sin t,\ \ y = \sin t - t\cos t,\ \ 0 \le t < 2\pi .$$

b) Find an expression for ρ on C, giving the answer in terms of t. **(6)**

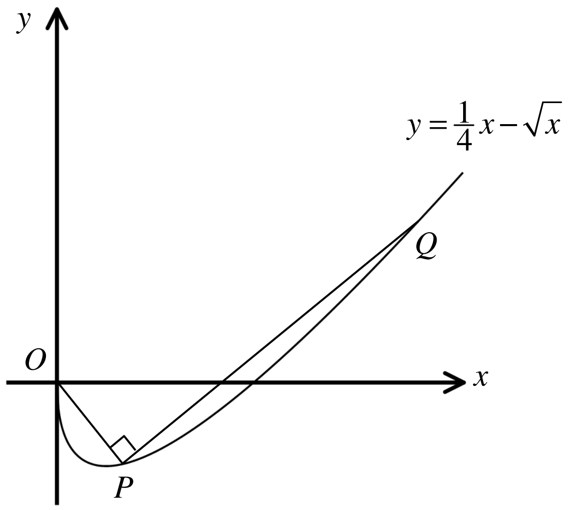
Question 18

Solve the cubic equation

$$16x^3 - 48x^2 + 60x - 31 = 0,\ \ x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root. **(14)**

Question 19



The figure above show the curve with equation

$$y=\frac{1}{4}x-\sqrt{x}\,,\ x\in\mathbb{R}\,,\ x\geq 0\,.$$

The points P and Q lie on the curve, so that $\measuredangle OPQ = 90°$, where O is the origin.

Determine the range of possible values of the x coordinate of P. **(14)**

Question 20

By expressing the integrand in the form $\text{sech}^2 x\, f(\tanh x)$, or otherwise, find the value of the following integral.

$$\int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\sqrt{2\,\text{sech}\,x}}{\sqrt[4]{\sinh 2x\cosh x}-\sqrt[4]{2\sinh^3 x}}\,dx\,.$$ **(12)**