IYGB

Special Paper C

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

The functions f and g are defined by

$$f(x) \equiv 3\sin x, \; x \in \mathbb{R}, \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2, \; x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}g(x)$. **(2)**

b) Determine with justification the domain of $f^{-1}g(x)$. **(5)**

Question 2

The sum of the first k terms of a geometric progression is 180.

It is further given that the sum of the first k terms of this geometric progression is **twelve less** than its sum to infinity.

If the sum to infinity of the geometric progression is four times as large as its second term, use algebra to determine the value of k. **(10)**

Question 3

The point P lies on the straight line L_1, which is parallel to the vector $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and passes through the point with coordinates $(10,3,7)$, relative to an origin at $(0,0,0)$.

The point Q lies on another straight line L_2, which is in the direction of the vector $4\mathbf{i} - \mathbf{j} + \mathbf{k}$ and passes through the point with coordinates $(9,1,0)$.

The straight line L_3 is perpendicular to both L_1 and L_2, and meets L_1 and L_1 at the points P and Q, respectively.

Find the coordinates of P and Q. **(10)**

Question 4

Find the finite solution of the equation

$$\arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\arctan x .$$

(10)

Question 5

$$I = \int_{1.5}^{2} \frac{(x-2)\left(2x^2-5x-1\right)}{(x-1)(x-3)}\ dx .$$

Use appropriate integrations techniques to show that

$$I = \frac{5}{4} - \ln k ,$$

where k is a positive integer. **(11)**

Question 6

The island state of Trigland has declared an exclusive economic zone into the sea, which is within 6 miles from every point of its coastline.

The island of Trigland is a rectilinear triangle of sides 13, 14 and 15 miles.

Determine, in exact form, the total economic zone of Trigland, which consists of land and sea. **(7)**

Question 7

$$f(x,y) \equiv \sqrt{\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5} .$$

Simplify $f(x,y)$ in a form not involving square roots. **(8)**

Question 8

The straight line l has equation

$$(2+a)x+(2-a)y=2-5a,$$

where a is a constant.

Show that l passes through a fixed point P for all values of a. **(8)**

Question 9

A curve is defined in its largest real domain by the equation

$$y=\arccos\left[\frac{a\cos x+b}{a+b\cos x}\right],$$

where a and b are constants such $a>b>0$.

Show that y increases with x at a rate which lies between

$$\sqrt{\frac{a-b}{a+b}} \quad \text{and} \quad \sqrt{\frac{a+b}{a-b}}.$$

You may assume that $\frac{d}{dx}[\arccos x]=-\frac{1}{\sqrt{1-x^2}}$. **(10)**

Question 10

By considering the solution of trigonometric equation

$$\sin(x-30)°=\cos(x-45)°,$$

find, in degrees, the exact value of $\arctan\left[\frac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right]$. **(10)**

Question 11

At time $t = 0$, one litre of a certain liquid chemical is added to a tank containing 20 litres of water. The chemical reacts with the water forming a gas, and as a result of this reaction both the volumes of the water and the chemical are reduced.

At time t minutes since the chemical reaction started, the respective volumes of the chemical and the water used in the reaction, are $(1-v)$ litres and $4(1-v)$ litres.

The rate at which the volume of the chemical in the tank reduces, is proportional to the product of the volume of the chemical and the volume of the water, still left in the tank.

Given that 2 minutes after the reaction started the volume of the chemical remaining is $\frac{4}{19}$ of a litre, show that

$$2^t = \frac{v+4}{5v}.$$

(14)

Question 12

The sum to infinity S of the convergent geometric series is given by

$$S = 1 + x + x^2 + x^3 + x^4 + \dots, \quad |x| < 1,$$

By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty} \left[\frac{1}{r \times 2^r} \right].$$

You may assume that integration and summation commute. **(8)**

Question 13

The distinct points A, B and C lie on the curve with equation

$$xy = p^2,$$

where p is a positive constant.

Given that ABC is a right angle, show that the tangent to the curve at B, is perpendicular to AC.

(10)

Question 14

Sketch the graph of

$$y = x|x-1| - x|x+4|, \quad x \in \mathbb{R}.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of any cusps of the curve.

[*No credit will be given to non analytical sketches based on plotting coordinates*] **(10)**

Question 15

$$I = \int_{-\frac{5}{2}}^{\frac{7}{2}} \frac{4x+1}{\sqrt{35+4x-4x^2}} \, dx.$$

By writing $35+4x-4x^2$ in completed the square form, followed by a suitable trigonometric substitution, show that

$$I = \frac{3}{2}\pi.$$

(10)

Question 16

Solve the following simultaneous equations

$$(91-2x)^3 = 216xy^2 \qquad \text{and} \qquad (37-2y)^3 = 216x^2y,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$. **(12)**

Question 17

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k} [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

By showing a detailed method prove that

$$\prod_{k=1}^{n} \left[\frac{2k-1}{2k+2} \right] = \binom{2n+1}{n} \frac{1}{4^n (2n+1)} .$$ **(12)**

Question 18

A spherical cap of depth a is removed from a sphere of radius na, where n is a positive constant, such that $n > \frac{1}{2}$. The volume of the spherical cap is less than half the volume of the sphere.

The remainder of the sphere is moulded to a right circular cone whose base is equal to that of the circular plane face of the spherical cap removed.

Given that the height of the cone is ma, where m is a positive constant, show that

$$m = (n+p)(2n+q),$$

where p and q are integers to be found. **(13)**

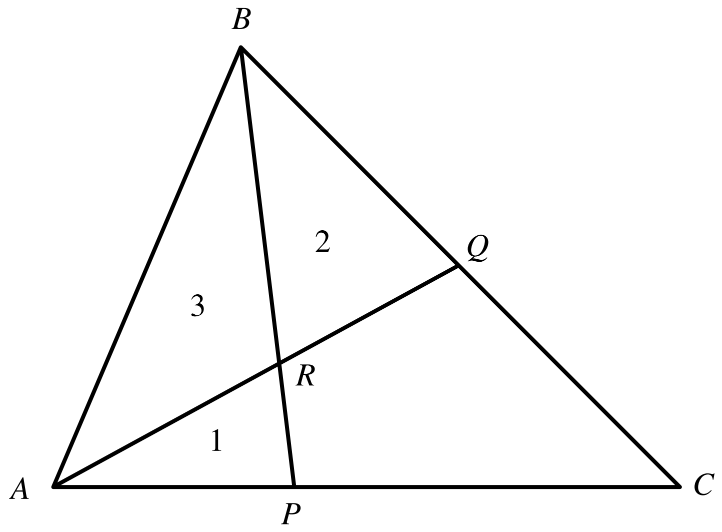
Question 19

Solve the following trigonometric equation.

$$2\cot 2x = \sec x \sec\left(\frac{2\pi}{15}\right) + 2\tan\left(\frac{2\pi}{15}\right), \quad 0 \le x < 2\pi .$$

Give the answers in the form $k\pi$, where k is rational. **(12)**

Question 20



The figure above shows a triangle ABC.

The point P lies on AC and the point Q lies on BC.

The point R is the intersection of BP and AQ.

Given that the respective areas of the triangles APR, BQR and ABR are 1, 2 and 3 square units, determine the exact area of the quadrilateral $CPRQ$.

(8)
