IYGB

Mathematical Methods 4

Practice Paper E

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 7 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 4

- Finding General Solutions of First Order Partial Differential Equations, Without Context.

  Solutions by Transformations or Lagrange's Method.

- Finding General Solutions of Second Order Partial Differential Equations, Without Context.

- Elementary Boundary Value Problems.

- The One Dimensional Wave Equation

  (D'Alembert's Solution in an Infinite Domain)

- The One Dimensional Wave Equation in a Finite Domain.

- The One Dimensional Heat Equation in a Finite Domain.

- The Two Dimensional Laplace Equation in a Finite Domain, in Cartesian or Polar Coordinates.

- Solutions of Partial Differential Equations by Fourier Transforms.

- Solutions of Partial Differential Equations by Laplace Transforms.

Question 1

It is given that $z = z(x, y)$ satisfies the partial differential equation

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z .$$

Given further that $z = y$ at $x = 1$ for all y, find the solution of the above partial differential equation. **(7)**

Question 2

$$(x + y)\frac{\partial z}{\partial x} + (y - x)\frac{\partial z}{\partial y} = 0 .$$

Transform the above partial differential equation using the equations

$$u = \frac{1}{2}\ln\left(x^2 + y^2\right) \quad \text{and} \quad v = \arctan\left(\frac{y}{x}\right) .$$ **(12)**

Question 3

Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad c > 0,$$

for $u = u(x,t)$, $0 \le x \le 1$, $t \ge 0$,

subject to the following boundary and initial conditions.

$$u(0,t) = 0, \quad u(1,t) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = \sin(5\pi x) + 2\sin(7\pi x).$$ **(15)**

Question 4

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \ c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$. **(6)**

It is further given further that

$$F(x) = 0 \quad \text{and} \quad G(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

b) Use the result of part **(a)** with the method of characteristics to determine expressions for

$$z(x,t), \quad \text{for } t = \frac{1}{2c}, \quad t = \frac{1}{c} \quad \text{and } t = \frac{3}{2c}$$ **(10)**

c) Sketch the wave profiles for $t = \frac{n}{2c}$, $n = 0, 1, 2, 3$. **(2)**

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Question 5

$$\theta(x) = 8\sin(2\pi x),\ 0 \le x \le 1$$

The above equation represents the temperature distribution θ °C, maintained along the 1 m length of a thin rod.

At time $t = 0$, the temperature θ is suddenly dropped to $\theta = 0$ °C at both the ends of the rod at $x = 0$, and at $x = 1$, and the source which was previously maintaining the temperature distribution is removed.

The new temperature distribution along the rod $\theta(x,t)$, satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t},\quad 0 \le x \le 1,\quad t \ge 0.$$

Use Laplace transforms to determine an expression for $\theta(x,t)$. **(15)**

Question 6

The temperature $u(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{9}\frac{\partial u}{\partial t},\ t \ge 0,\ 0 \le x \le 2$$

where x is a spatial coordinate and t is time.

It is further given that

$$u(0,t) = 0, \qquad u(2,t) = 8, \qquad u(x,0) = 2x^2$$

Determine an expression for $u(x,t)$. **(17)**

Question 7

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that

$$\varphi(x,0) = f(x)$$

$$\varphi(x, y) \to 0 \text{ as } \sqrt{x^2 + y^2} \to \infty$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2 + y^2} \, du. \qquad \textbf{(16)}$$

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