IYGB - MATHEMATICAL METHODS 3 - PAPER D - QUESTION 1

BY THE DEFINITION OF THE FOURIER TRANSFORM

$$\boxed{f(x) = xe^{-2x}, \quad x > 0}$$

$$\Longrightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

$$\Longrightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} xe^{-2x} e^{-ikx} \, dx$$

$$\Longrightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{(-2-ik)x} \, dx$$

PROCEED BY INTEGRATION BY PARTS (IGNORING THE $\frac{1}{\sqrt{2\pi}}$)

$$\Longrightarrow \hat{f}(k) = \cancel{\left[\frac{xe^{(-2-ik)x}}{-2-ik} \right]_0^{\infty}} - \int_0^{\infty} \frac{e^{(-2-ik)x}}{-2-ik} \, dx$$

x	1
$\frac{e^{(-2-ik)x}}{-2-ik}$	$e^{(-2-ik)x}$

$$\Longrightarrow \hat{f}(k) = \frac{-1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{(-2-ik)x}}{-2-ik} \, dx$$

$$\Longrightarrow \hat{f}(k) = \frac{-1}{\sqrt{2\pi}(-2-ik)^2} \left[e^{(-2-ik)x} \right]_0^{\infty}$$

$$\Longrightarrow \hat{f}(k) = \frac{-1}{\sqrt{2\pi}(2+ik)^2} \left[0 - 1 \right]$$

$$\Longrightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}(2+ik)^2}$$

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STARTING WITH THE GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS, BY REPLACING x WITH $-x$

$$\Rightarrow (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

$$\Rightarrow (1+2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(-x)\right]$$

NEXT WE REPLACE t WITH $-t$

$$\Rightarrow (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[(-t)^n P_n(-x)\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[t^n P_n(x)\right] = \sum_{n=0}^{\infty}\left[(-1)^n t^n P_n(-x)\right]$$

EQUATE COEFFICIENTS OF t^n IN BOTH SERIES

$$\Rightarrow P_n(x) = (-1)^n P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = (-1)^n(-1)^n P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = (-1)^{2n} P_n(-x)$$

$$\Rightarrow (-1)^n P_n(x) = P_n(-x)$$

$$\therefore \quad P_n(-x) = (-1)^n P_n(x)$$

AS REQUIRED

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IT BEST TO FIND THE RESIDUE BY EXPANSION FOR $f(z) = \dfrac{\cot z \coth z}{z^3}$

$$f(z) = \frac{1}{z^3} \times \frac{\cos z}{\sin z} \times \frac{\cosh z}{\sinh z}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{z + \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)}$$

$$= \frac{1}{z^5} \times \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^5} \times \frac{1 + \cancel{\frac{z^2}{2}} + \frac{z^4}{24} - \cancel{\frac{z^2}{2}} - \frac{z^4}{4} + \frac{z^4}{24} + O(z^6)}{1 + \cancel{\frac{z^2}{6}} + \frac{z^4}{120} - \cancel{\frac{z^2}{6}} - \frac{z^4}{36} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^5} \times \frac{1 - \frac{1}{6}z^4 + O(z^6)}{1 - \frac{1}{90}z^4 + O(z^6)}$$

REWRITE IN ORDER TO COMPLETE THE EXPANSION

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + O(z^6)\right]\left[1 - \frac{1}{90}z^4 + O(z^6)\right]^{-1}$$

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + O(z^6)\right]\left[1 + \frac{1}{90}z^4 + O(z^6)\right]$$

$$= \frac{1}{z^5}\left[1 + \frac{1}{90}z^4 - \frac{1}{6}z^4 - \frac{1}{540}z^8 + O(z^{10})\right]$$

