IYGB

Special Paper K

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

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Question 1

A rectangle has perimeter P and area A.

Show that

$$A \le f(P),$$

where $f(P)$ is a simplified expression to be found. **(6)**

Question 2

A cubic curve C has equation

$$y = ax^3 + bx, \quad x \in \mathbb{R},$$

where a and b are non zero constants with $a > 0$.

The curve C' is the reflection of C about the straight line with equation $y = x$.

The straight line with equation $y = -x$ is a tangent to both C and C' at the origin O.

Given that the finite region bounded by C and C' has area 9, find the value of a.

(10)

Question 3

The function $y = f(x)$, $x \in \mathbb{R}$ satisfies

$$f(x) + 2f(2-x) = x^2, \quad t \in \mathbb{R}, \quad t \ge 0.$$

Determine a simplified expression for $y = f(x)$. **(7)**

Question 4

$$y^2 - x^2 = 4, \ |y| \ge 2.$$

Use differentiation to show that

$$\frac{d^2y}{dx^2} = \frac{4}{y^3}. \qquad (7)$$

Question 5

Find in exact form the two real solutions of the equation

$$\frac{\left(x^3 - 3x^2 + 3x - 3\right)^2}{(x-1)^6} = 225. \qquad (10)$$

Question 6

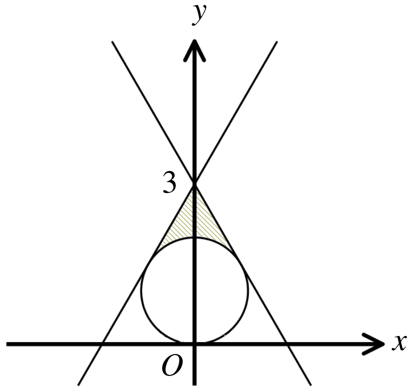
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k} [u_i] \ = \ u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{n=2}^{\infty} \left[1 - \frac{1}{2 - 2^n} \right]. \qquad (7)$$

Question 7



A circle with equation

$$x^2 + (y-1)^2 = 1.$$

Two tangents to the circle are drawn so both are passing though the point $(0,3)$.

Determine in exact simplified form the value of the finite region between the circle and the two tangents, shown shaded in the figure above. **(10)**

Question 8

Prove the validity of the following trigonometric identity.

$$\frac{\sin 4\theta - 8\sin^3\theta\cos\theta}{\sin\theta} \equiv 4\cos 3\theta, \quad \theta \neq n\pi. \qquad \textbf{(6)}$$

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Question 9

A family of straight lines, has equation

$$y = m(x-1)+2, \quad x \in \mathbb{R},$$

where m is a parameter.

From the above family of straight lines, determine the equations of any straight lines whose distance from the origin O is 1 unit. **(11)**

Question 10

The points A and B, have respective position vectors $\mathbf{a}$ and $\mathbf{b}$, relative to a fixed origin O.

The point C lies on AB produced such that $|AB| : |AC| = 1 : 4$

The point D lies on OB produced such that $|OB| : |OD| = 1 : k$, where $|OB| : |OD| = 1 : k$ is a scalar constant.

Given that AB is perpendicular to CD show that

$$k = \frac{3|\mathbf{a}|^2 - 7\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}.$$ **(10)**

Question 11

A curve has Cartesian equation

$$2y-1 = (x-1)(y-1)^2, \quad x \geq 0.$$

Make y the subject of the above equation, to show that

$$y = \frac{\sqrt{x}}{\sqrt{x} \pm 1}.$$ **(10)**

Question 12

Sketch the graph of the curve with equation

$$4\left[\log_{10} x\right]^2 + \left[\log_{10} y\right]^2 = 1, \ x > 0, \ y > 0.$$

The sketch must include the coordinates of any points where the tangent to the curve is parallel to the coordinate axes. **(8)**

Question 13

Find an exact value for

$$\int_0^{\pi} \frac{x \sin x}{\sqrt{4 - \cos^2 x}} \, dx .$$

You my assume without proof that

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + \text{constant} .$$ **(11)**

Question 14

A curve C has equation

$$y = \frac{3|x| - 1}{2x^2 + 2 - |x+2|}, \quad x \in \mathbb{R}, \quad x \neq 0, \quad x \neq \frac{1}{2} .$$

Find, in exact simplified surd form, the y coordinate of the stationary point of C. **(13)**

Question 15

The finite region R is bounded by the curve with equation $x = \cos y^2$, the y axis and the straight line with equation $y = \frac{1}{2}\sqrt{\pi}$.

Determine, in exact simplified form, the volume of the solid formed by revolving R by a full turn in the x axis.

(12)

Question 16

$$\sin\left(3x - \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = \sin x, \quad 0 \le x < \pi.$$

Determine the solutions of the above trigonometric equation, giving the answers in terms of π.

(11)

Question 17

$$I = \int_0^{\frac{1}{2}\pi} 4\sin x\sqrt{\cos 2x} \; dx.$$

By using an appropriate substitution or substitutions, show that

$$I = 2 - \sqrt{2}\ln\left(1 + \sqrt{2}\right).$$

(12)

Question 18

The points P and Q are two distinct points which lie on the curve with equation

$$y=\frac{1}{x}, \; x\in\mathbb{R}, \; x\neq 0.$$

P and Q are free to move on the curve so that the straight line segment PQ is a normal to the curve at P.

The tangents to the curve at P and Q meet at the point R.

Show that R is moving on the curve with Cartesian equation

$$\left(y^2-x^2\right)^2+4xy=0.$$ **(16)**

Question 19

The non zero function $f(x)$ satisfies the integral equation

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx, \quad f(0)=\tfrac{1}{4}.$$

Use the substitution $f(x)=\left(\frac{dy}{dx}\right)^2$, to find a simplified expression for $f(x)$. **(11)**

Question 20

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$ **(12)**