IYGB

Special Paper J

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

The gradient function of a curve satisfies the following relationship

$$(x+1)\frac{dy}{dx}+16=4(2x+y).$$

The normal to the curve at the point P has equation $x+3y=6$.

Determine the coordinates of P. **(7)**

Question 2

Solve the following exponential equation

$$16+8^{x+1}-4^{x+1}-2^{x+5}=0, \quad x \in \mathbb{R}.$$ **(7)**

Question 3

The function f is defined, in terms of the real constant k, by

$$f(x) \equiv x^3+kx^2+x+1, \quad x \in \mathbb{R}.$$

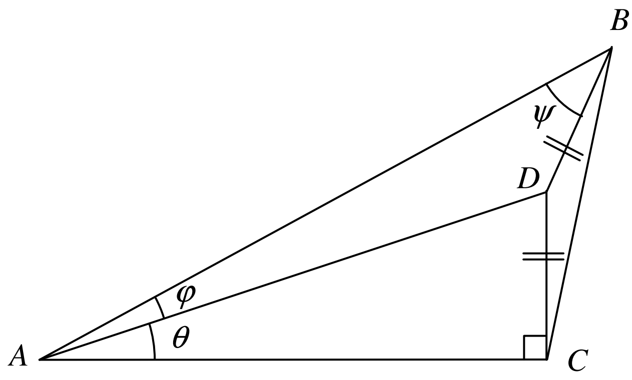
Investigate the number of turning points of f for different values of k, further distinguishing which ones are stationary **(8)**

Question 4

Solve the following logarithmic equation, over the largest real domain.

$$\log_{2-x}\left[2x^2-1\right]=2, \quad x \in \mathbb{R}.$$ **(6)**

Question 5



The point D lies inside the triangle ABC, so that $|DB| = |DC|$ and $\measuredangle DCA = \frac{1}{2}\pi$.

Let $\theta = \measuredangle DAC$, $\varphi = \measuredangle BAD$ and $\psi = \measuredangle ABD$.

Show that

$$\sin\psi = \sin\theta \sin\varphi. \qquad \textbf{(6)}$$

Question 6

If $\tan 3y = 3\tan x$ show clearly that

$$\frac{dy}{dx} = \frac{1}{1+8\sin^2 x}. \qquad \textbf{(8)}$$

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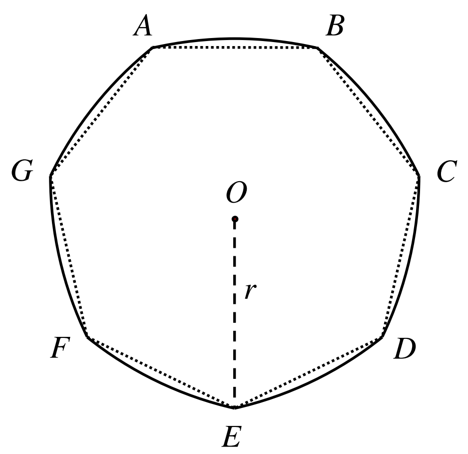
Question 7

The function $y = f(t)$ is defined by the integral

$$f(t) \equiv \int_0^1 (x-t)^2 + t^2 \ dx, \quad t \in \mathbb{R}, \ t \geq 0.$$

Determine the range of y. **(9)**

Question 8



The figure above shows a Reuleaux heptagon, $ABCDEFG$, constructed as follows.

Firstly a regular heptagon $ABCDEFG$ with centre at O and radius r is constructed. This is shown dotted in the figure.

A circular arc $\overset{\frown}{AB}$ is drawn with centre at E and radius EA. A second circular arc $\overset{\frown}{BC}$ is drawn with centre at F and radius FB.

A third circular arc $\overset{\frown}{CD}$ is drawn with centre at G and radius GC and the process is repeated , forming a curved heptagon known as a Reuleaux heptagon.

Show that the area of this Reuleaux heptagon is

$$r^2\left[2\pi\cos^2\left(\frac{\pi}{14}\right) - \sin\left(\frac{\pi}{7}\right)\right].$$ **(9)**

Question 9

Solve the following simultaneous equations.

$$x + y = \tfrac{1}{5}\pi \,, \quad \cos x + \cos y = 0 \,, \quad 0 \le x < 2\pi \,. \qquad (9)$$

Question 10

Solve the following system of equations for $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$.

$$xy + 2yz - xz = 5 \,,$$

$$2xy - 2yz - xz = 9 \,,$$

$$3xy + 4yz + xz = 0 \,. \qquad (10)$$

Question 11

It is given that a, b and c are consecutive terms of an arithmetic progression.