$$= \frac{1}{z^5}\left[1 - \frac{7}{45}z^4 + O(z^8)\right]$$

$$= \frac{1}{z^5} - \frac{7}{45z} + O(z^3)$$

$\therefore$ RESIDUE IS $-\frac{7}{45}$

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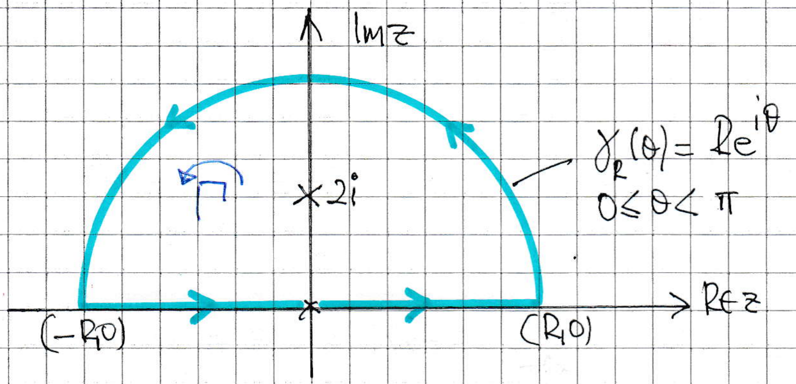
CONSIDER $\displaystyle\int_{\Gamma} \frac{1}{(z^2+4)^2}\,dz$, OVER THE SEMICIRCULAR CONTOUR Γ, SHOWN BELOW



$$\boxed{\begin{aligned} z &= Re^{i\theta} \\ dz &= iRe^{i\theta}\,d\theta \end{aligned}}$$

THE INTEGRAND HAS DOUBLE POLES AT $z = \pm 2i$; ONLY THE ONE AT $+2i$ IS INSIDE Γ — CALCULATE THE RESIDUE OF THIS POLE

$$\lim_{z \to 2i}\left[\frac{d}{dz}\left[(z-2i)^2 f(z)\right]\right] = \lim_{z \to 2i}\left[\frac{d}{dz}\left[\cancel{(z-2i)^2} \times \frac{1}{\cancel{(z-2i)^2}(z+2i)^2}\right]\right]$$

$$= \lim_{z \to 2i}\left[\frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]\right] = \lim_{z \to 2i}\left[-\frac{2}{(z+2i)^3}\right]$$

$$= -\frac{2}{(4i)^3} = -\frac{2}{-64i} = \frac{1}{32i}$$

BY RESIDUE THEOREM

$$\Rightarrow \int_{\Gamma} \frac{1}{(z^2+4)^2}\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \int_{\gamma} \frac{1}{(z^2+4)^2}\,dz + \int_{-R}^{R} \frac{1}{(z^2+4)^2}\,dz = 2\pi i \times \frac{1}{32i}$$

$$\Rightarrow \int_0^{\pi} \frac{1}{\left[(Re^{i\theta})^2+4\right]^2}\left(iRe^{i\theta}\,d\theta\right) + \int_{-R}^{R} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

$$\Rightarrow \int_0^{\pi} \frac{iRe^{i\theta}}{(R^2e^{2i\theta}+4)^2}\,d\theta + \int_{-R}^{R} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

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NEXT CONSIDER THE CONTRIBUTION ALONG γ, AS $R \to \infty$

$$\left| \int_0^{\pi} \frac{iRe^{i\theta}}{(R^2e^{2i\theta}+4)^2}\,d\theta \right| = \left| \int_0^{\pi} \frac{iRe^{i\theta}}{R^4e^{4i\theta}+8R^2e^{2i\theta}+16}\,d\theta \right|$$

$$\leqslant \int_0^{\pi} \left| \frac{iRe^{i\theta}}{R^4e^{4i\theta}+8R^2e^{2i\theta}+16} \right| d\theta = \int_0^{\pi} \frac{|iRe^{i\theta}|}{|R^4e^{4i\theta}+8R^2e^{2i\theta}+16|}\,d\theta$$

$$|z \pm w| \geqslant \big||z|-|w|\big|$$

$$\frac{1}{|z \pm w|} \leqslant \frac{1}{\big||z|-|w|\big|}$$

$$\frac{1}{|z \pm w \pm v|} \leqslant \frac{1}{\big||z|-|w|-|v|\big|}$$

$$= \int_0^{\pi} \frac{|i||R||e^{i\theta}|}{\big||R^4e^{4i\theta}|-|8R^2e^{2i\theta}|-|16|\big|}\,d\theta$$

