IYGB

Special Extension Paper C

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

A triangle, ABC has $|BC| = 4$ cm, $|AC| = 8$ cm and $\measuredangle ACB = 60°$.

Determine, in degrees, the size of $\measuredangle BAC$. **(5)**

Question 2

A curve C is defined by the parametric equations

$$x = t^3 + 2 \ , \ y = t^2 + 3, \ t \in \mathbb{R}.$$

Show clearly that

$$\frac{d^2y}{dx^2} = f(y),$$

where f must by explicitly stated. **(5)**

Question 3

$$x = 4\operatorname{arcosh}\left(\tfrac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}, \ y \geq 4.$$

Use differentiation to show that

$$\frac{d^2y}{dx^2} = \frac{2}{y^2}.$$ **(8)**

Question 4

Solve the quadratic equation

$$\mathrm{i}z^2 - 2\sqrt{2}\,z - 2\sqrt{3} = 0, \ z \in \mathbb{C}.$$

Give the answers in the form $x + \mathrm{i}y$, where x and y are exact real constants. **(9)**

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Question 5

It is given that

$$I_{n,m} = \int_0^1 (1-x)^n x^m \, dx ,$$

where $n, m \in \mathbb{Z}$, with $n, m \geq 0$.

a) Show that …

i. … $I_{n,m} - I_{n-1,m} = -I_{n-1,m+1}$. **(4)**

ii. … $I_{n,m} = \frac{n}{m} I_{n-1,m+1}$ **(3)**

b) Hence derive an expression of $I_{n,m}$ and use it to find

$$\int_0^1 7x^{\frac{1}{2}} (1-x)^3 \, dx .$$ **(4)**

Question 6

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$

Determine the exact value of u_{61}. **(8)**

Question 7

Determine a simplified expression, in the form $\ln\left[f(n)\right]$, for the following sum.

$$\sum_{r=2}^{N} \left[\int_2^r \frac{2}{x^2 - 1} \, dx \right].$$ **(10)**

Question 8

A rotation R, acting in the $\boldsymbol{x}$ - $\boldsymbol{y}$ plane is given by the following 3×3 matrix.

$$\mathbf{R} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Using a detailed method, find the centre and angle of this rotation. **(12)**

Question 9

A curve C is defined in the largest real domain by the equation

$$y = \frac{4x^2 - 25}{(2x-1)(x-2)(x+3)}.$$

a) Sketch the graph of C. **(7)**

The sketch must include the equations of any asymptotes of C and the coordinates of any point where C meets the coordinate axes.

Any turning points, including points of inflexion, must be clearly indicated but their coordinates need **not** be found.

b) Hence sketch on separate set of axes the graph of ...

i. ... $y^2 = \dfrac{4x^2 - 25}{(2x-1)(x-2)(x+3)}$. **(2)**

ii. ... $y = \dfrac{4x - 25}{(2\sqrt{x}-1)(\sqrt{x}-2)(\sqrt{x}+3)}$. **(2)**

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Question 10

The roots of the cubic equation

$$8x^3 + 12x^2 + 2x - 3 = 0$$

are denoted in the usual notation by α, β and γ.

An integer function S_n, is defined as

$$S_n \equiv (2\alpha+1)^n + (2\beta+1)^n + (2\gamma+1)^n, \quad n \in \mathbb{Z}.$$

Determine the value of S_3 and the value of S_{-2}. **(11)**

Question 11

$$f(x) \equiv \frac{1}{x^{100} + 100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \ x \in \mathbb{R}.$$

Use a formal method to find

$$\lim_{x \to \infty} f(x).$$ **(6)**

Question 12

Use appropriate integration techniques to find an exact answer for the following definite integral.

$$\int_{\frac{1}{2}\pi}^{2\pi - \arccos\frac{1}{8}} \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \quad dx$$ **(10)**

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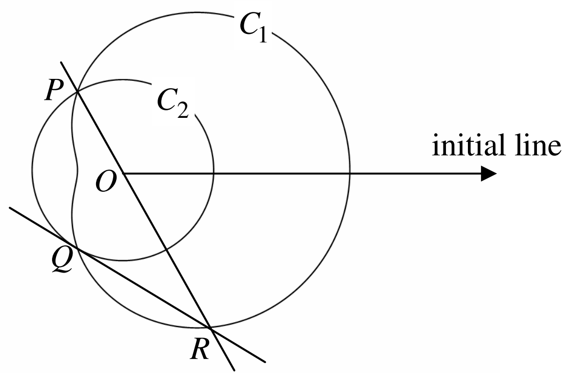
Question 13



The figure above shows the curves C_1 and C_2 with respective polar equations

$$r_1 = 3 + 2\cos\theta,\ 0 \le \theta < 2\pi \qquad \text{and} \qquad r_2 = 2.$$

The two curves intersect at the points P and Q.

A straight line passing through P and the pole O intersects C_1 again at the point R.

Show that RQ is a tangent of C_1 at Q. **(7)**

Question 14

It is given that

$$\sum_{r=1}^{n} u_r = \frac{1 + 3^{2n+2} - 2 \times 5^{n+1}}{8},$$

where u_n is the n^{th} term of a sequence.

Find a simplified expression for u_n. **(7)**

Question 15

The point $P\left(p+\frac{1}{p}, p-\frac{1}{p}\right)$, $p \neq 0$, lies on the rectangular hyperbola, with Cartesian equation

$$x^2 - y^2 = 4 .$$

The normal to the hyperbola at P meets the y axis at the point $Q(0,-k)$, $k > 0$.

The area of the triangle OPQ, where O is the origin, is $\frac{15}{4}$.

Determine the two possible sets of coordinates for P. **(12)**

Question 16

$$4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} = 1 .$$

By using the substitution $t = \sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y = A - \sqrt{x} + \ln\left[1 + B\mathrm{e}^{2\sqrt{x}}\right],$$

where A and B are arbitrary constants. **(15)**

Question 17

By showing a detailed method involving complex numbers, sum the following series.

$$\sum_{n=0}^{\infty}\left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n}\right].$$ **(15)**

Question 18

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$n^{n+1} > (n+1)^n,$$

and hence deduce that if $n \in \mathbb{N}$, $n \geq 3$, then **(13)**

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}$$

Question 19

Use the substitution $v = \dfrac{y-x}{y+x}$, $y + x \neq 0$, to solve the following differential equation

$$x\frac{dy}{dx} - y = \frac{(1-x)\left(x^2 - y^2\right)}{x^3 + x^2 + x + 1}, \quad y(0) = 1.$$

Give the answer in the form $y = f(x)$.

Question 20

Consider the following inequality

$$kx^2 + 2x + 1 \leq (x+1)(x-3),$$

where k is a real constant.

Find, in terms of k where appropriate, the solution intervals of the above inequality for all possible values of k. **(10)**
