IYGB

Special Extension Paper F

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \ge 70$, Merit $55 \le P \le 69$, Pass $40 \le P \le 54$

Question 1

Prove by induction that every positive integer power of 5 can be written as the sum of squares of two distinct positive integers.
(6)

Question 2

The straight line L has Cartesian equation

$$x-9=\frac{y-a}{2}=\frac{z-1}{b},$$

where a and b are non zero constants.

The plane Π has Cartesian equation

$$x+y-2z=12.$$

a) If L is contained by Π, determine the value of a and the value of b. **(3)**

b) Given instead that L meets Π at the point where $x=0$, and is inclined at an angle $\arcsin\frac{5}{6}$ to Π, determine the value of a. **(8)**

Question 3

The curve C has equation

$$y+2=\left[\ln(4x+1)\right]^2, \quad x\in\mathbb{R}, \quad x\geq -\frac{1}{4}.$$

Sketch a detailed graph of C. **(7)**

Question 4

A parabola C has Cartesian equation

$$y^2 = 4ax \ ,$$

where a is a positive constant.

The points $P\left(ap^2, 2ap\right)$ and $Q\left(aq^2, 2aq\right)$ are distinct and lie on C.

The tangent to C at P and the tangent to C at Q meet at the point R.

Show that

$$|SR|^2 = |SP||SQ|,$$

where S is the focus of the parabola. **(12)**

Question 5

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] \ = \ u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \ .$$

Simplify, showing a clear method

$$\prod_{r=1}^{n}\left[\frac{r}{2r+1}\right].$$

(9)

Question 6

By using the substitution $u = 1 + e^{-x}\tan x$, or otherwise, show that the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{2 - \sin 2x}{(1 + \cos 2x)e^x + \sin 2x}\, dx,$$

can be written as

$$\ln\left[2e^{-\frac{1}{8}\pi}\cosh\left(\tfrac{1}{8}\pi\right)\right].$$ **(12)**

Question 7

It is given that

$$(a + bx)^n = 512 + 576x + 288x^2 + \ldots,$$

where a, b and n are non zero constants.

Use algebra to determine the values of a, b and n.

No credit will be given to solutions by inspection and/or verification **(11)**

Question 8

Solve the following logarithmic equation.

$$\log_{4x}\left(\tfrac{1}{2}\right) + \log_x 8 + \log_{\frac{1}{2}x}\left(\tfrac{1}{2}\right) = \tfrac{1}{4}, \quad x > 0, \quad x \neq 1.$$

Give the answers in exact simplified form where appropriate. **(12)**

Question 9

The function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = k\left[\frac{1}{h} - \frac{1}{x}\right],$$

where k and h are non zero constants.

It is further given that $y = -1$, $\frac{dy}{dx} = -1$, $\frac{d^2y}{dx^2} = 2$ at $x = -1$.

Solve the differential equation to show that

$$e^{y-x} = (y+2)^2.$$ **(10)**

Question 10

The complex number z satisfies the relationship

$$5(z+i)^n = (4+3i)(1+iz)^n, \ n \in \mathbb{R}.$$

Show that z is a real number. **(12)**

Question 11

The following convergent series S is given below

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos\theta}\sin(\sin\theta).$$ **(9)**

Question 12

A system of simultaneous equations is given below

$$x + y + z = 1$$
$$x^2 + y^2 + z^2 = 21$$
$$x^3 + y^3 + z^3 = 55.$$

By forming an auxiliary cubic equation find the solution to the above system.

You may find the identity

$$x^3 + y^3 + z^3 \equiv (x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right) + 3xyz,$$

useful in this question. **(10)**

Question 13

Differentiate $\arctan\left[\frac{\sqrt{1-x^2}}{x}\right]$ with respect to $\arccos\left[2x\sqrt{1-x^2}\right]$, and hence sketch the graph of the resulting gradient function. **(12)**

Question 14

By using a suitable transformation, or otherwise, find a general solution for the following differential equation

$$\frac{dy}{dx} = \frac{1+y^2}{-x + \arctan y}.$$ **(10)**

I.Y.G.B. www.madasmaths.com

Question 15

Use appropriate integration techniques to show that

$$\int_{0}^{1} \frac{1}{x+\sqrt{1-x^2}} \, dx = \frac{\pi}{4}. \qquad (11)$$

Question 16

$$S_n = (2\times 1!) + (5\times 2!) + (10\times 3!) + (17\times 4!) + \ldots + \left(n^2+1\right)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)! \qquad (10)$$

Question 17

Use the properties of determinants to express the following determinant in fully factorized form.

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} \qquad (8)$$

Question 18

A square jewellery design is made of gold and silver.

The amount of gold used is proportional to the side of the square but the amount of silver used is proportional to the area of the square.

If the side of the square was to be enlarged by a factor of 8, the cost of the jewellery design would increase by a factor of 8.

Given that gold is 18 times more expensive than silver, determine the percentage of gold used in the standard design.

(8)

Question 19

Consider the following infinite series, S.

$$S = \frac{5}{18} - \frac{5\times8}{18\times24} + \frac{5\times8\times11}{18\times24\times30} - \frac{5\times8\times11\times14}{18\times24\times30\times36} + \frac{5\times8\times11\times14\times17}{18\times24\times30\times36\times42} - \dots$$

Given that S converges, show that

$$S = 9A - 41,$$

where A is an exact simplified surd. **(10)**

Question 20

By using the substitution $\sqrt[3]{20 \pm 14\sqrt{2}} = u \pm \sqrt{v}$, where $u \in \mathbb{Q}$, $v \in \mathbb{Q}$, simplify fully the following cubic radical expression.

$$\sqrt[3]{20+14\sqrt{2}}.$$ **(10)**

I.Y.G.B. www.madasmaths.com