$$= \int_0^{\pi} \frac{1 \times R \times 1}{\big||R|^4|e^{4i\theta}|-|8R^2||e^{2i\theta}|-|16|\big|}\,d\theta$$

$$= \int_0^{\pi} \frac{R}{\big||R^4 \times 1|-|8R^2 \times 1|-16\big|}\,d\theta = \int_0^{\pi} \frac{R}{\big||R^4|-8|R^2|-16\big|}\,d\theta$$

$$= \frac{R}{\big||R^4|-8|R^2|-16\big|} \int_0^{\pi} 1\,d\theta = \frac{\pi R}{\big||R^4|-8|R^2|-16\big|} = O\left(\frac{1}{R^3}\right)$$

WHICH VANISHES AS $R \to \infty$

HENCE AS $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

EVEN INTEGRAND

$$2\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

$$\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{32}$$

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START BY USING THE RULE

$$\boxed{\mathcal{L}[t\,g(t)] = -\frac{d}{ds}\left(\bar{g}(s)\right)}$$

$$\Rightarrow \mathcal{L}[t\,f(t)] = \frac{1}{s^3+s}$$

$$\Rightarrow -\frac{d}{ds}\left(\bar{f}(s)\right) = \frac{1}{s(s^2+1)}$$

$$\Rightarrow -\bar{f}(s) = \int \frac{1}{s(s^2+1)}\,ds$$

PROCEED BY PARTIAL FRACTIONS – EASY TO GUESS BY INSPECTION

$$\Rightarrow -\bar{f}(s) = \int \frac{1}{s} - \frac{s}{s^2+1}\,ds$$

$$\Rightarrow \bar{f}(s) = \int \frac{s}{s^2+1} - \frac{1}{s}\,ds$$

$$\Rightarrow \bar{f}(s) = \frac{1}{2}\ln(s^2+1) - \ln s + \cancel{C}$$ NOT POSSIBLE

$$\Rightarrow \bar{f}(s) = \frac{1}{2}\left[\ln(s^2+1) - 2\ln s\right]$$

$$\Rightarrow \bar{f}(s) = \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)$$

THERE IS NO NEED TO FIND THE $f(t)$ – PROCEED AS FOLLOWS

$$\mathcal{L}[g(at)] = \frac{1}{a}\,\bar{g}\left(\frac{s}{a}\right)$$

$$\Rightarrow \bar{f}(s) = \mathcal{L}[f(t)] = \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)$$

$$\Rightarrow \mathcal{L}[f(2t)] = \frac{1}{2} \times \frac{1}{2}\ln\left[\frac{\left(\frac{s}{2}\right)^2+1}{\left(\frac{s}{2}\right)^2}\right]$$

$$\Rightarrow \mathcal{L}[f(2t)] = \frac{1}{4}\ln\left[\frac{\frac{s^2}{4}+1}{\frac{s^2}{4}}\right]$$

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$$\Rightarrow \mathcal{L}[f(2t)] = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

FINALLY WE CAN APPLY ANOTHER RULE

$$\boxed{\mathcal{L}\left[e^{-bt}g(t)\right] = \bar{g}(s+a)}$$

$$\Rightarrow \mathcal{L}\left[e^{-t}f(2t)\right] = \frac{1}{4}\ln\left[\frac{(s+1)^2+4}{s^2}\right]$$

$$\Rightarrow \mathcal{L}\left[e^{-t}f(2t)\right] = \frac{1}{4}\ln\left[\frac{s^2+2s+5}{s^2}\right]$$

OR

$$\Rightarrow \mathcal{L}\left[e^{-t}f(2t)\right] = \frac{1}{2}\ln\left[\frac{\sqrt{s^2+2s+5}}{s}\right]$$

