IYGB

Special Extension Paper B

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

$$\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+\ldots}}}}}$$

Given that the above nested radical converges, determine its limit. **(6)**

Question 2

A triangle, ABC has $|AB| = 2\sqrt{3}$ cm, $\measuredangle BAC = 45°$ and $\measuredangle ACB = 60°$.

Determine, in exact simplified surd form, the area of this triangle. **(6)**

Question 3

The 2×2 matrix $\mathbf{P}$ is given below.

$$\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

The points on the x-y plane which lie on the curve with equation

$$13x^2 - 16xy + 5y^2 + 8x - 6y = 4,$$

are transformed by $\mathbf{P}$ onto the points which lie on another curve C.

Determine an equation for C and hence describe it geometrically. **(8)**

Question 4

A curve C is defined in the largest real domain by the equation

$$y = \log_x 2.$$

Sketch a detailed graph of C. **(4)**

Question 5

Solve the following inequality in the largest real domain.

$$\frac{x^2-2|x|-8}{6|x|^3-5x^2+12|x|}\le 0.$$

(8)

Question 6

In a standard Argand diagram the complex number $\sqrt{3}+\mathrm{i}$, represents one of the vertices of a regular hexagon, with centre at the origin O.

The complex numbers that represent these 6 vertices are all raised to the power of 4, creating a closed shape with straight sides.

Determine the area of S. **(10)**

Question 7

It is given that 2^{10} is approximately 1000.

a) Given further that $\ln 2$ is approximately 0.7, find the approximate value of $\ln 10$, giving the answer in the form $\frac{a}{b}$, where a and b are positive integers. **(4)**

b) Given further that e^3 is approximately 20, show that the approximate value of $\ln 2$ is $\frac{9}{13}$. **(5)**

Question 8

$$\mathbf{A}=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n=n\mathbf{A}-(n-1)\mathbf{I},\ n\ge 1,\ n\in\mathbb{N}.$$

(8)

Question 9

$$I = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{\sqrt{1+x^2}}{x^4} \, dx .$$

a) Use a trigonometric substitution to show that

$$I = \frac{2}{3}\left(a + b\sqrt{2}\right),$$

where a and b are integers to be found. **(8)**

b) Use a hyperbolic substitution to verify the answer of part **(a)**. **(8)**

Question 10

$$2xy\frac{dy}{dx} + y^2 = 2x - 3x^2,$$

a) Use the substitution $z = x^2 + y^2$ to solve the above differential equation, subject to the condition $y = 1$ at $x = 1$. **(8)**

b) Verify the answer of part **(a)** by a using a different substitution. **(8)**

Question 11

A cycloid is given by the parametric equations

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad 0 < \theta < \pi .$$

The gradient at the point P on this cycloid is $\frac{1}{2}$.

Show that at P, $\tan\theta = -\frac{4}{3}$. **(10)**

Question 12

Evaluate the following expression

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1}. \qquad (9)$$

Question 13

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{2x-1}\left(1-\frac{dy}{dx}\right), \quad y(0)=1, \ \frac{dy}{dx}(0)=-1.$$

Solve the above differential equation giving the answer in the form $y = f(x)$. **(10)**

Question 14

The complex number z satisfies the equation

$$z+1+8\mathrm{i} = |z|(1+\mathrm{i}).$$

Show clearly that

$$|z|^2 - 18|z| + 65 = 0,$$

and hence find the possible values of z. **(10)**

I.Y.G.B. www.madasmaths.com

Question 15

$$I = \int_{\arcsin\frac{3}{5}}^{\arccos\frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)} \, dx .$$

Use appropriate integration techniques to show that

$$I = \ln\left(\frac{a}{b}\right),$$

where a and b are positive integers to be found. **(11)**

Question 16

The gradient at every point on a curve C is given by

$$\frac{dy}{dx} = \frac{1}{2}s ,$$

where s is the arc length along C measured from the point P whose Cartesian coordinates are $(0,2)$.

It is further given that $\psi = 0$ at P, where ψ is the angle the tangent to C makes with the positive x axis.

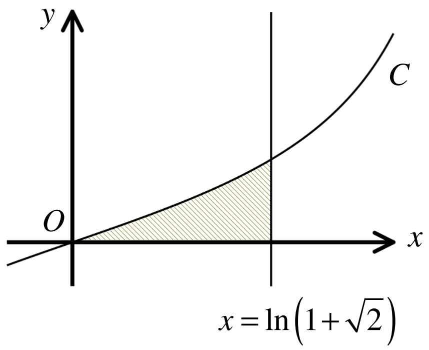
a) Show clearly that

$$x = 2\ln|\sec\psi + \tan\psi| , \qquad y = 2\sec\psi .$$ **(9)**

b) Eliminate ψ to show further that

$$y = 2\cosh\left(\tfrac{1}{2}x\right).$$ **(6)**

Question 17



The figure above shows the curve C whose parametric equations are

$$x = \operatorname{artanh}(\sin t), \quad y = \sec t \tan t, \quad -\frac{1}{2}\pi < t < \frac{1}{2}\pi .$$

Find the area of the finite region bounded by the x axis, the curve and the straight line with equation $x = \ln\left(1+\sqrt{2}\right)$. **(12)**

Question 18

A curve has equation

$$f(x) \equiv \begin{cases} x^2 - 6x + 8, & x \in \mathbb{R},\ 2 \le x \le 4 \\ f(x) + f(4-x) = 0, & x \in \mathbb{R} \\ f(x) - f(4+x) = 0, & x \in \mathbb{R} \end{cases}$$

Sketch a detailed graph of $f(1-2x)$, $x \in \mathbb{R}$, $0 \le x \le 4$. **(5)**

Question 19

$$I(m,n) = \int_a^b (b-x)^m (x-a)^n \ dx, \quad m \in \mathbb{N}, \ n \in \mathbb{N}.$$

Show that

$$I(m,n) = \frac{m!\,n!}{(m+n+1)!}(b-a)^{m+n+1},$$

where a and b are real constants such that $b > a$ **(12)**

Question 20

Given that p and q are positive, shown that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\text{artanh}\left(x^2\right)$ useful in this question. **(15)**