It is further given that

$$a\cos^2\frac{x}{2} - (2a+c)\sin^2\frac{x}{2} = a\cos x - b(1+\sin x)\,, \qquad x \in \mathbb{R}\,.$$

Show clearly that

$$\tan x = -1 \,. \qquad (10)$$

Question 12

A sequence of positive integers is generated by the formula

$$u_n = 2n^3 - 57n^2 - 120n + 9200 \,, \quad n \in \mathbb{N} \,.$$

Determine the largest value of n, such that $u_n > u_{n+1}$. **(10)**

Question 13

$$y = \arccos x, \ -1 \le x \le 1.$$

a) Show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}. \qquad \textbf{(2)}$$

The Chebyshev polynomials of the first kind $T_n(x)$ is a family of functions defined as

$$T_n(x) = \cos(n \arccos x), \ -1 \le x \le 1, \ n \in \mathbb{N}.$$

b) Show further that

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}} \frac{d}{dx}\left[T_n(x)\right]\right] = \frac{-n^2 \, T_n(x)}{\sqrt{1-x^2}}. \qquad \textbf{(10)}$$

Question 14

$$I = \int_{-\infty}^{\infty} \left| x^3 \left(2^{-x^2}\right) \right| \, dx$$

It is given that $I \approx 2$.

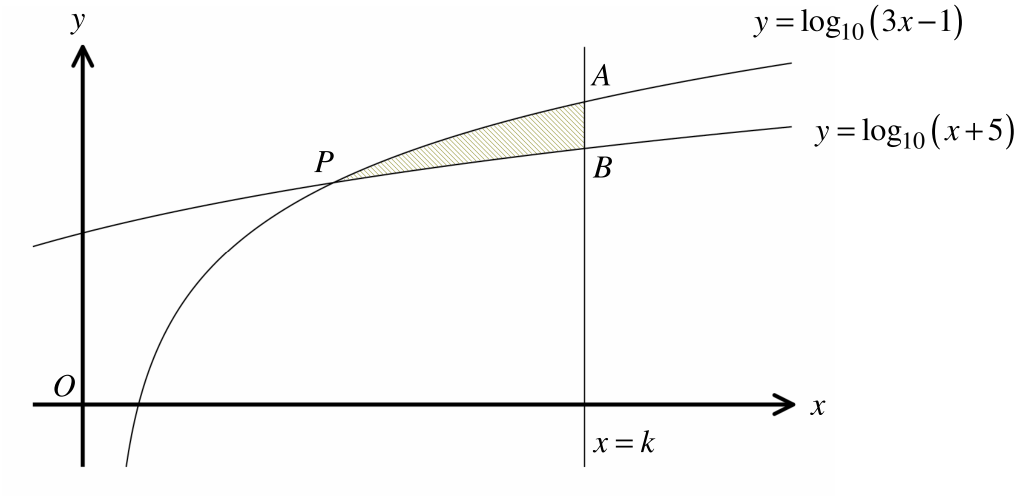
Use this fact to estimate the value of $\ln 2$ correct to 1 significant figure. **(11)**

Question 15

By using the substitution $y = xu$, where $u = f(x)$, or otherwise, find a simplified general solution for the following differential equation.

$$x\frac{dy}{dx} = 2x^2 + 2xy + y. \qquad \textbf{(9)}$$

Question 16



The figure above shows the graphs of the curves with equations

$$y = \log_{10}(3x-1), \quad \text{and} \quad y = \log_{10}(x+5).$$

The two curves intersect at the point P

The straight line with equation $x = k$, $k > 3$, meets the graph of $y = \log_{10}(3x-1)$ at the point A and the graph of $y = \log_{10}(x+5)$ at the point B, so that $|AB| = \frac{1}{2}$.

Determine the value of the area of the finite region, bounded by the two curves and the straight line $x = k$, shown shaded in the above figure. **(14)**

Question 17

Relative to a fixed origin O located at the point with coordinates $(0,0,0)$, the points $A(8,1,4)$ and $B(4,-1,8)$ are given.

A circle, with centre at the point P and radius r, is drawn so that the three sides of the triangle OAB are tangents to this circle.

Determine the coordinates of P and the exact value of r. **(15)**

Question 18

$$f(x) \equiv \frac{1-7x}{(1+x)(1-3x)}, \ -\tfrac{1}{3} < x < \tfrac{1}{3}.$$

Show that $f(x)$ can be written in the form

$$f(x) = 1 - \sum_{r=1}^{\infty} \left[x^r g(r) \right],$$

where $g(r)$ is a simplified function to be found. **(12)**

Question 19

A curve C has equation

$$x^2 + xy + y^2 = 1. \quad , \quad 0 \le x \le 3.$$

By seeking a suitable parameterization of C in the form

$$x = A\cos\theta + B\sin\theta \quad \text{and} \quad y = A\cos\theta - B\sin\theta,$$

where A and B are suitable constants,

determine the area of the finite region in the first quadrant, bounded by the curve and the coordinate axes.

You may assume that the curve does not intersect itself. **(14)**

Question 20

Use partial fractions followed by integration by parts to show that

$$\int_0^{\infty} \left[\frac{x^2+3x+3}{(x+1)^3} \right] e^{-x} \sin x \, dx = \frac{1}{2}.$$ **(14)**

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