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START WITH A SUBSTITUTION

$t = \tan x$ & THE LIMITS

$dt = \sec^2 x \, dx$ $\quad x = 0 \longmapsto t = 0$

$dx = \dfrac{dt}{\sec^2 x}$ $\quad x = \frac{\pi}{2} \longmapsto t = \infty$

$dx = \cos^2 x \, dt$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{(\cos x + \sin x)^2} \, dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x \left(1 + \frac{\sin x}{\cos x}\right)^2} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x (1 + \tan x)^2} \, dx$$

$$= \int_0^{\infty} \frac{t^{\frac{1}{2}}}{\cancel{\cos^2 x}(1+t)^2} \left(\cancel{\cos^2 x} \, dt\right)$$

$$= \int_0^{\infty} \frac{t^{\frac{1}{2}}}{(1+t)^2} \, dt$$

NOW USING AN ALTERNATIVE DEFINITION OF THE BETA FUNCTION

$$B(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} \, dt$$

AND BY REWRITING THE ABOVE INTEGRAL TO "FIT" THIS DEFINITION

$$= \int_0^{\infty} \frac{t^{\frac{3}{2}-1}}{(1+t)^{\frac{3}{2}+\frac{1}{2}}} \, dt = B\left(\tfrac{3}{2}, \tfrac{1}{2}\right)$$

SWITCHING INTO GAMMA FUNCTIONS

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1!} = \frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1} = \frac{\pi}{2}$$

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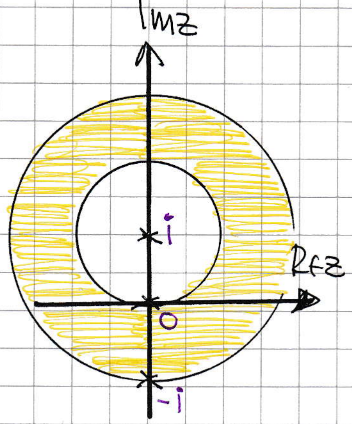
PROCEED BY PARTIAL FRACTIONS (COVER UP)

$$f(z) = \frac{5z+3i}{z(z+i)} = \frac{\frac{3i}{i}}{z} + \frac{\frac{-2i}{-i}}{z+i} = \frac{3}{z} + \frac{2}{z+i}$$

THE SINGULARITIES OF $f(z)$ ARE SHOWN OPPOSITE, AT THE ORIGIN & AT $-i$



EXPANDING $\frac{3}{z}$ FOR $|z-i|>1$

$$\frac{3}{z} = \frac{3}{i+(z-i)} = \frac{3}{z-i}\left(\frac{1}{1+\frac{i}{z-i}}\right)$$

$$= \frac{3}{z-i}\left[1+\frac{i}{z-i}\right]^{-1} \leftarrow \left|\frac{i}{z-i}\right|<1$$

$$= \frac{3}{z-i}\left[1 - \frac{i}{z-i} + \frac{i^2}{(z-i)^2} - \frac{i^3}{(z-i)^3} + \frac{i^4}{(z-i)^4} - \dots\right]$$

$$= \frac{3}{z-i}\sum_{r=0}^{\infty}\left[\frac{i^r}{(z-i)^r}(-1)^r\right] = 3\sum_{r=0}\frac{(-i)^r}{(z-i)^{r+1}}$$

EXPANDING $\frac{2}{z+i}$ FOR $|z-i|<2$

$$\frac{2}{z+i} = \frac{2}{2i+(z-i)} = \frac{2}{2i\left(1+\frac{z-i}{2i}\right)} = \frac{1}{i}\left(1+\frac{z-i}{2i}\right)^{-1}$$

$$\left|\frac{z-i}{2i}\right|<1$$

$$= -i\left[1 - \frac{z-i}{2i} + \frac{(z-i)}{(2i)^2} - \frac{(z-i)^3}{(2i)^3} + \dots\right]$$

$$= -i\sum_{r=1}^{\infty}\left(\frac{z-i}{2i}\right)^r(-1)^r$$

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ADDING THE EXPANSIONS FOR $1 < |z-i| < 2$

$$f(z) = \sum_{r=0}^{\infty} \frac{3(-i)^r}{(z-i)^{r+1}} - i \sum_{r=0}^{\infty} \left(\frac{z-i}{-2i}\right)^r$$

NEGATIVE POWERS

POSITIVE POWERS.

