IYGB

Special Paper Y

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

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Question 1

A curve C has implicit equation

$$x^2 - 2y^2 + 4xy - 4x - 6y + 4 = 0.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{2y + x + a}{2y - 2x + b},$$

where a and b are integers to be found. **(3)**

The straight line l_1 with equation $y = 2x - 3$ is a tangent to C at the point P.

The straight line l_2 is parallel to l_1 and is also a tangent to C at a different point Q.

b) Find an equation of l_2. **(7)**

Question 2

The circle C_1 has equation

$$x^2 + y^2 - 4x - 4y + 6 = 0.$$

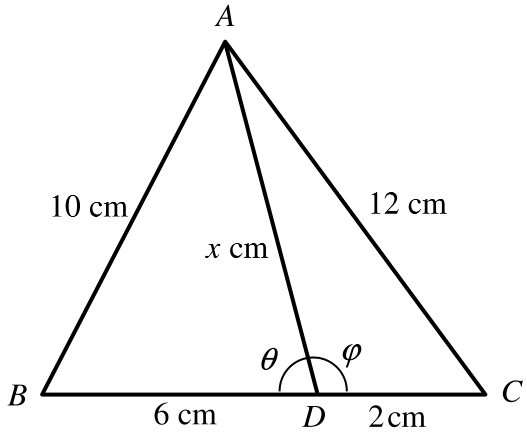
The circle C_2 has equation

$$x^2 + y^2 - 10x - 10y + k = 0,$$

where k is a constant.

Given that C_1 and C_2 have exactly two common tangents, determine the range of possible values of k. **(6)**

Question 3



The figure above shows the triangle ABC, where $|AB| = 10$ cm, $|AC| = 12$ cm and $|AB| = 8$ cm.

The point D lies on BC so that $|AD| = 6$ cm, $|DC| = 2$ cm and $|AD| = x$ cm.

The angle BDA is denoted by θ and the angle CDA is denoted by φ.

a) Express $\cos\theta$ and $\cos\varphi$ in terms of x. **(3)**

b) Use part **(a)** to find the length of AD. **(3)**

c) Hence show that the area of the triangle ABD is exactly $\frac{45}{4}\sqrt{7}$ cm^2. **(5)**

Question 4

It is given that

$$a^{\log b} \equiv b^{\log a}, \; a > 0, \; b > 0.$$

a) Prove the validity of the above result. **(3)**

b) Hence, or otherwise, solve the equation

$$3^{\log x} + 3 \times x^{\log 3} = 36$$ **(5)**

Question 5

The function f is defined

$$f(x) = \sqrt{4-x}, \ x \in \mathbb{R}, \ x \le 4.$$

It is further given that

$$fg(x) = \sqrt{4+2x}, \ x \in \mathbb{R}, \ x \ge -2,$$

$$hf(x) = x-4, \ x \in \mathbb{R}, \ x \le 4.$$

for some functions $g(x)$, $x \in \mathbb{R}$ and $h(x)$, $x \in \mathbb{R}$.

Find simplified expressions for …

a) … $g(x)$. **(3)**

b) … $h(x)$. **(3)**

Question 6

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 9\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} + \lambda(4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$$

$$\mathbf{r}_2 = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} + \mu(3\mathbf{i} - 4\mathbf{j} + a\mathbf{k}),$$

where λ and μ are scalar parameters and a is a scalar constant.

The point A is the intersection of l_1 and l_2.

a) Find in any order …

- **i.** … the value of a.
- **ii.** … the coordinates of A. **(4)**

The acute angle between l_1 and l_2 is θ.

b) Show that $\theta = 60°$. **(2)**

The point B lies on l_1 and the point C lies on l_2.

The triangle ABC is equilateral with sides of length $15\sqrt{2}$.

c) Find the **two** possible pairings for the coordinates of B and C. **(6)**

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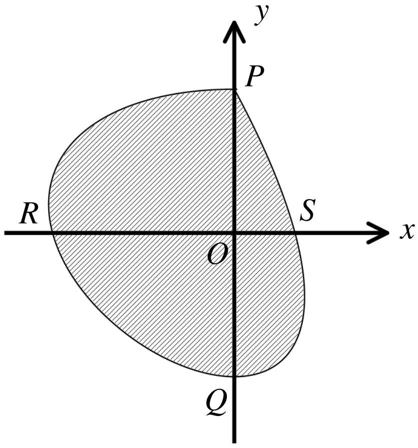
Question 7



The figure above shows the curve C with parametric equations

$$x = t\sin t,\ \ y = \cos t,\ \ 0 \le t < 2\pi .$$

The curve meets the coordinate axes at the points P, Q, R and S.

a) Find the value of the parameter t at each the points P, Q, R and S. **(4)**

The finite region bounded by the curve C is shown shaded in the above figure.

b) Show that the area of this region is exactly π^2 square units. **(8)**

Question 8

Given that

$$y = \frac{e^x}{1+e^x},$$

show that $\frac{dy}{dx} = f(y)$, where $f(y)$ is a simplified function to be determined. **(7)**

Question 9

It is given that for $\theta \neq (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$,

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \equiv \tan\theta + \sec\theta .$$

a) Prove the validity of the above trigonometric identity. **(4)**

b) Hence find a similar expression for $\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$. **(1)**

