IYGB

Special Paper I

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

Find in simplified form, in terms of n, the value of

$$\sum_{r=1}^{2n}\left[(3r-2)(-1)^r\right].$$

(4)

Question 2

The four sides of a square, $ABCD$, are tangents to a circle of radius $\sqrt{2}$.

The diagonal BD intersects the circle at the points P and Q.

Determine in exact simplified form the length of AP. **(6)**

Question 3

The point P lies on the curve with equation $y = x^2$, $x > 0$.

The finite region bounded by the curve, the tangent to the curve at P and the y axis has area of 72 square units.

Determine the x coordinate of P. **(7)**

Question 4

A curve has equation

$$y = \ln(4-x), \quad x \in \mathbb{R}, \ x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $x = 2$, is revolved by 2π radians in the y axis.

Find the exact volume of the solid formed. **(10)**

Question 5

Solve the inequality

$$x^2 - x + \frac{1}{x} > 1 .$$

(8)

Question 6

The function $y = f(x)$ satisfies the differential equation

$$\frac{d}{dx}\left(yx^2\right) = \frac{dy}{dx}\frac{d}{dx}\left(x^2\right), \quad x > 0$$

subject to the condition $y = 4$ at $x = 3$.

Find a simplified expression for $y = f(x)$. **(10)**

Question 7

$$f(x) \equiv 4^{x+1} \times 3^{1-2x}, \quad x \in \mathbb{R} .$$

Determine the value of $f(a)$, where $a = \dfrac{\log_{10} 2}{\log_{10} 4 - \log_{10} 9}$. **(10)**

Question 8

Show clearly that

$$\tan\frac{3\pi}{8} - \tan\frac{\pi}{8} - \tan\frac{3\pi}{8}\tan\frac{\pi}{8} = 1 .$$

You may not use verification in this question. **(9)**

Question 9

Relative to a fixed origin O, the position vectors of two points A and B are denoted by $\mathbf{a}$ and $\mathbf{b}$. The point P is the foot of the perpendicular from O to the straight line through A and B.

Show that if $\mathbf{p}$ denotes the position vector of P, then

$$\mathbf{p} = \mathbf{a} - \frac{\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{b})}{|\mathbf{a}-\mathbf{b}|^2}. \qquad (10)$$

Question 10

The real functions f and g have a common domain $0 \le x \le 4$, and defined as

$$f(x) \equiv (x-1)(x-2)(x-3) \quad \text{and} \quad g(x) \equiv \int_0^x f(t)\, dt.$$

Use a detailed algebraic method to determine the range of g. **(13)**

Question 11

A curve in the x-y plane has equation

$$x^2 + y^2 + 6x\cos\theta - 18y\sin\theta + 45 = 0,$$

where θ is a parameter such that $0 \le \theta < 2\pi$.

Given that curve represents a circle determine the range of possible values of θ. **(10)**

Question 12

Solve the exponential equation

$$\frac{3^{2x} + 5^{2x}}{34} = 15^{x-1}. \qquad (11)$$

Question 13

Find an exact value for the following integral.

$$\int_0^{\pi} x\sin^3 x \, dx .$$

(11)

Question 14

It is given that the three angles of a triangle α, β and γ satisfy the relationship

$$\tan\frac{\alpha}{2} = \left(1+\tan^2\frac{\alpha}{2}\right)\sin(\beta-\gamma) .$$

Assuming that the triangle is not right angled, show that

$$3\tan\gamma = \tan\beta .$$

(11)

Question 15

The variables x, y and w are related by the equations

$$y = xy + 1 \qquad \text{and} \qquad w = x^3 + wx .$$

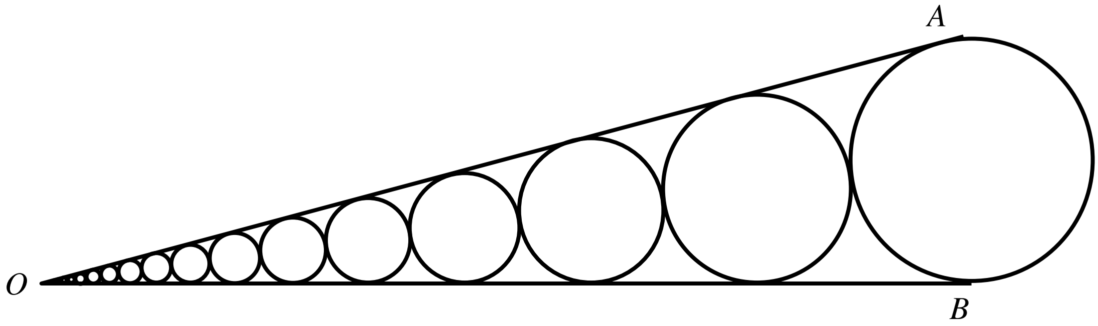
At a certain instant the rate of change of y with respect to t is increasing at the constant rate of 2, in suitable units.

At the same instant the rate of change of w with respect to t is decreasing at the constant rate of 8, also in suitable units.

Determine the value of w at that instant.

(12)

Question 16



The figure above shows a infinite sequence of circles of decreasing radius, the radius of the larger circle being $\frac{4}{3}$.

The centres of these circles lie on a straight line. The straight lines OA and OB are tangents to every circle in the sequence, the angle AOB denoted by 2θ.

Given that the total area of these circles is 2π , determine the value of θ. **(14)**

Question 17

Determine the two real roots of the equation

$$(x-7)(x-3)(x+5)(x+1)=1680.$$ **(10)**

Question 18

A family of functions, known as the Chebyshev polynomials of the first kind $T_n(x)$, is defined as

$$T_n(x) = \cos(n \arccos x), \ -1 \le x \le 1, \ n \in \mathbb{N}.$$

Evaluate the following integral

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}} \, dx. \qquad \textbf{(10)}$$

Question 19

The function f maps points from a Cartesian x-y plane onto the same Cartesian x-y plane by

$$f:(x,y) \mapsto \left(\frac{1-x^2-y^2}{x^2+(1-y)^2}, \ \frac{-2x}{x^2+(1-y)^2} \right), \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad (x,y) \ne (0,1).$$

The set of points, S, which lie on the x axis are mapped by f onto a new set of points S', which in turn are mapped by f onto a new set of points S''.

Use algebra to determine the equation of S''. **(12)**

Question 20

A curve, defined in the largest real domain, has equation

$$y = \ln\big||x+4|-6\big|.$$

Determine, in its simplest form, an expression for $\frac{dy}{dx}$. **(12)**