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START BY MANIPULATING THE GIVEN SERIES

$$\Longrightarrow J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(r+n)!}\left(\frac{x}{2}\right)^{2r+n}\right]$$

$$\Longrightarrow J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(r+n)!}\ \frac{x^{2r}\times x^n}{2^{2r}\times 2^n}\right]$$

$$\Longrightarrow J_n(x) = \frac{x^n}{2^n}\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(r+n)!}\left(\frac{x^2}{4}\right)^r\right]$$

DIVIDE THROUGH BY x^n & WRITE OUT THE FIRST FEW TERMS OF THE SERIES

$$\Longrightarrow \frac{J_n(x)}{x^n} = \frac{1}{2^n}\left[\frac{1}{n!} - \frac{1}{1!(n+1)!}\left(\frac{x^2}{4}\right) + \frac{1}{2!(n+2)!}\left(\frac{x^4}{16}\right) - \frac{1}{3!(n+3)!}\left(\frac{x^6}{64}\right) + \dots\right]$$

TAKING LIMITS AS $x \to 0$ IN THE ABOVE EQUATION YIELDS

$$\lim_{x\to 0}\left[\frac{J_n(x)}{x^n}\right] = \frac{1}{2^n n!}$$

AS REQUIRED

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a) START BY NOTING THAT $\frac{d}{dx}(a^x) = a^x \ln a$

$$\int_0^\infty e^{-\lambda t} t^{x-1} \ln t \, dt = \int_0^\infty \frac{\partial}{\partial x}\left[e^{-\lambda t} t^{x-1}\right] dt$$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-\lambda t} t^{x-1} \, dt$$

AS THIS ALMOST LOOKS LIKE A GAMMA FUNCTION WE USE A SIMPLE LINEAR SUBSTITUTION

$$\boxed{\begin{array}{l} u = \lambda t \\ \frac{du}{dt} = \lambda \\ dt = \frac{1}{\lambda} du \\ \text{LIMITS UNCHANGED} \end{array}}$$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-u}\left(\frac{u}{\lambda}\right)^{x-1}\left(\frac{1}{\lambda}du\right)$$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-u} u^{x-1} \times \frac{1}{\lambda^x} \, du$$

$$= \frac{\partial}{\partial x}\left[\lambda^{-x}\int_0^\infty e^{-u} u^{x-1} \, du\right]$$

$$= \frac{\partial}{\partial x}\left[\lambda^{-x}\Gamma(x)\right]$$

DIFFERENTIATING W.R.T. x, USING PRODUCT RULE

$$= \lambda^{-x}(-1)\ln\lambda \, \Gamma(x) + \lambda^{-x}\Gamma'(x)$$

$$= \lambda^{-x}\left[\Gamma'(x) - \Gamma(x)\ln\lambda\right]$$ // AS REQUIRED

b) START BY COMPUTING SIMPLIFIED EXPRESSIONS, IN TERMS OF γ, FOR THE DERIVATIVES OF GAMMA FOR $x = 1, 2, 3$

USING $\boxed{\Gamma'(x) = \Gamma(x)\left[-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right]}$

- $\Gamma'(1) = \Gamma(1)\left[-\gamma - 1 + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right]$

$$= 0!\left[-\gamma - 1 + 1 \cancel{-\frac{1}{2}} \cancel{+\frac{1}{2}} \cancel{-\frac{1}{3}} \cancel{+\frac{1}{3}} \cancel{-\frac{1}{4}} + \dots\right]$$

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$$= 1\left[-\gamma - 1 + 1\right] = -\gamma$$

- $\Gamma'(2) = \Gamma(2)\left[-\gamma - \frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right)\right]$

$$= 1!\left[-\gamma - \frac{1}{2} + 1 + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \dots - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \dots\right]$$

$$= 1\left[-\gamma - \cancel{\frac{1}{2}} + 1 + \cancel{\frac{1}{2}}\right] = 1 - \gamma$$

- $\Gamma'(3) = \Gamma(3)\left[-\gamma - \frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+3}\right]$

$$= 2!\left[-\gamma - \frac{1}{3} + 1 + \frac{1}{2} + \frac{1}{3} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \cancel{\frac{1}{6}} + \dots - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} - \dots\right]$$

$$= 2\left[-\gamma - \cancel{\frac{1}{3}} + 1 + \frac{1}{2} + \cancel{\frac{1}{3}}\right] = 3 - 2\gamma$$

