IYGB

Mathematical Methods 4

Practice Paper B

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 7 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 4

- Finding General Solutions of First Order Partial Differential Equations, Without Context.

  Solutions by Transformations or Lagrange's Method.

- Finding General Solutions of Second Order Partial Differential Equations, Without Context.

- Elementary Boundary Value Problems.

- The One Dimensional Wave Equation

  (D'Alembert's Solution in an Infinite Domain)

- The One Dimensional Wave Equation in a Finite Domain.

- The One Dimensional Heat Equation in a Finite Domain.

- The Two Dimensional Laplace Equation in a Finite Domain, in Cartesian or Polar Coordinates.

- Solutions of Partial Differential Equations by Fourier Transforms.

- Solutions of Partial Differential Equations by Laplace Transforms.

Question 1

It is given that $\psi = \psi(x, y)$ satisfies the partial differential equation

$$3\frac{\partial \psi}{\partial x} - 4\frac{\partial \psi}{\partial y} = x^2 .$$

Use the transformation equations

$$\xi = Ax + By \quad \text{and} \quad \eta = Cx + Dy , \quad AD - BC \neq 0$$

with suitable values of A, B, C and D, in order to determine a general solution of the above partial differential equation. **(7)**

Question 2

The surface S, with equation $z = z(x, y)$, satisfies the partial differential equation

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0 .$$

The plane with equation $z = 1$ meets S on the curve with equation $xy = x + y$.

Find a Cartesian equation of S, in the form $z = f(x, y)$. **(11)**

Question 3

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$. **(7)**

It is further given further that

$$F(x) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad G(x) = 0.$$

b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$. **(7)**

c) Sketch the wave profiles for $t = 0$ and $t = \frac{2}{c}$. **(3)**

Question 4

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x,0) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{k} \mathrm{e}^{-ky} \sin k \cos kx \ dk,$$

and hence deduce the value of $\varphi(\pm 1, 0)$. **(12)**

Question 5

A taut string of length L is fixed at its endpoints at $x = 0$ and at $x = L$, and rests in a horizontal position along the x axis. The midpoint of the string is pulled by a small distance h and released from rest.

If the vertical displacement of the string z satisfies the standard wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2}, \ c > 0,$$

show that

$$z(x,t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{(2n-1)^2} \sin\left[\frac{(2n-1)\pi x}{L}\right] \cos\left[\frac{(2n-1)\pi ct}{L}\right] \right].$$

(12)

Question 6

The temperature $\theta = \theta(x, y)$ for a **steady** two-dimensional heat flow in the semi-infinite region for which $y \geq 0$ satisfies Laplace's Equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0,$$

subject to the boundary conditions

- $\frac{\partial \theta}{\partial x}(0, y) = 0, \quad y \in [0, \infty)$
- $\frac{\partial \theta}{\partial x}(L, y) = 0, \quad y \in [L, \infty)$
- $\lim_{y \to \infty} [\theta(x, y)] \leq |M|, \; M \in \mathbb{R}, \quad x \in (-\infty, \infty), \; y \in [x, \infty)$
- $\theta(x, 0) = \frac{x(L-x)}{L^2}, \quad x \in (-\infty, \infty), \quad f(-x) = f(x), \quad f(x) = f(x + 2L)$

Solve the equation and hence show that

$$\theta\left(\tfrac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \exp\left[-\frac{2n\pi y}{L}\right]}{n^2}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

(20)

Question 7

The one dimensional heat equation for the temperature, $T(x,t)$, satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}, \quad t \geq 0,$$

where t is the time, x is a spatial dimension and σ is a positive constant.

The temperature $T(x,t)$ is subject to the following conditions.

i. $\lim_{x \to \infty} \left[T(x,t) \right] = 0$

ii. $T(0,t) = 1$

iii. $T(x,0) = 0$

a) Use Laplace transforms to show that

$$\mathcal{L}\left[T(x,t)\right] = \bar{T}(x,s) = \frac{1}{s}\exp\left[-\sqrt{\frac{s}{\sigma}}\, x\right]. \qquad \textbf{(5)}$$

b) Use contour integration on the Laplace transformed temperature gradient $\frac{\partial}{\partial x}\left[\bar{T}(x,s)\right]$ to show further that

$$T(x,t) = 1 - \text{erf}\left[\frac{x}{\sqrt{4\sigma t}}\right]. \qquad \textbf{(16)}$$

You may assume without proof that

- $\displaystyle\int_0^{\infty} e^{-ax^2} \cos kx \, dx = \sqrt{\frac{\pi}{4a}}\exp\left[-\frac{k^2}{4a}\right]$
- $\displaystyle\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-\xi^2} \, d\xi$