IYGB

Mathematical Methods 3

Practice Paper C

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This paper follows the most common syllabi of Mathematical Methods used in the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to ALL questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 11 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 3

- Complex Variables including Residues, Laurent Series and Calculus
- Series Solutions of Differential Equations
- Gamma Functions
- Beta Functions
- Laplace Transform
- Fourier Transform
- Special Functions and Polynomials, including Bessel, Legendre, Chebyshev etc.

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Question 1

Determine a Laurent series for

$$f(z) = \frac{1}{z+4},$$

which is valid for $|z| > 4$. **(4)**

Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} \, d\theta .$$ **(9)**

Question 3

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2} + \left[1 - \frac{1}{2x}\right]\frac{dy}{dx} + \frac{y}{2x^2} = 0 .$$

Give the final answer in simplified Sigma notation. **(12)**

Question 4

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\ln\left(1 + \frac{1}{s^2}\right)\right].$$ **(8)**

Question 5

Prove the validity of Legendre's duplication formula for the Gamma function, which states that

$$\Gamma\left(n+\tfrac{1}{2}\right) \equiv \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}, \quad n \in \mathbb{N}. \qquad \textbf{(7)}$$

Question 6

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

By differentiating the above relationship with respect to t, prove that

$$(2n+1)\,x\,P_n(x)-(n+1)P_{n+1}(x)+n\,P_{n-1}(x)=0. \qquad \textbf{(8)}$$

Question 7

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}\, dx. \qquad \textbf{(8)}$$

Question 8

$$f(z) \equiv \frac{z\,e^{kz}}{z^4+1}, \quad z \in \mathbb{C},\ k \in \mathbb{R},\ k>0.$$

Show that the sum of the residues of the four poles of $f(z)$, is

$$\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right). \qquad \textbf{(10)}$$

Question 9

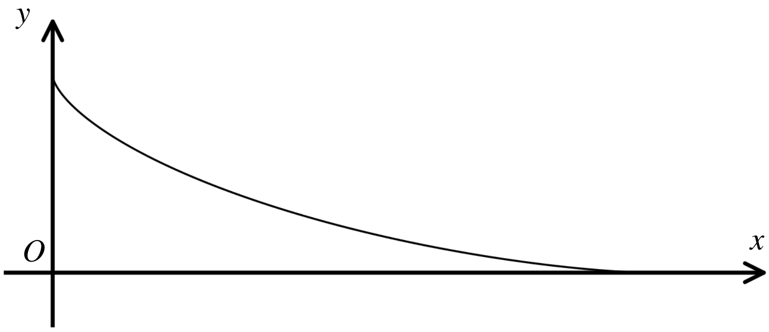
The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \ n \in \mathbb{Z}.$$

Use the generating function relation, to show that

$$J_n(x+y) = \sum_{m=-\infty}^{\infty}\left[J_n(x)\, J_{n-m}(x)\right]. \qquad \textbf{(10)}$$

Question 10



The figure above shows the curve with parametric equations

$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \tfrac{1}{2}\pi.$$

The finite region bounded by the curve and the coordinate axes is revolved fully about the x axis, forming a solid of revolution S.

Determine the x coordinate of the centre of mass of S. **(11)**

Question 11

The function f is defined by

$$f(x) = \frac{1}{\left(x^2 + a^2\right)^2},$$

where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$. **(13)**