FINALLY WE OBTAIN

$$I(\lambda,1) = \lambda^{-1}\left[\Gamma'(1) - \Gamma(1)\ln\lambda\right] = \frac{1}{\lambda}\left[-\gamma - 0!\ln\lambda\right] = -\frac{1}{\lambda}(\gamma + \ln\lambda)$$

$$I(\lambda,2) = \lambda^{-2}\left[\Gamma'(2) - \Gamma(2)\ln\lambda\right] = \frac{1}{\lambda^2}\left[(1-\gamma) - 1!\ln\lambda\right] = \frac{1}{\lambda^2}(1 - \gamma - \ln\lambda)$$

$$I(\lambda,3) = \lambda^{-3}\left[\Gamma'(3) - \Gamma(3)\ln\lambda\right] = \frac{1}{\lambda^3}\left[(3-2\gamma) - 2!\ln\lambda\right] = \frac{1}{\lambda^3}\left[3 - 2\gamma - 2\ln\lambda\right]$$

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ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty} a_r x^{r+k}$, $a_0 \neq 0$, $k \in \mathbb{R}$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (r+k) x^{r+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (r+k)(r+k-1) x^{r+k-2}$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r(r+k)(r+k-1)x^{r+k-1} - \sum_{r=0}^{\infty} 3a_r(r+k)x^{r+k-1} + \sum_{r=0}^{\infty} a_r x^{r+k} = 0$$

WHEN $r=0$, THE LOWEST POWER OF x IN THESE SUMMATIONS IS x^{k-1} AND THE HIGHEST IS x^k, SO WE PULL x^{k-1} OUT OF THE SUMMATIONS IN ORDER TO FORM AN INDICIAL EQUATION

$$\Rightarrow a_0 k(k-1)x^{k-1} + \sum_{r=1}^{\infty} a_r(r+k)(r+k-1)x^{r+k-1} - 3a_0 k x^{k-1} - \sum_{r=1}^{\infty} 3a_r(r+k)x^{r+k-1} + \sum_{r=0}^{\infty} a_r x^{r+k} = 0$$

$$\Rightarrow \underbrace{[k(k-1)-3k]a_0 x^{k-1}}_{\text{INDICIAL EQUATION}} + \underbrace{\sum_{r=1}^{\infty} a_r(r+k)(r+k-1)x^{r+k-1} - \sum_{r=1}^{\infty} 3a_r(r+k)x^{r+k-1} + \sum_{r=0}^{\infty} a_r x^{r+k}}_{\text{ADJUST THE SUMMATIONS SO THEY ALL START FROM } r=0} = 0$$

INDICIAL EQUATION

$$k^2-k-3k=0$$
$$k^2-4k=0$$
$$k(k-4)=0$$
$$k = \begin{cases} 0 \\ 4 \end{cases}$$

DISTINCT SOLUTIONS, WHICH DIFFER BY AN INTEGER AND THERE ARE NO SPARE COEFFICIENTS IN ORDER TO SEE IF ANY OF THEM IS UNDETERMINED

ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty} a_{r+1}(r+1+k)(r+k)x^{r+k} - \sum_{r=0}^{\infty} 3a_{r+1}(r+k+1)x^{r+k} + \sum_{r=0}^{\infty} a_r x^{r+k} = 0$$

$$\sum_{r=0}^{\infty} \Big[\big[(r+k)(r+k+1) - 3(r+k+1)\big]a_{r+1} + a_r\Big]x^{r+k} = 0$$

$$\sum_{r=0}^{\infty} \Big[(r+k+1)(r+k-3)a_{r+1} + a_r\Big]x^{r+k} = 0$$

$$(r+k+1)(r+k-3)a_{r+1} = -a_r$$

$$\boxed{a_{r+1} = -\frac{1}{(r+k+1)(r+k-3)}a_r}$$

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NOW IF $k=0$ THE RECURRENCE RELATION FAILS TO PRODUCE A VALUE FOR a_4

