IYGB

Mathematical Methods 3

Practice Paper D
Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This paper follows the most common syllabi of Mathematical Methods used in the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to ALL questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 10 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 3

- Complex Variables including Residues, Laurent Series and Calculus
- Series Solutions of Differential Equations
- Gamma Functions
- Beta Functions
- Laplace Transform
- Fourier Transform
- Special Functions and Polynomials, including Bessel, Legendre, Chebyshev etc.

Question 1

$$f(x) = x\mathrm{e}^{-2x}, \ x > 0.$$

Find, by direct integration, the Fourier transform of $f(x)$. **(7)**

Question 2

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this relationship to prove that

$$P_n(-x) = (-1)^n P_n(x).$$ **(7)**

Question 3

$$f(z) \equiv \frac{\cot z \coth z}{z^3}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$ at $z = 0$. **(9)**

Question 4

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{\left(x^2+4\right)^2}\, dx.$$ **(11)**

Question 5

It is given that

$$\mathcal{L}\left[t\,f(t)\right] = \frac{1}{s^3+s}, \quad t \geq 0.$$

Determine a simplified expression for

$$\mathcal{L}\left[\mathrm{e}^{-t} f(2t)\right]. \qquad \mathbf{(12)}$$

Question 6

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x + 4\sin^2 x}\,dx. \qquad \mathbf{(10)}$$

Question 7

Determine a Laurent series for

$$f(z) = \frac{5z+3\mathrm{i}}{z(z+\mathrm{i})},$$

which is valid in the annulus $1 < |z-\mathrm{i}| < 2$. **(10)**

Question 8

The Bessel function of the first kind is defined by the series

$$J_n(x) = \sum_{r=0}^{\infty} \left[\frac{(-1)^r}{(n+r)!\,r!} \left(\frac{x}{2} \right)^{2r+n} \right], \quad n \in \mathbb{Z}.$$

Use the above definition to show

$$\lim_{x \to 0} \left[\frac{J_n(x)}{x^n} \right] = \frac{1}{2^n \ n!}, \quad n \in \mathbb{Z}. \qquad (7)$$

Question 9

$$I(\lambda, x) = \int_0^{\infty} e^{-\lambda t} t^{x-1} \ln t \ dt.$$

a) By carrying a suitable differentiation over the integral sign, show that

$$I(\lambda, x) = \lambda^{-x} \left[\Gamma'(x) - \Gamma(x) \ln \lambda \right].$$

b) Find simplified expressions for $I(\lambda, 1)$, $I(\lambda, 2)$ and $I(\lambda, 3)$.

You may assume that $\Gamma'(x) = \Gamma(x) \left[-\gamma + \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k} \right) \right]$. **(12)**

Question 10

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + y = 0. \qquad (15)$$