You are now given the equation

$$\tan x - \tan(x-\alpha) = 2\tan x ,$$

where α is a constant.

c) Express $\tan x$ in terms of trigonometric functions involving α only. **(6)**

d) Hence solve the trigonometric equation

$$\tan x - \tan\left(x-\frac{3\pi}{5}\right) = 2\tan\frac{3\pi}{5}, \quad 0 \leq x < 2\pi ,$$

giving the answers in terms of π. **(5)**

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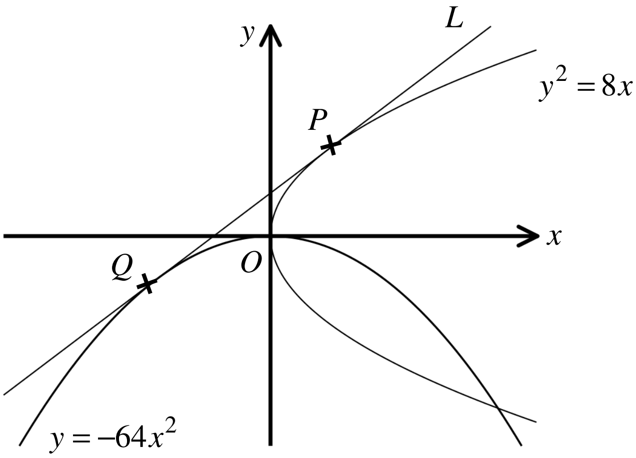
Question 10



The straight line L is a tangent at the point P to the curve with equation

$$y^2 = 8x.$$

The straight line L is also a tangent at the point Q to the curve with equation

$$y = -64x^2.$$

Determine the exact area of the triangle POQ, where O is the origin. **(12)**

Question 11

A sequence u_1, u_2, u_3, u_4, u_5 … is given by the recurrence formula

$$u_{n+2} = \frac{3u_n + u_{n+1}}{2}, \quad u_1 = 1, \; u_2 = 1.$$

It is further given that in this sequence **the ratio of consecutive terms** converges to a limit L.

Determine the value of L. **(7)**

Question 12

$$J = \int_0^1 \frac{(x^2+1)e^x}{(x+1)^2} \, dx .$$

Show that $J = 1$ **(10)**

Question 13

A curve C has equation

$$y^3 - y^2 = x, \; x \in \mathbb{R}, \; y \in \mathbb{R} .$$

Sketch the graph of C. **(8)**

The graph must include the coordinates …

… of any points where the graph of C meets the coordinate axes.

… of the three turning points of C, of which one is a point of inflection.

Question 14

The mass of a radioactive isotope decays at a rate proportional to the mass of the isotope present at that instant.

The half life of the isotope is 12 days.

Show that the proportion of the original amount of the isotope left after a period of 30 days is $\frac{1}{8}\sqrt{2}$. **(11)**

Question 15

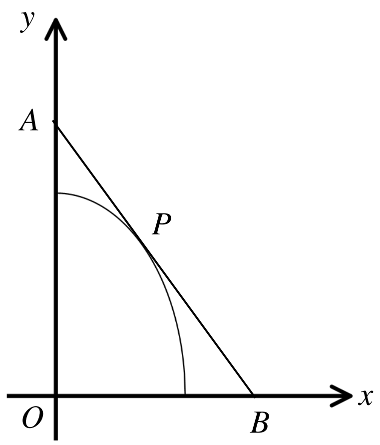
It given that

$$\arctan x + \arctan y + \arctan z = \frac{\pi}{2}.$$

Show that x, y and z satisfy the relationship

$$xy + yz + zx = 1. \qquad \textbf{(10)}$$

Question 16



The figure above shows the curve C with equation

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad x \geq 0,$$

where a and b are constants such that $b > a > 0$.

The point P lies on C and the tangent to C at P meets the coordinate axes at the points A and B, as shown in the figure.

Show with full justification that the minimum area of the triangle AOB, where O is the origin, is ab.

(13)

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Question 17

Find a simplified expression for the following sum

$$\frac{1}{100!}+\frac{1}{99!}+\frac{1}{2!98!}+\frac{1}{3!97!}+\frac{1}{4!96!}+\ldots+\frac{1}{2!98!}+\frac{1}{99!}+\frac{1}{100!}. \qquad \textbf{(6)}$$

Question 18

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x+\cos x}\,dx = \frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right). \qquad \textbf{(11)}$$

Question 19

The angle θ satisfies the equation

$$\tan\theta\tan 2\theta = \sum_{r=0}^{\infty} 2\cos^r 2\theta .$$

Given that $0<\theta<\frac{\pi}{2}$, find the exact value of $\tan\theta$. **(10)**

Question 20

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

The function $f(n,k)$ is defined as

$$f(n,k) = \left[\prod_{m=1}^{n}\left[\frac{2n-2m+1}{n!}\right]\right]\left[\prod_{r=1}^{k}\left[\frac{n-r+1}{2n-2r+1}\right]\right]\left[\prod_{l=1}^{k}\left[\frac{1}{(2l)}\right]\right], \ n \geq 2k .$$

Show by a detailed method that

$$f(n,k) = \frac{(2n-2k)!}{2^n \ k!(n-k)! \ (n-2k)!}$$

(14)
