# IYGB

## Special Extension Paper E

**Time: 3 hours 30 minutes**

**Candidates may NOT use any calculator.**

### Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

### Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

### Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

**Question 1**

A curve has equation

$$f(x) \equiv 3^{ax} + b, \ x \in \mathbb{R},$$

where $a$ and $b$ are non zero constants.

Find the value of $a$ and the value of $b$, given further that

$$f(2) = 3 - \sqrt{3} \quad \text{and} \quad f(3) = 2\sqrt{3}. \qquad \textbf{(5)}$$

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**Question 2**

The positive integer functions $f$ and $g$ are defined as

$$f(n) = \sum_{r=1}^{n} r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^{n} (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[ \frac{f(n)}{g(n)} \right]. \qquad \textbf{(6)}$$

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**Question 3**

A curve with equation $y = f(x)$ passes through the point with coordinates $(0,1)$ and satisfies the differential equation

$$y^2 \frac{dy}{dx} + y^3 = 4e^x.$$

Solve the differential equation, by finding a suitable integrating factor, to show that

$$y^3 = 3e^x - 2e^{-3x}. \qquad \textbf{(8)}$$

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**Question 4**

$$I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cot^3 x}{\operatorname{cosec} x} \, dx .$$

Use appropriate integration techniques to show that

$$I = \frac{1}{6}\left[a + b\sqrt{3}\right],$$

where $a$ and $b$ are integers to be found. $(7)$

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**Question 5**

Two joggers, $A$ and $B$ ran a standard route of $5 \text{ km}$, which consists of a downhill section to start with, a flat section in the middle of the run and an uphill section all the way to the finish line.

$A$ ran the three sections with respective speeds $2.4 \text{ ms}^{-1}$, $3.2 \text{ ms}^{-1}$ and $2 \text{ ms}^{-1}$.

$A$ took $31$ minutes and $40$ seconds to complete the run.

$B$ ran the three sections with respective speeds $3.6 \text{ ms}^{-1}$, $3 \text{ ms}^{-1}$ and $2.5 \text{ ms}^{-1}$.

$A$ took exactly $27$ minutes to complete the run.

Assuming that both runners started at the same time, determine the distance between $A$ and $B$, as $B$ crosses the finish line. $(7)$

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**Question 6**

The quadratic equation

$$x^2 - 4x - 2 = 0,$$

has roots $\alpha$ and $\beta$ in the usual notation, where $\alpha > \beta$.

It is further given that

$$f_n \equiv \alpha^n - \beta^n .$$

Determine the value of

$$\frac{f_{10} - 2f_8}{f_9} . \qquad \mathbf{(7)}$$

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**Question 7**

It is given that

$$I_n = \int_0^a x^{n+\frac{1}{2}} \sqrt{a-x} \; dx, \; n \in \mathbb{Z}, \; n \geq 0$$

where $a$ is a positive constant.

**a)** Use integration by parts to show

$$I_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{I_0}{n+1}, \quad n \geq 1. \qquad \mathbf{(8)}$$

**b)** Determine the value of

$$\int_0^2 x^{10} \sqrt{4 - x^2} \; dx . \qquad \mathbf{(4)}$$

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**Question 8**

Solve the following inequality.

$$(5-x)(5-|x|)>9,\ x \in \mathbb{R}$$ **(7)**

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**Question 9**

The variable point $P$ lies on the rectangular hyperbola, with Cartesian equation

$$xy = a^2,$$

where $a$ is a positive constant.

The normal to the hyperbola at $P$ meets the hyperbola again at the point $Q$.

The point $M$ is the midpoint of $PQ$.

Determine, in the form $f(x,y)=0$, an equation of the locus of $M$, for all the possible positions of $P$. **(10)**

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**Question 10**

$$z^3-(2+4\mathrm{i})z^2-3(1-3\mathrm{i})z+14-2\mathrm{i}=0,\ z \in \mathbb{C}.$$

Find the three solutions of the above equation given that one of these solutions is purely imaginary. **(11)**

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**Question 11**

Determine, as exact simplified natural logarithms, the solutions of the following simultaneous equations

$$\cosh x + \cosh y = 4 \qquad \text{and} \qquad \sinh x + \sinh y = 2.$$ **(11)**

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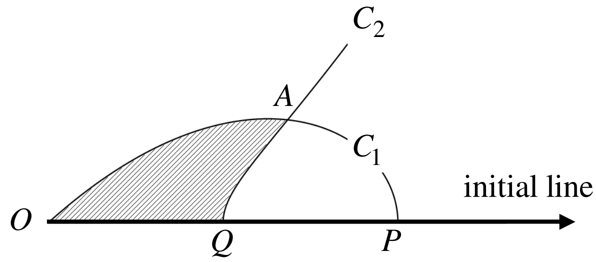
**Question 12**




The figure above shows the curves $C_1$ and $C_2$ with respective polar equations

$$r_1 = \sec\theta\left(1-\tan^2\theta\right) \quad \text{and} \quad r_2 = \tfrac{1}{2}\sec^3\theta, \qquad 0 \le \theta < \tfrac{1}{4}\pi.$$

The points $P$ and $Q$ are the respective points where $C_1$ and $C_2$ meet the initial line, and the point $A$ is the intersection of $C_1$ and $C_2$.

