IYGB

Special Paper H

Time: 3 hours 30 minutes

Candidates may use any calculator allowed by the Regulations of the Joint Council for Qualifications.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $=T$, Number of non attempted questions $=N$, Percentage score $=P$.

$P=\frac{1}{2}T+N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

The cubic equation

$$x^3 + kx + 4 = 0,$$

where k is a constant has 2 distinct real roots.

Determine the exact value of k. **(7)**

Question 2

The function f satisfies the following three relationships

i. $f(3n-2) \equiv f(3n) - 2, \ n \in \mathbb{N}$.

ii. $f(3n) \equiv f(n), \ n \in \mathbb{N}$.

iii. $f(1) = 25$.

Determine the value of $f(25)$. **(4)**

Question 3

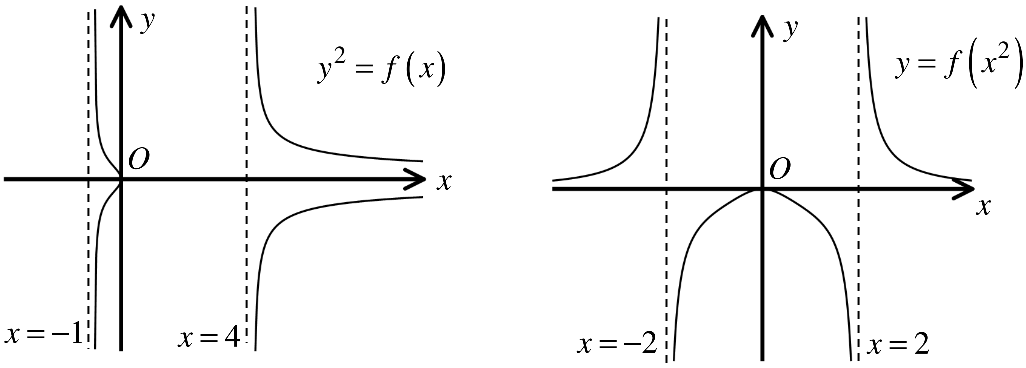
A family of circles is passing through the points with coordinates $(2,1)$ and $(4,5)$

Show that the equation of every such circle has equation

$$x^2 + y^2 + 2x(2k-9) + 2ky = 6k - 41,$$

where k is a parameter. **(9)**

Question 4



The figures above show two transformations of a function with equation $y = f(x)$, the graph of $y^2 = f(x)$ in the first set of axes, and the graph of $y = f(x^2)$ in the second set of axes.

The equations of the vertical asymptotes for each graph are included in the figures.

The x axis is a horizontal asymptote for both graphs.

Sketch a possible graph of $y = f(x)$, showing all relevant details. **(5)**

Question 5

Determine, in the form $y = f(x)$, a simplified solution for the following differential equation.

$$\frac{dy}{dx}\cos x + 4y^2 \sin x = \sin x, \quad y = \frac{15}{34} \text{ at } x = \frac{1}{3}\pi .$$ **(10)**

Question 6

It is given that for $x > 0$, $x \neq 1$ and $y > 0$, $y \neq 1$

$$\log_x y = \log_y x \qquad \text{and} \qquad \log_x (x - y) = \log_y (x + y).$$

Show that

$$x^4 - x^2 - 1 = 0.$$ **(9)**

Question 7

The straight line L_1, has gradient m, $m > 0$ and passes through the point $A(2,3)$.

Another straight line L_2 is perpendicular to L_1 and passes through the point $B(2,k)$, $k \neq 3$.

The point C is the intersection of L_1 and L_2.

Determine the y coordinate of C, in terms of k and m, and given further that the triangle ABC is isosceles, prove that $m = 1$. **(10)**

Question 8

Determine, in exact simplified form, the area of the finite region bounded by the curves with equations

$$y = 1 + \sqrt{x}, \ x \in \mathbb{R}, \ x \geq 0.$$

$$y = 4^{\frac{x}{9}}, \ x \in \mathbb{R}.$$ **(10)**

Question 9

The function f is defined as

$$f(x) = \sin\left(x + \frac{7\pi}{12}\right)\sin\left(x + \frac{\pi}{12}\right), \quad 0 \le x < 2\pi$$

Solve the equation

$$f(x) + f(-x) = f\left(\frac{\pi}{4} - x\right).$$ **(10)**

Question 10

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] \;=\; u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Show in detail that

$$\prod_{r=1}^{n}\left(\frac{1}{n-r+\frac{1}{2}}\right) \;=\; \frac{2^{2n+1} \times n!}{(2n+1)!}.$$ **(9)**

