IYGB

Special Paper X

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

Find the coordinates of the points of intersections between

$$x^2 + y^2 = 25 \qquad \text{and} \qquad 3y = 15 + 14x - 5x^2,$$

given further that the x coordinate of one of these points is 4. **(7)**

Question 2

$$y = 16e^{-\frac{t}{3}\ln 2}, \; t \in \mathbb{R}.$$

Show clearly that when $t = 10$, $y = \sqrt[3]{4}$ **(5)**

Question 3

The first two terms of a geometric series are 10 and $(10 - x)$.

Given that the series is convergent determine …

a) … the range of values of x. **(4)**

b) … the range of the sum to infinity of the series. **(4)**

Question 4

Show that for all positive real numbers a and b

$$a^3 + 2b^3 \geq 3ab^2.$$ **(7)**

Question 5

$$f(x) \equiv \frac{(2x+1)^2}{x(x+1)^4}, \quad x \in \mathbb{R}, \quad x \neq 0, x \neq -1$$

Express $f(x)$ into partial fractions in their simplest form. **(7)**

Question 6

A finite region R is defined by the inequalities

$$y^2 \leq 4ax, \quad 0 \leq x \leq a, \quad y \geq 0,$$

where a is a positive constant.

The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine, in terms of π and a, the exact volume of this solid. **(8)**

Question 7

A curve C has equation

$$y = x^x, \quad x \in \mathbb{R}, \quad x > 0.$$

Show that y is a solution of the equation

$$\frac{d^2y}{dx^2} = x^x(1 + \ln x)^2 + x^{x-1}.$$ **(6)**

Question 8

The points $A(3,2,14)$, $B(0,1,13)$ and $C(5,6,8)$ have the given coordinates with respect to a fixed origin O.

a) Show that the cosine of the angle ABC is $\frac{3}{\sqrt{33}}$. **(3)**

The straight line L passes through A and it is parallel to the vector $\overrightarrow{BC}$.

b) Find a vector equation of L. **(2)**

The point D lies on L so that $ABCD$ is a parallelogram.

c) Find the coordinates of D. **(2)**

d) If instead $ABCD$ is an isosceles trapezium and the point D still lies on L, determine the new coordinates of D. **(5)**

Question 9

A right circular cone of radius r and height h is to be cut out of a sphere of radius R.

It is a requirement that the circumference of the base of the cone and its vertex lie on the surface of the sphere.

Determine, in exact form in terms of R, and with full justification, the maximum volume of the cone that can be cut out of this sphere. **(11)**

Question 10

By using the substitution $x = 2\tan^2\theta$, or otherwise, find

$$\int \frac{2-x}{\sqrt{x}(x+2)^2}\,dx.$$ **(10)**

Question 11

By considering a sequence of four transformations, or otherwise, sketch the graph of

$$y = -\left| |x-2|^2 - 4|x-2| - 5 \right|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

(7)

Question 12

It is given that

- a and x are positive real numbers such that $x \neq a$, $x \neq 1$ and $a > 1$.
- n is a positive integer such that $n > 1$.

Show that the equation

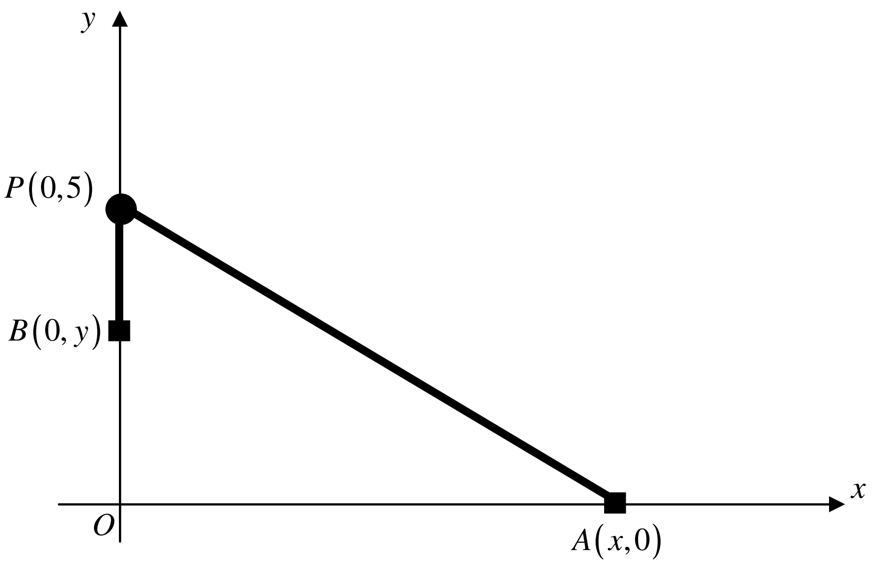
$$\sum_{r=1}^{n} \log_a x^r = \sum_{r=1}^{n} (\log_a x)^r,$$

can be written as

$$2(\log_a x)^n - n(n+1)\log_a x + (n-1)(n+2) = 0.$$

(12)

Question 13



Two particles, A and B, can move on the positive x axis and positive y axis respectively. They are connected with a rope which remains taut at all times.