**a)** Find the exact area of the curvilinear triangle $OAQ$, where $O$ is the pole. **(11)**

The angle $OAP$ is denoted by $\psi$.

**b)** Show that $\tan\psi = -3\sqrt{3}$. **(4)**

**You may assume without proof**

$$\int \sec^6 x \; dx = \tfrac{1}{15}\left(8+4\sec^2 x+3\sec^4 x\right)\tan x + C$$

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**Question 13**

$$f(x) \equiv (2+\mathrm{e})(2\mathrm{e})^{x-1} - \mathrm{e}^{2x-1}\,4^{x-\frac{1}{2}}, \quad x \in \mathbb{R}.$$

**a)** Find in exact simplified form the solution of the equation $f(x)=0$. **(6)**

**b)** Determine, in terms of $\ln 2$, the two solutions of the equation $f(x)=1$. **(6)**

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**Question 14**

The part of the graph of the exponential curve

$$y = e^x \ , \ \ln\left(\tfrac{3}{4}\right) \le x \le \ln\left(\tfrac{4}{3}\right),$$

is rotated by $2\pi$ radians in the $x$ axis, forming a surface of revolution $S$.

Show that area of $S$ is

$$\pi\left[\frac{185}{144} + \ln\left(\frac{3}{2}\right)\right].$$

**(10)**

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**Question 15**

Find the sum to infinity of the following series.

$$\frac{1}{1} - \frac{1}{1+4} + \frac{1}{1+4+9} - \frac{1}{1+4+9+16} + \frac{1}{1+4+9+16+25} + \ldots$$

You may find the series expansion of $\arctan x$ useful in this question. **(12)**

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**Question 16**

Sketch the graph of the curve with equation

$$y = x\sqrt{\left|\ln|x|\right|}, \ x \in \mathbb{R}.$$

The sketch must include the coordinates of ...

... any points where the curve meets the coordinate axes.

... any stationary points. **(10)**

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**Question 17**

The point $P$ in an Argand diagram represents the complex number $z$, which satisfies

$$\arg\left[\frac{z-1-\mathrm{i}}{z-2\mathrm{i}}\right], \ z \neq 2\mathrm{i}.$$

It further given that $P$ lies on the arc $AB$ of a circle centred at $C$ and of radius $r$.

**a)** Sketch in an Argand diagram the circular arc $AB$, stating the coordinates of $C$ and the value of $r$. **(10)**

**b)** Given further that $|PA| = |PB|$, find the complex number represented by $P$. **(6)**

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**Question 18**

Three circles, $C_1$, $C_2$ and $C_3$, have their centres at $A$, $B$ and $C$, respectively, so that $|AB| = 5$, $|AC| = 4$ and $|BC| = 3$.

The positive $x$ and $y$ axis are tangents to $C_1$.

The positive $x$ axis is a tangent to $C_2$.

$C_1$ and $C_2$ touch each other externally at the point $M$.

Given further that $C_3$ touches externally both $C_1$ and $C_2$, find, in exact simplified form, an equation of the straight line which passes through $M$ and $C$. **(10)**

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**Question 19**

The integral $I$ is defined as

$$I = \int_{0}^{\pi} \frac{\sin^2 x}{1+\cos^2 x} \, dx .$$ (4)

**a)** Show by a detailed method that

$$I + \pi = \int_{o}^{\frac{1}{2}\pi} \frac{4}{1+\cos^2 x} \, dx .$$

**b)** Hence, find the value of $I$ in exact simplified form. (7)

**c)** Verify the answer obtained in part **(b)** by an alternative method by first writing the integrand of $I$ as a function of $\cot^2 x$. (8)

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**Question 20**

It is given that $11a+13b$ is a multiple of $13-a$, where $a \in \mathbb{N}$, $b \in \mathbb{N}$.

It is then asserted that $(13+a)(11+b)$ is also a multiple of $13-a$.

Prove the validity of this assertion. (5)

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