(SINCE $a_4 = \frac{-a_3}{4\times 0}$)

HOWEVER IF $k=4$ WE HAVE

$$\boxed{a_{r+1} = -\frac{a_r}{(r+5)(r+1)}}$$

- $r=0$ $\quad a_1 = -\frac{a_0}{5\times 1}$
- $r=1$ $\quad a_2 = -\frac{a_1}{6\times 2} = \frac{a_0}{(6\times 5)(2\times 1)}$
- $r=2$ $\quad a_3 = -\frac{a_2}{7\times 3} = -\frac{a_0}{(7\times 6\times 5)(3\times 2\times 1)}$
- $r=3$ $\quad a_4 = -\frac{a_3}{8\times 4} = \frac{a_0}{(8\times 6\times 5\times 4)(4\times 3\times 2\times 1)}$

E.T.C.

HENCE WE NOW HAVE

$$y_1 = x^k\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\right]$$

$$y_1 = x^4\left[a_0 - \frac{a_0x}{5\times 1} + \frac{a_0x^2}{(6\times 5)(2\times 1)} - \frac{a_0x^3}{(7\times 6\times 5)(3\times 2\times 1)} + \frac{a_0x^4}{(8\times 6\times 5\times 4)(4\times 3\times 2\times 1)} - \dots\right]$$

LOOKING FOR A PATTERN, SAY BY CLOSELY EXAMINING THE 5^{TH} TERM, IGNORING a_0 & $\pm$

$$\frac{x^4}{(8\times 7\times 6\times 5)(4\times 3\times 2\times 1)} = \frac{4\times 3\times 2\ x^4}{(8\times 7\times 6\times 5\times 4\times 3\times 2)(4\times 3\times 2\times 1)} = \frac{24x^{\overset{f(r)}{4}}}{\underset{f(r)}{8!}\ \underset{f(r)}{4!}}$$

$$\therefore y_1 = x^4\sum_{r=0}^{\infty}\frac{a_0(-1)^r x^r\times 24}{(r+4)!\ r!} = 24a_0x^4\sum_{r=0}^{\infty}\frac{(-x)^r}{r!(r+4)!}$$

$$y_1 = A\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!}$$

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TO GET A SECOND INDEPENDENT SOLUTION WE RETURN TO THE RECURRENCE RELATION BEFORE WE SUBSTITUTED $k=4$

$$a_{r+1} = -\frac{a_r}{(r+k+1)(r+k-3)}$$

$$a_1 = -\frac{a_0}{(k+1)(k-3)}$$

$$a_2 = -\frac{a_1}{(k+2)(k-2)} = \frac{a_0}{(k+2)(k+1)(k-3)(k-2)}$$

$$a_3 = -\frac{a_2}{(k+3)(k-1)} = -\frac{a_0}{(k+3)(k+2)(k+1)(k-3)(k-2)(k-1)}$$

$$a_4 = -\frac{a_3}{(k+4)k} = \frac{a_0}{(k+4)(k+3)(k+2)(k+1)(k-3)(k-2)(k-1)k} \quad \text{ETC.}$$

BECAUSE OF THE PROBLEM WITH $k=0$, MULTIPLY EACH COEFFICIENT BY $(k-0)=k$, BEFORE DIFFERENTIATING WITH RESPECT TO k, AND THEN SUBSTITUTE $k=0$

- $\frac{d}{dk}(a_0 k) = a_0$ EVALUATED AT $k=0$ GIVES a_0 ← CONSTANT TERM

- $\frac{d}{dk}\left(\frac{-a_0 k}{(k+1)(k-3)}\right) = -a_0 \frac{d}{dk}\left(\frac{k}{k^2-2k-3}\right) = -a_0 \frac{k^2-2k-3-k(2k-2)}{(k^2-2k-3)^2}$

