IYGB

Special Paper R

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

Show with a detailed method that

$$\frac{d}{dx}\left[\ln\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right)\right]=\frac{1}{\sqrt{e^x+1}}.$$

(7)

Question 2

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{r=2}^{\infty}\left[\frac{r^3-1}{r^3+1}\right].$$

(7)

Question 3

The vectors **a** and **b** are such so that

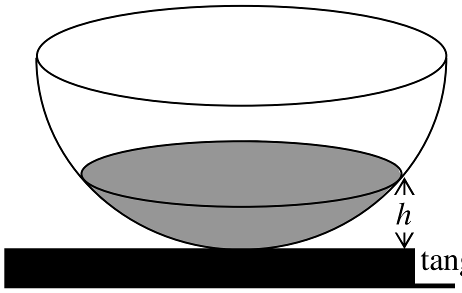
$$|\mathbf{a}|=3, \ |\mathbf{b}|=12 \quad \text{and} \quad \mathbf{a}\cdot\mathbf{b}=18 .$$

Show clearly that

$$|\mathbf{a}-\mathbf{b}|=3\sqrt{13}.$$

(7)

Question 4



The figure above shows a hemispherical bowl of radius r containing water up to a height h. The water in the bowl is in the shape of a minor spherical segment. It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h.

Show by integration that the volume V of the minor spherical segment is given by

$$V = \frac{1}{3}\pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

(8)

Question 5

Use trigonometric algebra to fully simplify

$$\arctan\left[\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right], \; 0 < x < \frac{\pi}{4},$$

giving the final answer in terms of x.

(8)

Question 6

Show that for all positive real numbers a and b

$$a^3 + b^3 \geq a^2 b + ab^2 .$$ **(7)**

Question 7

A man walked from his village to the nearby town in 2 hours and 14 minutes.

His return journey over the same route took him 2 hours and 2 minutes.

It is further known that the man always walks at 5 km h^{-1} uphill, at 6 km h^{-1} on flat ground and at 7 km h^{-1} downhill.

Given that the distance between the village and the town is 12.5 km, determine how long the flat distance between the village and the nearby town is. **(10)**

Question 8

Water is leaking out of a hole at the base of a cylindrical barrel with constant cross sectional area and a height of 1 m.

It is given that t minutes after the leaking started, the volume of the water left in the barrel is $V \text{ m}^3$, and its height is $h \text{ m}$.

It is assumed that the water is leaking out, in m^3 per minute, at a rate proportional to the square root of the volume of the water left in the barrel. The barrel was initially full and 5 minutes later half its contents have leaked out.

If T is the time taken for the barrel to empty, find h when $t = \frac{1}{2}T$. **(14)**

Question 9

$$f(x) \equiv \frac{x-k}{x^2-4x-k}, \quad x \in \mathbb{R}, \quad x^2-4x-k \neq 0,$$

where k is a constant.

Given that the range of the function is all the real numbers determine the range of possible values of k. **(12)**

Question 10

Sum the following series of infinite terms.

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\ldots$$ **(7)**

Question 11

Determine the value of x and the value of y in the following equation

$$15^{3x-2} \times 6^{1-2y} = 6.25, \quad (3x-2) \in \mathbb{N}.$$ **(8)**

Question 12

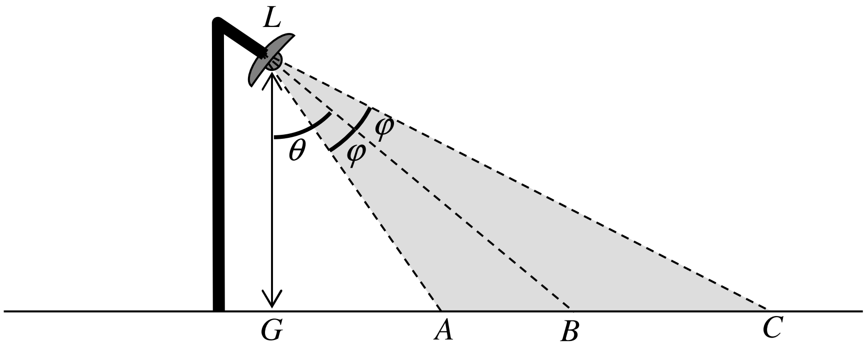
Use integration by parts to find a simplified exact value for

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) \; dx.$$ **(12)**

You may assume that

$$\int \operatorname{cosec} x \; dx = \ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \text{constant}.$$

Question 13



The figure above shows a spotlight L, beaming down on level ground, where L is mounted at a height of 12 metres and the point G is directly below L.

The bulb emits light in the shape of a cone, whose axis of symmetry LB is angled at $\theta°$ to the vertical.

The beam is $\varphi°$ wide all the way round the axis of symmetry of the light cone.

a) Show that the length of AB is

$$|AB| = \frac{|LB|^2}{12(\cot\varphi + \tan\theta)}.$$ **(10)**

The lengths of LB and AB are $8\sqrt{3}$ metres and 8 metres, respectively.

b) Show further that

$$\tan\varphi = \frac{1}{11}\left(6+\sqrt{3}\right).$$ **(4)**

Question 14

The straight parallel lines l_1 and l_2 have respective equations

$$y = 3x + 2 \quad \text{and} \quad y = 3x - 4 .$$

The straight line l_3, passing through the point $P(9,13)$, intersects l_1 and l_2 at the points A and B respectively.

Given that $|AB| = \sqrt{18}$ determine the possible equation of l_3. **(15)**

Question 15

$$y = \frac{1}{\sqrt{ax+b}}, \quad x \geq 0 ,$$

where a and b are positive constants.

Show, by a detailed method, that

$$\frac{d^n y}{dx^n} = \frac{(-1)^n y \ (2n)!}{n!} \left(\frac{a}{4(ax+b)} \right)^n . \qquad \textbf{(10)}$$

Question 16

The trapezium rule with n equally spaced intervals is to be used to estimate the value of the following integral

$$\int_0^1 2^x \ dx .$$

Show that the value of this estimate is given by

$$\frac{1}{2n} \left[\frac{2^{\frac{1}{n}} + 1}{2^{\frac{1}{n}} - 1} \right]. \qquad \textbf{(9)}$$

Question 17

The curve C has equation

$$y = A\ln|x| + Bx^2 + x \,,\ \ x \in \mathbb{R} \,,$$

where A and B are non zero constants.

The curve has stationary points at $x = -1$ and at $x = 2$.

Sketch the graph of C. **(10)**

The sketch must include …

- … the coordinates of all the stationary points.
- … the equations of the asymptotes of the curve.

You need not find any intercepts with the coordinate axes.

Question 18

$$f(x) = 1 + 2x - x^3 + \tfrac{1}{4}x^4 \ , \quad x \in \mathbb{R} \,.$$

a) Extract the square roots of $f(x)$. **(5)**

b) Hence, or otherwise, solve the equation

$$x^4 - 4x^3 + 8x = 32 \,,\ \ x \in \mathbb{R} \,.$$ **(7)**

Question 19

Solve the trigonometric equation

$$8\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) = \cos 3x \,, \qquad 0 \le x < \frac{\pi}{2} \,,$$

giving the answer in terms of π. **(12)**

Question 20

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x \, dx \, . \qquad (12)$$
