IYGB

Special Extension Paper D

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $=T$, Number of non attempted questions $=N$, Percentage score $=P$.

$P=\frac{1}{2}T+N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

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Question 1

A parabola has Cartesian equation

$$y = x^2 \ , \ x \in \mathbb{R} .$$

A chord of the parabola is defined as the straight line segment joining any two distinct points on the parabola.

Find the equation of the locus of the midpoints of parallel chords of the parabola whose gradient is m. **(10)**

Question 2

$$f(x) \equiv 4x(x-2)(x+1)(x-3), \quad x \in \mathbb{R} .$$

Evaluate $f\left(1+\frac{1}{2}\sqrt{10}\right)$.

You must show detailed workings in this question. **(6)**

Question 3

Prove by induction that

$$\frac{d^n}{dx^n}\left(\mathrm{e}^x \cos x\right) = 2^{\frac{1}{2}n}\,\mathrm{e}^x \cos\left(x+\frac{n\pi}{4}\right), \ \ n \geq 1, n \in \mathbb{N} \ . \qquad \textbf{(9)}$$

Question 4

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} \left(4x^2 - 16x + 15\right)^4 \ dx .$$

(10)

Question 5

The complex numbers z and w are such so that $|z| = |w| = 1$.

Show clearly that $\dfrac{z+w}{1+zw}$ is real. **(9)**

Question 6

A curve has equation

$$y = \frac{1}{4}\left[(2x+1)\sqrt{4x^2+4x} - \text{arcosh}(2x+1)\right], \quad 0 \le x \le 4$$

Show that the length of the curve is 20 units. **(10)**

Question 7

The point T lies on the curve with equation

$$x^2 + y^2 - 5xy = 15.$$

The tangent to the curve at T passes through the point with coordinates $(2,6)$.

Determine the two possible sets of coordinates for T. **(9)**

Question 8

The function f is defined as

$$f(x) \equiv \frac{\left(1+4\sin^2 x\right)^{\frac{1}{2}}\left(8+\sec^2 x\right)^{\frac{3}{2}}}{\tan^3 x}, \quad x \in \mathbb{R}, \quad x \ne \tfrac{1}{2}n\pi.$$

Find in exact simplified form the value of $f'\left(\tfrac{1}{3}\pi\right)$. **(8)**

Question 9

With respect to a fixed origin O, the points A, B and C have respective position vectors

$$\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}\,, \quad \mathbf{b} = 6\mathbf{i} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = 3\mathbf{j} + 5\mathbf{k}\,,$$

so that the plane Π contains A, B and C.

The straight line L is **parallel** to Π and has vector equation

$$\mathbf{r} = (13\mathbf{i} - 9\mathbf{j}) + \lambda(-7\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})\,,$$

where λ is a scalar parameter.

The point P lies outside the plane so that PC is perpendicular to Π.

The point Q lies on L so that PQ is perpendicular to L.

Given further that P is equidistant from Π and L, find the position vector of P and the position vector of Q. **(11)**

Question 10

A parabola has the following equation

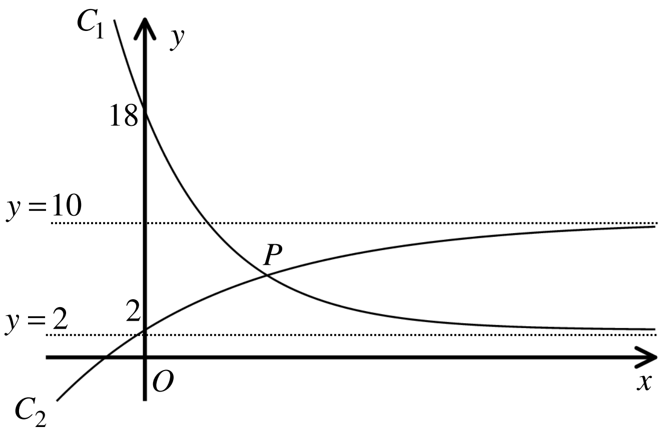
$$y^2 = Ax\,,\ x \geq 0\,,\ A > 0\,.$$

The parabola is rotated about O onto a new parabola with equation

$$16x^2 - 24xy + 9y^2 + 30x + 40y = 0\,.$$

Use algebra to determine the value of A. **(10)**

Question 11



Two exponential curves, C_1 and C_2, intersect at the point $P(\ln 8, 6)$.

- C_1 meets the y axis at $(0,18)$ and the straight line with equation $y=2$ is an asymptote to C_1.
- C_2 meets the y axis at $(0,2)$ and the straight line with equation $y=10$ is an asymptote to C_2.

Show that at P, C_1 and C_1 cross each other at an acute angle of $\arctan\left(\frac{36}{23}\right)$. **(10)**

Question 12

It is given that a curve with equation $x = f(y)$ passes through the point $\left(0,\frac{1}{2}\right)$ and satisfies the differential equation

$$(2y+3x)\frac{dy}{dx} = y.$$

Find an equation for the curve in the form $x = f(y)$. **(7)**

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Question 13

A curve C is defined parametrically by

$$x = \frac{1}{t} + \arctan t, \quad y = \frac{1}{t} - \arctan t, \quad t \in \mathbb{R}, \ t \neq 0.$$

Sketch the graph of C.

Indicate the equations of any asymptotes, stationary points and any endpoints.

You need not mark the coordinates of any intersections with the axes. **(10)**

Question 14

Determine, in terms of n, a simplified expression

$$\sum_{r=1}^{n} \left[\frac{r^2 + 9r + 19}{(r+5)!} \right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right].$$ **(14)**

Question 15

Find, in exact trigonometric form where appropriate, the real solutions of the following polynomial equation

$$x^7 - 7x^6 - 21x^5 + 35x^4 + 35x^3 - 21x^2 - 7x + 1 = 0.$$

(11)

Question 16

The point P lies on the ellipse with polar equation

$$r(5-3\cos\theta)=8,\ 0\le\theta<2\pi.$$

The ellipse has foci at $O(0,0)$ and at $T(3,0)$.

Show that $|OP|+|PT|$ is constant for all positions of P. **(11)**

Question 17

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1\times u_2\times u_3\times u_4\times\ldots\times u_{k-1}\times u_k.$$

a) By considering the sine double angle identity show that

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left[\cos\left(\frac{x}{2^k}\right)\right].$$ **(7)**

b) Deduce that

$$\frac{2}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{32}\right)\ldots} = \pi.$$ **(3)**

Question 18

Solve the cubic equation

$$16x^3+96x^2+180x+99=0,\ x\in\mathbb{R}.$$

You may assume that this cubic equation only has one real root. **(11)**

Question 19

The definite integral I is defined in terms of the constant k, where $k \neq 0$, $k \neq \pm 1$.

$$I = \int_0^{\frac{1}{2}\pi} \frac{1}{1+k^2 \tan^2 x} \, dx .$$

Use appropriate integration techniques to show that **(14)**

$$I = \frac{\pi}{2(k+1)} .$$

Question 20

A curve C has Cartesian equation $y = f(x)$.

The same curve has intrinsic equation $s = g(\psi)$, where s is measured from an arbitrary point and ψ is the angle the tangent to C makes with the positive x axis.

The radius of curvature ρ at any point on C is defined as $\dfrac{ds}{d\psi}$.

a) Show clearly that $\rho = \dfrac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ **(4)**

b) Given that C can be suitably parameterized as $x = h_1(t)$, $y = h_2(t)$, for some parameter t, show further that

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} ,$$

where a dot above a variable denotes differentiation with respect to t. **(6)**