  EVALUATED AT $k=0$ GIVES $-a_0 \times \frac{-3}{9} = \frac{1}{3}a_0$ ← x TERM

- $\frac{d}{dk}\left(\frac{a_0 k}{(k+2)(k+1)(k-2)(k-3)}\right) = a_0 \frac{d}{dk}(t)$ WHERE $t = \frac{k}{(k+2)(k+1)(k-2)(k-3)}$

$$\ln t = \ln k - \ln(k+2) - \ln(k+1) - \ln(k-2) - \ln(k-3)$$

$$\frac{1}{t}\frac{dt}{dk} = \frac{1}{k} - \frac{1}{k+2} - \frac{1}{k+1} - \frac{1}{k-2} - \frac{1}{k-3}$$

$$\frac{dt}{dk} = t\left(\frac{1}{k} - \frac{1}{k+2} - \frac{1}{k+1} - \frac{1}{k-2} - \frac{1}{k-3}\right) = \frac{k}{(k+2)(k+1)(k-2)(k-3)}\left(\frac{1}{k} - \frac{1}{k+2} - \frac{1}{k+1} - \frac{1}{k-2} - \frac{1}{k-3}\right)$$

$$\frac{dt}{dk} = \frac{1}{(k+2)(k+1)(k-2)(k-3)} - \frac{k}{(k+2)(k+1)(k-2)(k-3)}\left(\frac{1}{k+2} + \frac{1}{k+1} + \frac{1}{k-2} + \frac{1}{k-3}\right)$$

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$$\left.\frac{dt}{dk}\right|_{k=0} = \frac{1}{2\times1\times(-2)(-3)} = \frac{1}{12}$$ ∴ WE OBTAIN $\frac{1}{12}a_0$ ← x^2 TERM

◇ $$\frac{d}{dk}\left[\frac{-a_0k}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}\right] = -a_0\frac{d}{dk}(t) \text{ WITH } t = \frac{k}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}$$

$$\Rightarrow \ln t = \ln k - \ln(k+3) - \ln(k+2) - \ln(k+1) - \ln(k-1) - \ln(k-2) - \ln(k-3)$$

$$\Rightarrow \frac{1}{t}\frac{dt}{dk} = \frac{1}{k} - \frac{1}{k+3} - \frac{1}{k+2} - \frac{1}{k+1} - \frac{1}{k-1} - \frac{1}{k-2} - \frac{1}{k-3}$$

$$\Rightarrow \frac{dt}{dk} = t\left(\frac{1}{k} - \frac{1}{k+3} - \frac{1}{k+2} - \frac{1}{k+1} - \frac{1}{k-1} - \frac{1}{k-2} - \frac{1}{k-3}\right)$$

$$\Rightarrow \frac{dt}{dk} = \frac{1}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)} - \frac{k}{(k+3)(k+2)(k+1)(k-1)(k-2)(k-3)}\left[\frac{1}{k+3} + \frac{1}{k+2} + \frac{1}{k+1} + \frac{1}{k-1} \cdots\right]$$

$$\Rightarrow \left.\frac{dt}{dk}\right|_{k=0} = \frac{1}{3\times2\times1\times(-1)(-2)(-3)} = -\frac{1}{36}$$ ∴ WE OBTAIN $+\frac{1}{36}a_0$ ← x^3 TERM

AND IN A SIMILAR FASHION WE MAY CONTINUE...

THUS THE SECOND INDEPENDENT SOLUTION IS GIVEN BY

$$y_2 = B\left[y_1 \times \ln x + x^k \times (\text{SERIES WHOSE COEFFICIENTS WE JUST FOUND})\right]$$

$$y_2 = B\left[y_1 \ln x + x^0\left(1 + \frac{1}{3}x + \frac{1}{12}x^2 + \frac{1}{36}x^3 + \dots\right)\right]$$

NOTE THAT a_0 WAS ABSORBED INTO B

FINALLY THE GENERAL SOLUTION IS

$$y = y_1 + y_2$$

$$y = A\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!} + B\left[\ln x\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!} + \left(1 + \frac{1}{3}x + \frac{1}{12}x^2 + \frac{1}{36}x^3 + \dots\right)\right]$$

$$y = (A + B\ln x)\sum_{r=0}^{\infty}\frac{(-x)^{r+4}}{r!(r+4)!} + B\left(1 + \frac{1}{3}x + \frac{1}{12}x^2 + \frac{1}{36}x^3 + \dots\right)$$