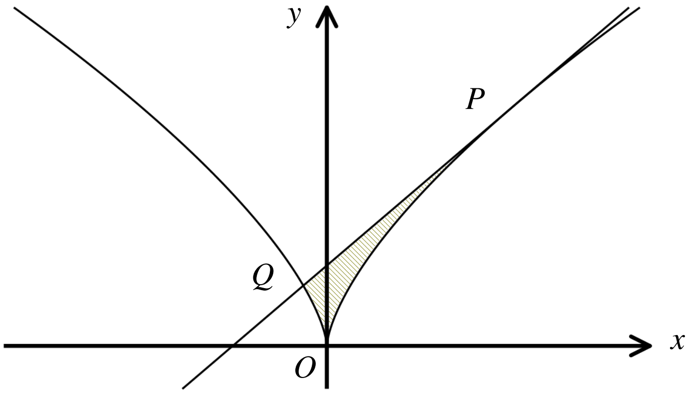
Question 11

Use algebra to solve the following simultaneous equations

$$\sqrt{\frac{x+y}{x}} + \sqrt{\frac{x}{x+y}} = \frac{5}{2} \qquad \text{and} \qquad 2x^2 + y^2 = 176,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$. **(10)**

Question 12



The figure above shows the curve with parametric equations

$$x = t^3, \quad y = t^2, \quad t \in \mathbb{R} .$$

The tangent to the curve at the point P meets the curve again at the point Q .

Given that the area of the finite region bounded by the curve and the tangent, shown shaded in the above figure, is $2\frac{7}{10}$ square units, determine the coordinates of P . **(12)**

Question 13

The point P lies on the curve C with equation

$$y = \sqrt{1 + 2e^{2x^2}} , \quad x \in \mathbb{R} .$$

Given that the tangent to C at P passes through the origin, determine the coordinates of P , correct to 3 significant figures. **(11)**

Question 14

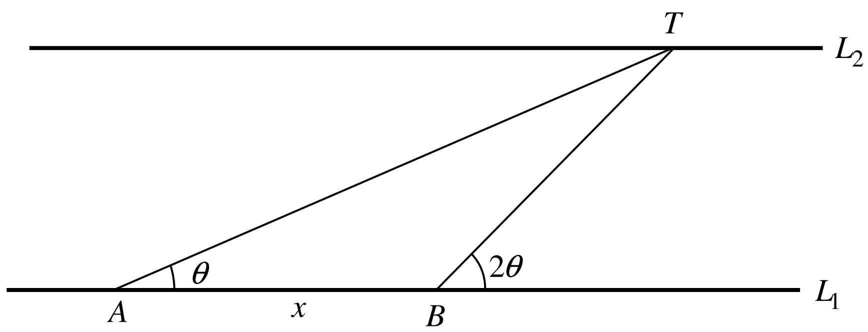
Find the value of

$$\sum_{r=0}^{\infty}\left[\frac{\sin^4\left(\pi\times 2^{r-2}\right)}{4^r}\right].$$

Hint: Express $\sin^4\theta$ in terms of $\sin^2\theta$ and $\sin^2 2\theta$ only. **(10)**

Question 15

In the following question you may not use the sine or the cosine rule.



The figure above shows the plan of a river whose banks are modelled as straight parallel lines L_1 and L_2.

The points A and B lie on L_1, so that $|AB| = x$.

A tree is positioned at the point T on L_2, so that AT and BT subtend angles of θ and 2θ, respectively.

The tree located at T has height h.

The angle of elevation of the top of the tree as viewed from A is θ.

Show that

$$h = 2x\sin\theta.$$

(10)

Question 16

$$I = \int_0^\infty \frac{1}{\left(x+\sqrt{x^2+1}\right)^2}\, dx .$$

a) Use the substitution $u = x+\sqrt{x^2+1}$ to find the value of I. **(9)**

b) Verify the answer to part **(a)** by a trigonometric substitution. **(7)**

Question 17

The points $A(14,1,15)$, $B(8,1,0)$ and $C(-16,7,-18)$ are three of the vertices of the kite $ABCD$. A circle of radius r is circumscribed by the kite.

Find the area of the kite and hence or otherwise determine, in exact simplified surd form, the value of r. **(11)**

Question 18

It is given that

$$I = \int_{\frac{1}{2}\pi}^{\pi} \frac{3+\cos x}{13+3\cos x+2\sin x}\, dx \quad \text{and} \quad J = \int_{\frac{1}{2}\pi}^{\pi} \frac{2+\sin x}{13+3\cos x+2\sin x}\, dx .$$

By considering two linear combinations in I and J, show that

$$I = \frac{1}{26}\left[3\pi - \ln\left(\frac{81}{16}\right)\right],$$

and find a similar expression for J. **(10)**

Question 19

A finite region in the x-y plane is defined by the inequalities

$$|x-1|+|y-1|<1 \quad \text{and} \quad |x(y-2)|>1.$$

Sketch in detail this region, showing clearly any relevant coordinates. **(11)**

Question 20

The point P has rational coordinates and lies on the curve C with equation

$$y = x^2 - 4x + 3, \quad x \in \mathbb{R}.$$

The straight line L is the normal to the C at P.

L meets the curve again at the point Q.

Given that $|PQ| = \sqrt{8}$, determine the possible coordinates of P and Q. **(16)**