Particle A has coordinates $(x,0)$ metres, where $x \geq 0$ and particle B has coordinates $(0,y)$ metres, where $0 \leq y \leq 5$.

The rope connecting the two particles has a length of 15 metres and passes over a small fixed pulley located at $P(0,5)$ metres.

a) Show that

$$\frac{dy}{dx} = \frac{x}{y+10}.$$ **(4)**

At a given instant the particle A is at the point with coordinates $(12,0)$ metres and moving away from O with a speed of 6.5 metres per second.

b) Find the rate at which the particle B is rising at that instant. **(5)**

Question 14

It is given that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=k^n,$$

where n and k are positive integer constants.

a) By considering the binomial expansion of $(1+x)^n$, find the value of k. **(3)**

b) By considering the coefficient of x^n in

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

simplify fully

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$ **(4)**

Question 15

A curve is given by the parametric equations

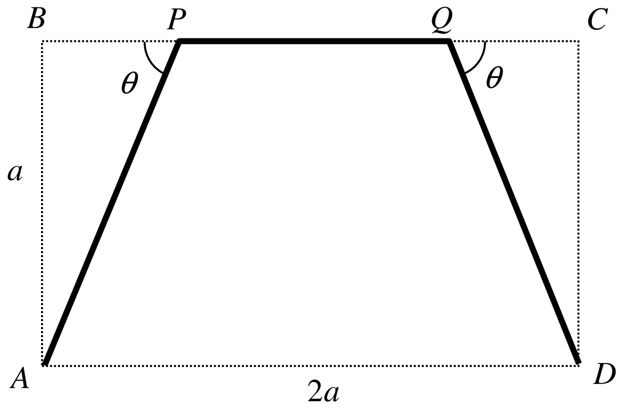
$$x=\sin\theta,\quad y=\theta\cos\theta,\quad -\pi<\theta<\pi.$$

The tangents to the curve, at the points where $\theta=-\frac{\pi}{4}$ and $\theta=\frac{\pi}{4}$, are parallel to one another , at a distance d apart.

Show that

$$d=\sqrt{\frac{8\pi^2-32\pi+32}{\pi^2-8\pi+32}}.$$ **(15)**

Question 16



The figure above shows a network $APQD$ inside a rectangle $ABCD$, where $|AB| = a$ and $|BC| = 2a$. The endpoints of the network A and D are fixed. The points P and Q are variable so that they always lie on BC with $|AP| = |QD|$. The angles BPA and CQD are both equal to θ. A particle travels with constant speed v on the sections AP and QD, and with constant speed $\frac{5}{3}v$ on the section PQ.

Let T be the total time for the journey $APQD$.

Given that the positions of the points P and Q can be varied as appropriate, show that the minimum value of T is $\dfrac{14a}{5v}$, fully justifying that this is the minimum value.

(15)

Question 17

It is given that the angles A, B and C are the three angles of a triangle ABC with $B \neq 90°$.

Given further that

$$\sin A - \sin(B - C) = \frac{\cos(B - C)}{\tan B},$$

show that the triangle ABC is right angled. **(10)**

Question 18

It is given that a function with equation $y = f(x)$ is a solution of the following differential equation.

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0.$$

Show with a clear method that

$$\int_1^x f(x)\ dx = \left(x^2-1\right)f'(x).$$ **(10)**

Question 19

$$I = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + \sin 2x}\ dx$$

Use appropriate integration techniques to show that

$$I = A\ln N + B\ln M\ ,$$

where A, B, N and M are integers to be found. **(12)**

Question 20

It is given that

$$(\arcsin x)^3 + (\arccos x)^3 = k\pi^3, \ |x| \le 1,$$

for some constant k.

a) Show that a necessary but not sufficient condition for the above equation to have solutions is that

$$k \ge \frac{1}{32}.$$ **(10)**

b) Solve the equation given that it only has one solution. **(4)**

c) Given instead that that $k = \frac{7}{96}$, find the two solutions of the equation, giving the answers in the form $x = \sin(a\pi)$, where $a \in \mathbb{Q}$. **(8)**

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