IYGB

Special Paper L

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

The function $y = f(x)$ satisfies the following relationships

$$\int_0^1 5f(x)\,dx + \int_1^2 3f(x)\,dx = 31$$

$$\int_0^2 \tfrac{1}{3}f(x)\,dx + \int_1^2 2f(x)\,dx = 17.$$

Determine the value of

$$\int_0^2 f(x)\,dx. \qquad \textbf{(6)}$$

Question 2

Two coplanar circles, with respective radii $\sqrt{2}$ and $\sqrt{6}$, intersect each other at the points A and B.

The tangent to one of the circles at A, intersects the tangent to the other circle at A at right angles.

Show that the total area enclosed by the two circles is

$$\tfrac{1}{2}\left(19\pi + 6\sqrt{3}\right). \qquad \textbf{(7)}$$

Question 3

Show that the following logarithmic equation has no real solutions.

$$\log_{x^2+2}\left[2x^4 - 2x^3 + 7x^2 - 2x + 5\right] = 2,\ x \in \mathbb{R}. \qquad \textbf{(9)}$$

Question 4

The point $P\left(\frac{1}{2},\frac{1}{2}\right)$ lies on the curve given parametrically as

$$x = \cos 2t\,, \qquad y = 4\sin^3 t\,, \quad 0 \le t < 2\pi\,.$$

The tangent to the curve at P meets the curve again at the point Q.

Determine the exact coordinates of Q. **(8)**

Question 5

It is given that θ and φ satisfy the relationship

$$\tan\theta = \frac{3\sin\varphi\cos\varphi}{1-3\sin^2\varphi}\,.$$

Show clearly that

$$\tan(\theta-\varphi) = 2\tan\varphi\,.$$ **(7)**

Question 6

Solve the equation

$$\frac{\sqrt{5x+6}+\sqrt{5x-6}}{\sqrt{5x+6}-\sqrt{5x-6}} = 3\,,\ x > \frac{6}{5}\,.$$ **(8)**

Question 7

By considering the binomial expansion of

$$\frac{1}{(1-\cos\theta)^2},$$

sum each of the following series.

- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right].$ **(4)**

- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right].$ **(3)**

Question 8

The function f is defined as

$$f(x) \equiv 4x^3 - 12x^2 + 8x, \quad x \in \mathbb{R}, \ 0 \le x \le 3.$$

Find the range of f, and hence sketch its graph, showing clearly the coordinates of any relevant points. **(10)**

Question 9

A cubic curve has equation

$$f(x) \equiv x^3 + x^2(1-2a-3b) + x(6ab-2a-b) + 6ab, \quad x \in \mathbb{R},$$

where a and b are constants.

Given that the equation $f(x) = 0$ has 3 equal real roots, determine the value of a and the value of b. **(7)**

Question 10

Prove the validity of each of the following trigonometric identities.

i. $\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta \equiv \tan\theta + \cot\theta$. (5)

ii. $\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta) \equiv \sec\theta + \operatorname{cosec}\theta$. (5)

Question 11

Show that the area of the triangular region bounded by

$$x^2 - y^2 + x + 3y = 2 \qquad \text{and} \qquad 2x - y = 2,$$

is $2\frac{1}{12}$ square units. (**12**)

Question 12

The sum of the first n terms of an arithmetic series is m, $m \in \mathbb{N}$.

The sum of the first m terms of the same arithmetic series is n.

Use algebra to show that the sum of the first $(m+n)$ terms of the series is $-m-n$. (**11**)

Question 13

The function $y = f(x)$ is defined in the largest possible real domain by

$$f(x) \equiv \ln\left[x^2 - 2x + 2\right].$$

Sketch the graph of $f(x)$ and determine an exact simplified value for the area of the finite region bounded by the graph of $f(x)$ and the coordinate axes. (**12**)

Question 14

A curve has equation

$$y^2 = \ln|3x-12|, \quad x \in \mathbb{R}, \quad x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $y=1$, is revolved by 2π radians in the x axis.

Find the exact volume of the solid formed. **(12)**

Question 15

Determine, in exact form where appropriate, the two real roots of the equation

$$(x+1)^6 - 2(x-1)^6 = \left(x^2-1\right)^3.$$ **(10)**

Question 16

A general curve C has equation

$$y = x^m (x-1)^n,$$

where $x \in \mathbb{R}$, $m \in \mathbb{N}$, $m \geq 2$, $n \in \mathbb{N}$, $n \geq 2$.

Sketch in four separate of axes, the 4 separate shapes which C can take, $m \geq 2$.

The sketches must contain the coordinates of any stationary points. **(10)**

Question 17

$$I = \int_{1}^{3} (3-x)^7 (x-1)^7 \ dx .$$

Show that

$$I = \frac{(7!)^2 \times 2^{15}}{15!} .$$ **(12)**

Question 18

From a thin sheet of metal, a circular sector of area A is removed.

The circular sector is folded without any overlapping into the curved surface of a right circular cone of volume V .

The measurements of the circular sector are such so that V is maximum.

Find the angle subtended by the circular sector at its centre and show further that the maximum value of V is

$$\frac{1}{9}\sqrt[4]{\frac{12A^6}{\pi^2}} .$$ **(14)**

Question 19

The points $A(-3,1,5)$, $B(1,1,1)$ and $C(-1,5,-1)$ are three of the vertices of the kite $ABCD$, which is circumscribed by a circle.

a) Given that $|AB| = |AD|$ and $|BC| = |DC|$, find the exact coordinates of D. **(6)**

A smaller circle is circumscribed by the kite, and a smaller kite similar to $ABCD$ is circumscribed by the smaller circle.

b) Determine in exact form the area of the smaller kite. **(8)**

Question 20

The non zero functions $u(x)$ and $v(x)$ satisfy the integral equations

$$\int u(x)\,dx = x^2 u(x) \qquad \text{and} \qquad \int u(x)v(x)\,dx = \left[\int u(x)\,dx\right]\left[\int v(x)\,dx\right].$$

Determine, in terms of an arbitrary constant, a simplified expression for $u(x)$ and a similar expression for $\left[v(x)\right]^2$. **(14)**
