IYGB

Mathematical Methods 4

Practice Paper D

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 7 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 4

- Finding General Solutions of First Order Partial Differential Equations, Without Context.

  Solutions by Transformations or Lagrange's Method.

- Finding General Solutions of Second Order Partial Differential Equations, Without Context.

- Elementary Boundary Value Problems.

- The One Dimensional Wave Equation

  (D'Alembert's Solution in an Infinite Domain)

- The One Dimensional Wave Equation in a Finite Domain.

- The One Dimensional Heat Equation in a Finite Domain.

- The Two Dimensional Laplace Equation in a Finite Domain, in Cartesian or Polar Coordinates.

- Solutions of Partial Differential Equations by Fourier Transforms.

- Solutions of Partial Differential Equations by Laplace Transforms.

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Question 1

The smooth function $z = z(x, y)$ satisfies

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1.$$

Find the general solution of the above partial differential equation by using the transformation equations

$$x = u^2 + v^2 \quad \text{and} \quad y = u^2 - v^2. \qquad \textbf{(8)}$$

Question 2

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \; c > 0,$$

subject to the initial conditions.

$$z(x,0) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = 0.$$

a) Display the values of $z(x,t)$ in the regions of an (x,t) plane diagram. **(8)**

b) Sketch the wave profiles for $t = 0$, $t = \frac{1}{2c}$, $t = \frac{1}{c}$ and $t = \frac{3}{2c}$. **(7)**

You may use without proof the standard D'Alembert's solution for the wave equation.

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Question 3

The function $u = u(t, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y\,,\ t \geq 0\,,\ y > 0\,,$$

subject to the following conditions

i. $u(0, y) = 1 + y^2\,,\ y > 0$

ii. $u(t,0) = 1\,,\ t \geq 0$

Use Laplace transforms in t to show that

$$u(t, y) = 1 + y - y\,\mathrm{e}^{-t} + y^2\,\mathrm{e}^{-2t}\,.$$ **(13)**

Question 4

The surface S, with equation $z = z(x, y)$, satisfies the partial differential equation

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} + xy = 0\,.$$

S contains the curve with equation

$$xy = 1\,,\ z = x\,,\ \forall x\ .$$

Find a Cartesian equation of S, in the form $z = f(x, y)$. **(10)**

Question 5

A thin rod of length 2 m has temperature $z = 20$ °C throughout its length.

At time $t = 0$, the temperature z is suddenly dropped to $z = 0$ °C at both its ends at $x = 0$, and at $x = 2$.

The temperature distribution along the rod $z(x,t)$, satisfies the standard heat equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}, \quad 0 \le x \le 2, \quad t \ge 0.$$

Assuming the rod is insulated along its length, determine an expression for $z(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

(13)

Question 6

A taut string of length 2 units is fixed at its endpoints at $x = \pm 1$ and rests in a horizontal position along the x axis.

At time $t = 0$, while the string is undisturbed, it is given a small transverse velocity $1 - x^2$ along its length. It is assumed that the displacement of the string

$$u(x,t), \ |x| \le 1, \ t \ge 0$$

satisfies a standard wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2},$$

Show that

$$u(x,t) = \frac{32}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{(2n-1)^4}\cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]\right],$$

and hence determine of the normal modes of the vibration of the string. **(17)**

Question 7

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0.$$

a) Derive the general solution of the above equation, in variable separable form **(10)**

The functions $\Phi_1 = \Phi_1(r,\theta)$, $\Phi_2 = \Phi_2(r,\theta)$ and $\Phi_3 = \Phi_3(r,\theta)$ satisfy

$$\nabla^2 \Phi_1 = 0, \ r > 2$$

$$\nabla^2 \Phi_2 = 0, \ 1 < r < 2$$

$$\nabla^2 \Phi_3 = 0, \ 0 < r < 1.$$

It is further given that

- $\lim_{r \to \infty} \left[\Phi_1(r,\theta) - r\cos\theta \right] = 0.$
- $\frac{\partial \Phi_1}{\partial r}(2,\theta) = \frac{\partial \Phi_2}{\partial r}(2,\theta) = 2\cos\theta.$
- $\Phi_2(1,\theta) = \Phi_3(1,\theta) = \cos\theta.$
- $\lim_{r \to 0} \left[r\frac{\partial \Phi_3}{\partial r}(r,\theta) \right] = 1.$

b) Determine expressions for Φ_1, Φ_2 and Φ_3. **(14)**

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